Received 25/03/14

VARIANCE JENSEN TYPE INEQUALITIES FOR GENERAL LEBESGUE INTEGRAL WITH APPLICATIONS

S. S. DRAGOMIR[1,2]

Abstract. Some inequalities similar to Jensen inequalities for general Lebesgue integral are obtained. Applications for functions of selfadjoint operators and functions of unitary operators on complex Hilbert spaces are provided as well.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a $\sigma -$ algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu = 1$. Consider the *Lebesgue space*

$$L(\Omega,\mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)|\, d\mu(t) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(t)\, d\mu(t)$.

Assume that $f, g \in L(\Phi, \mu)$ with $fg \in L(\Phi, \mu)$ and consider the *Čebyšev functional*

$$C(f,g) := \int_\Omega fg d\mu - \int_\Omega f d\mu \int_\Omega g d\mu.$$

It is known that if the function f, g are *synchronous*, i.e.

$$(f(t) - f(s))(g(t) - g(s)) \geq 0$$

for μ-almost every $t, s \in \Omega$, then we have the *Čebyšev inequality*

$$C(f,g) \geq 0. \tag{1.1}$$

If there exists the constants γ, Γ such that $\infty < \gamma \leq f \leq \Gamma < \infty$ μ-almost everywhere on Ω, then we have the following refinement of Grüss' inequality due to Cerone & Dragomir [2], which was obtained for univariate functions of real variable by Cheng & Sun in [3]:

$$\begin{aligned} |C(f,g)| &\leq \frac{1}{2}(\Gamma-\gamma)\int_\Omega \left| g - \int_\Omega g d\mu \right| d\mu \\ &\leq \frac{1}{2}(\Gamma-\gamma)\left[\int_\Omega g^2 d\mu - \left(\int_\Omega g d\mu\right)^2\right]^{1/2}. \end{aligned} \tag{1.2}$$

Key words and phrases. Jensen's inequality, Selfadjoint operators, Unitary operators.

If there exists the constants $\infty < \lambda \leq g \leq \Lambda < \infty$ μ-almost everywhere on Ω, then we have the sequence of inequalities

$$|C(f,g)| \leq \frac{1}{2}(\Gamma-\gamma)\int_{\Omega}\left|g-\int_{\Omega} g d\mu\right| d\mu \tag{1.3}$$
$$\leq \frac{1}{2}(\Gamma-\gamma)\left[\int_{\Omega} g^2 d\mu-\left(\int_{\Omega} g d\mu\right)^2\right]^{1/2} \leq \frac{1}{4}(\Gamma-\gamma)(\Lambda-\lambda).$$

The inequality between the first and last term in (1.3) is known in the literature as *Grüss' inequality.*

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, the author obtained in 2002 [4] the following result:

Theorem 1. *Let $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable convex function on (m,M) and $f:\Omega\to[m,M]$ so that $\Phi\circ f,\ f,\ \Phi'\circ f,\ (\Phi'\circ f)f\in L(\Omega,\mu)$. Then we have the inequality:*

$$0 \leq \int_{\Omega}\Phi\circ f d\mu-\Phi\left(\int_{\Omega} f d\mu\right) \tag{1.4}$$
$$\leq \int_{\Omega}(\Phi'\circ f) f d\mu-\int_{\Omega}\Phi'\circ f d\mu\int_{\Omega} f d\mu$$
$$\leq \frac{1}{2}[\Phi'(M)-\Phi'(m)]\int_{\Omega}\left|f-\int_{\Omega} f d\mu\right| d\mu.$$

In the case of discrete measure, we have:

Corollary 1. *Let $\Phi:[m,M]\to\mathbb{R}$ be a differentiable convex function on (m,M). If $x_i\in[m,M]$ and $w_i\geq 0$ $(i=1,\dots,n)$ with $W_n:=\sum_{i=1}^n w_i=1$, then one has the counterpart of Jensen's weighted discrete inequality:*

$$0 \leq \sum_{i=1}^n w_i\Phi(x_i)-\Phi\left(\sum_{i=1}^n w_i x_i\right) \tag{1.5}$$
$$\leq \sum_{i=1}^n w_i\Phi'(x_i)x_i-\sum_{i=1}^n w_i\Phi'(x_i)\sum_{i=1}^n w_i x_i$$
$$\leq \frac{1}{2}[\Phi'(M)-\Phi'(m)]\sum_{i=1}^n w_i\left|x_i-\sum_{j=1}^n w_j x_j\right|.$$

Remark 1. *We notice that the inequality between the first and the second term in (1.5) was proved in 1994 by Dragomir & Ionescu, see* [34].

On making use of the results (1.4) and (1.3), we can state the following string of reverse inequalities

$$
\begin{aligned}
0 &\le \int_\Omega \Phi\circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \\
&\le \int_\Omega (\Phi'\circ f) f d\mu - \int_\Omega \Phi'\circ f d\mu \int_\Omega f d\mu \\
&\le \frac{1}{2}\left[\Phi'(M) - \Phi'(m)\right]\int_\Omega \left| f - \int_\Omega f d\mu\right| d\mu \\
&\le \frac{1}{2}\left[\Phi'(M) - \Phi'(m)\right]\left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right]^{\frac{1}{2}} \\
&\le \frac{1}{4}\left[\Phi'(M) - \Phi'(m)\right](M-m),
\end{aligned}
\tag{1.6}
$$

provided that $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ is a differentiable convex function on (m, M) and $f : \Omega \to [m, M]$ so that $\Phi\circ f,\ f,\ \Phi'\circ f,\ (\Phi'\circ f) f \in L(\Omega, \mu)$, with $\int_\Omega d\mu = 1$.

The following reverse of the Jensen's inequality also holds [23]:

Theorem 2. *Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, where $\mathring{I}$ is the interior of I. If $f : \Omega \to \mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty < m \le f(t) \le M < \infty \text{ for } \mu\text{-a.e. } t \in \Omega$$

and such that $f,\ \Phi\circ f \in L(\Omega, \mu)$, then

$$
\begin{aligned}
0 &\le \int_\Omega \Phi\circ f d\mu - \Phi\left(\int_\Omega f d\mu\right) \\
&\le \left(M - \int_\Omega f d\mu\right)\left(\int_\Omega f d\mu - m\right)\frac{\Phi'_-(M) - \Phi'_+(m)}{M-m} \\
&\le \frac{1}{4}(M-m)\left[\Phi'_-(M) - \Phi'_+(m)\right],
\end{aligned}
\tag{1.7}
$$

where Φ'_- is the left and Φ'_+ is the right derivative of the convex function Φ.

For other reverses of Jensen inequality and applications to divergence measures see [23], [24] and [25].

Motivated by the above results we establish in this paper some new inequalities for convex functions. Applications for functions of selfadjoint operators and functions of unitary operators on complex Hilbert spaces are provided as well.

2. The Results

The following result holds:

Theorem 3. *Let $\Phi : I \to \mathbb{R}$ be a convex function on the interior $\mathring{I}$ of I and $[0, \infty) \subset \mathring{I}$, $f : \Omega \to \mathbb{C}$ a μ-measurable function and such that $|f|^2$, $\Phi'\circ |f|^2$ and*

$\Phi\circ|f|^2\in L(\Phi,\mu)$. *Then*

$$\left(\int_\Omega |f|^2 d\mu-\left|\int_\Omega f d\mu\right|^2\right)\Phi'_+\left(\left|\int_\Omega f d\mu\right|^2\right) \tag{2.1}$$
$$\le\int_\Omega \Phi\left(|f|^2\right)d\mu-\Phi\left(\left|\int_\Omega f d\mu\right|^2\right)$$
$$\le\int_\Omega \Phi'_-\left(|f|^2\right)\left(|f|^2-\left|\int_\Omega f d\mu\right|^2\right)d\mu$$
$$=C\left(\Phi'_-\left(|f|^2\right),|f|^2\right)+\left(\int_\Omega |f|^2 d\mu-\left|\int_\Omega f d\mu\right|^2\right)\int_\Omega \Phi'_-\left(|f|^2\right)d\mu$$

with

$$C\left(\Phi'_-\left(|f|^2\right),|f|^2\right)=\int_\Omega \Phi'_-\left(|f|^2\right)|f|^2 d\mu-\int_\Omega \Phi'_-\left(|f|^2\right)d\mu\int_\Omega |f|^2 d\mu\ge 0.$$

If there exists the constants M, m such that

$$\infty>M\ge|f|\ge m\ge 0\ \mu\text{-almost everywhere,} \tag{2.2}$$

then

$$C\left(\Phi'_-\left(|f|^2\right),|f|^2\right) \tag{2.3}$$
$$\le\frac{1}{2}\begin{cases}\left(M^2-m^2\right)\int_\Omega\left|\Phi'_-\left(|f|^2\right)-\int_\Omega \Phi'_-\left(|f|^2\right)d\mu\right|d\mu\\ \\ \left[\Phi'_-\left(M^2\right)-\Phi'_-\left(m^2\right)\right]\int_\Omega\left||f|^2-\int_\Omega|f|^2d\mu\right|d\mu\end{cases}$$
$$\le\frac{1}{2}\begin{cases}\left(M^2-m^2\right)\left[\int_\Omega\left[\Phi'_-\left(|f|^2\right)\right]^2d\mu-\left(\int_\Omega \Phi'_-\left(|f|^2\right)d\mu\right)^2\right]^{1/2}\\ \\ \left[\Phi'_-\left(M^2\right)-\Phi'_-\left(m^2\right)\right]\left[\int_\Omega|f|^4d\mu-\left(\int_\Omega|f|^2d\mu\right)^2\right]^{1/2}\end{cases}$$
$$\le\frac{1}{4}\left(M^2-m^2\right)\left[\Phi'_-\left(M^2\right)-\Phi'_-\left(m^2\right)\right].$$

Proof. Since $\Phi:I\to\mathbb{R}$ is a convex function on $\mathring{I}$, then

$$\Phi'_-(y)(y-x)\ge\Phi(y)-\Phi(x)\ge\Phi'_+(x)(y-x) \tag{2.4}$$

for any $x,y\in[0,\infty)$.

Now, by taking $y=|f(t)|^2$ and $x=\left|\int_\Omega f d\mu\right|^2$ we get

$$\Phi'_-\left(|f(t)|^2\right)\left(|f(t)|^2-\left|\int_\Omega f d\mu\right|^2\right) \tag{2.5}$$
$$\ge\Phi\left(|f(t)|^2\right)-\Phi\left(\left|\int_\Omega f d\mu\right|^2\right)$$
$$\ge\Phi'\left(\left|\int_\Omega f d\mu\right|^2\right)\left(|f(t)|^2-\left|\int_\Omega f d\mu\right|^2\right)$$

for any $t \in \Omega$.

If we integrate over $d\mu(t)$ on Ω the inequality (2.5) then we get the first part of (2.1).

Now observe that

$$\begin{aligned}
&\int_\Omega \Phi'_-\left(|f|^2\right)\left(|f|^2 - \left|\int_\Omega f d\mu\right|^2\right) d\mu \\
&= \int_\Omega \left[\Phi'_-\left(|f|^2\right) - \int_\Omega \Phi'_-\left(|f|^2\right) d\mu\right]\left(|f|^2 - \left|\int_\Omega f d\mu\right|^2\right) d\mu \\
&+ \int_\Omega \Phi'_-\left(|f|^2\right) d\mu \left(\left[\int_\Omega |f|^2 d\mu - \left|\int_\Omega f d\mu\right|^2\right]\right).
\end{aligned} \tag{2.6}$$

Since

$$\int_\Omega \left[\Phi'_-\left(|f|^2\right) - \int_\Omega \Phi'_-\left(|f|^2\right) d\mu\right]\left(|f|^2 - \left|\int_\Omega f d\mu\right|^2\right) d\mu = C\left(\Phi'_-\left(|f|^2\right), |f|^2\right)$$

and

$$C\left(\Phi'_-\left(|f|^2\right), |f|^2\right) \ge 0,$$

because the functions $\Phi'_-\left(|f|^2\right), |f|^2$ are synchronous on Ω (due to the fact that Φ'_- is monotonic nondecreasing almost everywhere on $[0,\infty)$), we obtain the last part of (2.1).

Utilising (1.3) we have either

$$\begin{aligned}
C\left(\Phi'_-\left(|f|^2\right), |f|^2\right) &\le \frac{1}{2}\left(M^2 - m^2\right)\int_\Omega \left|\Phi'_-\left(|f|^2\right) - \int_\Omega \Phi'_-\left(|f|^2\right) d\mu\right| d\mu \\
&\le \frac{1}{2}\left(M^2 - m^2\right)\left[\int_\Omega \left[\Phi'_-\left(|f|^2\right)\right]^2 d\mu - \left(\int_\Omega \Phi'_-\left(|f|^2\right) d\mu\right)^2\right]^{1/2} \\
&\le \frac{1}{4}\left(M^2 - m^2\right)\left(\Phi'_-\left(M^2\right) - \Phi'_-\left(m^2\right)\right)
\end{aligned}$$

or

$$\begin{aligned}
C\left(\Phi'_-\left(|f|^2\right), |f|^2\right) &\le \frac{1}{2}\left(\Phi'_-\left(M^2\right) - \Phi'_-\left(m^2\right)\right)\int_\Omega \left||f|^2 - \int_\Omega |f|^2 d\mu\right| d\mu \\
&\le \frac{1}{2}\left(\Phi'_-\left(M^2\right) - \Phi'_-\left(m^2\right)\right)\left[\int_\Omega |f|^4 d\mu - \left(\int_\Omega |f|^2 d\mu\right)^2\right]^{1/2} \\
&\le \frac{1}{4}\left(M^2 - m^2\right)\left(\Phi'_-\left(M^2\right) - \Phi'_-\left(m^2\right)\right).
\end{aligned}$$

This proves the inequality (2.3). □

Remark 2. *Let $\Phi : [0, \infty) \to \mathbb{R}$ be a convex function on $(0, \infty)$. If $x_i \in \mathbb{C}$ and $w_i \geq 0 \quad (i = 1, \ldots, n)$ with $W_n := \sum_{i=1}^{n} w_i = 1$, then*

$$\left(\sum_{i=1}^{n} w_i |x_i|^2 - \left| \sum_{i=1}^{n} w_i x_i \right|^2 \right) \Phi'_+ \left(\left| \sum_{i=1}^{n} w_i x_i \right|^2 \right) \tag{2.7}$$
$$\leq \sum_{i=1}^{n} w_i \Phi \left(|x_i|^2 \right) - \Phi \left(\left| \sum_{i=1}^{n} w_i x_i \right|^2 \right)$$
$$\leq \sum_{i=1}^{n} w_i \Phi'_- \left(|x_i|^2 \right) |x_i|^2 - \left| \sum_{i=1}^{n} w_i x_i \right|^2 \sum_{i=1}^{n} w_i \Phi'_- \left(|x_i|^2 \right).$$

If $0 \leq m \leq |x_i| \leq M$ for $i = 1, \ldots, n$; then

$$\sum_{i=1}^{n} w_i \Phi'_- \left(|x_i|^2 \right) |x_i|^2 - \left| \sum_{i=1}^{n} w_i x_i \right|^2 \sum_{i=1}^{n} w_i \Phi'_- \left(|x_i|^2 \right)$$
$$\leq \left(\sum_{i=1}^{n} w_i |x_i|^2 - \left| \sum_{i=1}^{n} w_i x_i \right|^2 \right) \sum_{i=1}^{n} w_i \Phi'_- \left(|x_i|^2 \right)$$
$$+ \frac{1}{2} \begin{cases} \left(M^2 - m^2 \right) \sum_{i=1}^{n} w_i \left| \Phi'_- \left(|x_i|^2 \right) - \sum_{j=1}^{n} w_j \Phi'_- \left(|x_j|^2 \right) \right| \\ \\ \left[\Phi'_- \left(M^2 \right) - \Phi'_- \left(m^2 \right) \right] \sum_{i=1}^{n} w_i \left| |x_i|^2 - \sum_{j=1}^{n} w_j |x_j|^2 \right| \end{cases}$$
$$\leq \left(\sum_{i=1}^{n} w_i |x_i|^2 - \left| \sum_{i=1}^{n} w_i x_i \right|^2 \right) \sum_{i=1}^{n} w_i \Phi'_- \left(|x_i|^2 \right)$$
$$+ \frac{1}{2} \begin{cases} \left(M^2 - m^2 \right) \left[\sum_{i=1}^{n} w_i \left[\Phi'_- \left(|x_i|^2 \right) \right]^2 - \left(\sum_{i=1}^{n} w_i \Phi'_- \left(|x_i|^2 \right) \right)^2 \right]^{1/2} \\ \\ \left[\Phi'_- \left(M^2 \right) - \Phi'_- \left(m^2 \right) \right] \left[\sum_{i=1}^{n} w_i |x_i|^4 - \left(\sum_{i=1}^{n} w_i |x_i|^2 \right)^2 \right]^{1/2} \end{cases}$$
$$\leq \left(\sum_{i=1}^{n} w_i |x_i|^2 - \left| \sum_{i=1}^{n} w_i x_i \right|^2 \right) \sum_{i=1}^{n} w_i \Phi'_- \left(|x_i|^2 \right)$$
$$+ \frac{1}{4} \left(M^2 - m^2 \right) \left[\Phi'_- \left(M^2 \right) - \Phi'_- \left(m^2 \right) \right].$$

We have the following particular cases of interest.

If we take $\Phi(t) = t^r$, $r \geq 1$ and $t \in [0, \infty)$, then we can state the following power inequalities:

Corollary 2. *Let $r \geq 1$. If $f : \Omega \to \mathbb{C}$ is a μ-measurable function and such that $|f|^2$, $|f|^{2(r-1)}$ and $|f|^{2r} \in L(\Phi, \mu)$, then*

$$
\begin{aligned}
(2.8) \qquad & r\left(\int_\Omega |f|^2 d\mu - \left|\int_\Omega f d\mu\right|^2\right)\left|\int_\Omega f d\mu\right|^{2(r-1)} \\
& \leq \int_\Omega |f|^{2r} d\mu - \left|\int_\Omega f d\mu\right|^{2r} \\
& \leq r\left[\int_\Omega |f|^{2r} d\mu - \int_\Omega |f|^{2(r-1)} d\mu \left|\int_\Omega f d\mu\right|^2 d\mu\right].
\end{aligned}
$$

If there exists the constants M, m such that (2.2) is valid, then

$$
\begin{aligned}
(2.9) \qquad & \int_\Omega |f|^{2r} d\mu - \left|\int_\Omega f d\mu\right|^{2r} \\
& \leq r\left[\int_\Omega |f|^{2r} d\mu - \int_\Omega |f|^{2(r-1)} d\mu \left|\int_\Omega f d\mu\right|^2 d\mu\right] \\
& \leq r\left(\int_\Omega |f|^2 d\mu - \left|\int_\Omega f d\mu\right|^2\right)\int_\Omega |f|^{2(r-1)} d\mu \\
& + \frac{1}{2} r \begin{cases} \left(M^2 - m^2\right) \int_\Omega \left| |f|^{2(r-1)} - \int_\Omega |f|^{2(r-1)} d\mu \right| d\mu \\ \\ \left[M^{2(r-1)} - m^{2(r-1)}\right] \int_\Omega \left| |f|^2 - \int_\Omega |f|^2 d\mu \right| d\mu \end{cases} \\
& \leq r\left(\int_\Omega |f|^2 d\mu - \left|\int_\Omega f d\mu\right|^2\right)\int_\Omega |f|^{2(r-1)} d\mu \\
& + \frac{1}{2} r \begin{cases} \left(M^2 - m^2\right)\left[\int_\Omega |f|^{4(r-1)} d\mu - \left(\int_\Omega |f|^{2(r-1)} d\mu\right)^2\right]^{1/2} \\ \\ \left[M^{2(r-1)} - m^{2(r-1)}\right]\left[\int_\Omega |f|^4 d\mu - \left(\int_\Omega |f|^2 d\mu\right)^2\right]^{1/2} \end{cases} \\
& \leq r\left(\int_\Omega |f|^2 d\mu - \left|\int_\Omega f d\mu\right|^2\right)\int_\Omega |f|^{2(r-1)} d\mu \\
& + \frac{1}{4} r \left(M^2 - m^2\right)\left[M^{2(r-1)} - m^{2(r-1)}\right].
\end{aligned}
$$

If we take $\Phi(t) = -\ln t$, $t \in (0, \infty)$, then we can state the following logarithmic inequalities:

Corollary 3. *If $f : \Omega \to \mathbb{C}$ a μ-measurable function and such that $|f|^2$, $|f|^{-2}$, $\ln |f|^2 \in L(\Phi, \mu)$ and $\int_\Omega f d\mu \neq 0$, then*

$$
\begin{aligned}
(2.10) \qquad \frac{\int_\Omega |f|^2 d\mu}{\left|\int_\Omega f d\mu\right|^2} - 1 & \geq \int_\Omega \ln\left(|f|^2\right) d\mu - \ln\left(\left|\int_\Omega f d\mu\right|^2\right) \\
& \geq 1 - \left|\int_\Omega f d\mu\right|^2 \int_\Omega |f|^{-2} d\mu.
\end{aligned}
$$

Finally, if we take $\Phi(t)=\exp(t)$, $t\in\mathbb{R}$, then we can state the following exponential inequalities:

Corollary 4. *If $f:\Omega\to\mathbb{C}$ is a μ-measurable function and such that $|f|^2$, $\exp\left(|f|^2\right)$, $|f|^2\exp\left(|f|^2\right)\in L(\Phi,\mu)$, then*

$$\begin{aligned}(2.11)\quad &\left(\int_\Omega |f|^2\,d\mu-\left|\int_\Omega f d\mu\right|^2\right)\exp\left(\left|\int_\Omega f d\mu\right|^2\right)\\ &\le \int_\Omega \exp\left(|f|^2\right)d\mu-\exp\left(\left|\int_\Omega f d\mu\right|^2\right)\\ &\le \int_\Omega |f|^2\exp\left(|f|^2\right)d\mu-\left|\int_\Omega f d\mu\right|^2 d\mu\int_\Omega \exp\left(|f|^2\right)d\mu.\end{aligned}$$

If there exists the constants M, m such that (2.2) is valid, then

$$\begin{aligned}(2.12)\quad &\int_\Omega \exp\left(|f|^2\right)d\mu-\exp\left(\left|\int_\Omega f d\mu\right|^2\right)\\ &\le \int_\Omega |f|^2\exp\left(|f|^2\right)d\mu-\left|\int_\Omega f d\mu\right|^2 d\mu\int_\Omega \exp\left(|f|^2\right)d\mu\\ &\le \left(\int_\Omega |f|^2\,d\mu-\left|\int_\Omega f d\mu\right|^2\right)\int_\Omega \exp\left(|f|^2\right)d\mu\\ &+\frac{1}{2}\begin{cases}\left(M^2-m^2\right)\int_\Omega\left|\exp\left(|f|^2\right)-\int_\Omega \exp\left(|f|^2\right)d\mu\right|d\mu\\ \\ \left[\exp\left(M^2\right)-\exp\left(m^2\right)\right]\int_\Omega\left||f|^2-\int_\Omega |f|^2\,d\mu\right|d\mu\end{cases}\end{aligned}$$

$$\begin{aligned}&\le \left(\int_\Omega |f|^2\,d\mu-\left|\int_\Omega f d\mu\right|^2\right)\int_\Omega \exp\left(|f|^2\right)d\mu\\ &+\frac{1}{2}\begin{cases}\left(M^2-m^2\right)\left[\int_\Omega \exp\left(2|f|^2\right)d\mu-\left(\int_\Omega \exp\left(|f|^2\right)d\mu\right)^2\right]^{1/2}\\ \\ \left[\exp\left(M^2\right)-\exp\left(m^2\right)\right]\left[\int_\Omega |f|^4\,d\mu-\left(\int_\Omega |f|^2\,d\mu\right)^2\right]^{1/2}\end{cases}\\ &\le \left(\int_\Omega |f|^2\,d\mu-\left|\int_\Omega f d\mu\right|^2\right)\int_\Omega \exp\left(|f|^2\right)d\mu\\ &+\frac{1}{4}\left(M^2-m^2\right)\left[\exp\left(M^2\right)-\exp\left(m^2\right)\right].\end{aligned}$$

We also have:

Theorem 4. *Let $\Phi:I\to\mathbb{R}$ be a convex function on the interior $\mathring{I}$ of I and $[0,\infty)\subset\mathring{I}$, $f:\Omega\to\mathbb{C}$ a μ-measurable function such that $|f|^2$, $\Phi'\circ|f|^2$ and*

$\Phi\circ|f|^2\in L(\Phi,\mu)$. *Assume that* $\int_\Omega \Phi'_+\left(|f|^2\right)d\mu\neq 0$ *and*

$$\frac{\int_\Omega |f|^2\,\Phi'_+\left(|f|^2\right)d\mu}{\int_\Omega \Phi'_+\left(|f|^2\right)d\mu}\geq 0, \tag{2.13}$$

then

$$\begin{aligned} 0 &\leq \Phi\left(\frac{\int_\Omega |f|^2\,\Phi'_+\left(|f|^2\right)d\mu}{\int_\Omega \Phi'_+\left(|f|^2\right)d\mu}\right)-\int_\Omega \Phi\left(|f|^2\right)d\mu \\ &\leq \frac{1}{\int_\Omega \Phi'_+\left(|f|^2\right)d\mu}\Phi'_-\left(\frac{\int_\Omega |f|^2\,\Phi'_+\left(|f|^2\right)d\mu}{\int_\Omega \Phi'_+\left(|f|^2\right)d\mu}\right)C\left(\Phi'_+\left(|f|^2\right),|f|^2\right), \end{aligned} \tag{2.14}$$

where

$$C\left(\Phi'_+\left(|f|^2\right),|f|^2\right)=\int_\Omega \Phi'_+\left(|f|^2\right)|f|^2\,d\mu-\int_\Omega \Phi'_+\left(|f|^2\right)d\mu\int_\Omega |f|^2\,d\mu\geq 0.$$

If there exists the constants M, m *such that (2.2) is valid, then*

$$\begin{aligned} &C\left(\Phi'_+\left(|f|^2\right),|f|^2\right) \\ &\leq \frac{1}{2}\begin{cases} \left(M^2-m^2\right)\int_\Omega \left|\Phi'_+\left(|f|^2\right)-\int_\Omega \Phi'_+\left(|f|^2\right)d\mu\right|d\mu \\ \\ \left[\Phi'_+\left(M^2\right)-\Phi_+\left(m^2\right)\right]\int_\Omega \left||f|^2-\int_\Omega |f|^2\,d\mu\right|d\mu \end{cases} \\ &\leq \frac{1}{2}\begin{cases} \left(M^2-m^2\right)\left[\int_\Omega \left[\Phi'_+\left(|f|^2\right)\right]^2d\mu-\left(\int_\Omega \Phi'_+\left(|f|^2\right)d\mu\right)^2\right]^{1/2} \\ \\ \left[\Phi'_+\left(M^2\right)-\Phi'_+\left(m^2\right)\right]\left[\int_\Omega |f|^4\,d\mu-\left(\int_\Omega |f|^2\,d\mu\right)^2\right]^{1/2} \end{cases} \\ &\leq \frac{1}{4}\left(M^2-m^2\right)\left[\Phi'_+\left(M^2\right)-\Phi'_+\left(m^2\right)\right]. \end{aligned} \tag{2.15}$$

Proof. Let $t\in\Omega$. By taking $x=|f(t)|^2$ in (2.4) we have

$$\Phi'_-(y)\left(y-|f(t)|^2\right)\geq \Phi(y)-\Phi\left(|f(t)|^2\right)\geq \Phi'_+\left(|f(t)|^2\right)\left(y-|f(t)|^2\right) \tag{2.16}$$

for any $y\in[0,\infty)$ and $t\in\Omega$.

If we integrate over $d\mu(t)$ on Ω the inequality (2.16) we have

$$\begin{aligned} \Phi'_-(y)\left(y-\int_\Omega |f|^2\,d\mu\right) &\geq \Phi(y)-\int_\Omega \Phi\left(|f|^2\right)d\mu \\ &\geq y\int_\Omega \Phi'_+\left(|f|^2\right)d\mu-\int_\Omega |f|^2\,\Phi'_+\left(|f|^2\right)d\mu \end{aligned} \tag{2.17}$$

for any $y\in[0,\infty)$.

If we take

$$y=\frac{\int_\Omega |f|^2\,\Phi'_+\left(|f|^2\right)d\mu}{\int_\Omega \Phi'_+\left(|f|^2\right)d\mu}\in[0,\infty)$$

in (2.17) then we get the first and the second inequalities in (2.16).

The last part of the proof follows in a similar way with the proof of Theorem 3 by replacing Φ'_- with Φ'_+. We omit the details. □

Remark 3. *Let $\Phi : [0,\infty) \to \mathbb{R}$ be a convex function on $(0,\infty)$, $x_i \in \mathbb{C}$ and $w_i \geq 0$ $(i = 1,\ldots,n)$ with $W_n := \sum_{i=1}^n w_i = 1$. Assume that $\sum_{i=1}^n w_i \Phi'_+\left(|x_i|^2\right) \neq 0$ and*

$$\frac{\sum_{i=1}^n w_i \Phi'_+\left(|x_i|^2\right)|x_i|^2}{\sum_{i=1}^n w_i \Phi'_+\left(|x_i|^2\right)} \geq 0,$$

then

$$\begin{aligned}
(2.18)\qquad 0 &\leq \Phi\left(\frac{\sum_{i=1}^n w_i \Phi'_+\left(|x_i|^2\right)|x_i|^2}{\sum_{i=1}^n w_i \Phi'_+\left(|x_i|^2\right)}\right) - \sum_{i=1}^n w_i \Phi\left(|x_i|^2\right) \\
&\leq \frac{1}{\sum_{i=1}^n w_i \Phi'_+\left(|x_i|^2\right)} \Phi'_-\left(\frac{\sum_{i=1}^n w_i \Phi'_+\left(|x_i|^2\right)|x_i|^2}{\sum_{i=1}^n w_i \Phi'_+\left(|x_i|^2\right)}\right) \\
&\times \left[\sum_{i=1}^n w_i \Phi'_+\left(|x_i|^2\right)|x_i|^2 - \sum_{i=1}^n w_i \Phi'_+\left(|x_i|^2\right)\sum_{i=1}^n w_i |x_i|^2\right] \\
&\leq \frac{1}{\sum_{i=1}^n w_i \Phi'_+\left(|x_i|^2\right)} \Phi'_-\left(\frac{\sum_{i=1}^n w_i \Phi'_+\left(|x_i|^2\right)|x_i|^2}{\sum_{i=1}^n w_i \Phi'_+\left(|x_i|^2\right)}\right) \\
&\times \frac{1}{2}\begin{cases} \left(M^2 - m^2\right)\sum_{i=1}^n w_i \left|\Phi'_+\left(|x_i|^2\right) - \sum_{j=1}^n w_j \Phi'_+\left(|x_j|^2\right)\right| \\ \\ \left[\Phi'_+\left(M^2\right) - \Phi_+\left(m^2\right)\right]\sum_{i=1}^n w_i \left||x_i|^2 - \sum_{j=1}^n w_j |x_j|^2\right| \end{cases} \\
&\leq \frac{1}{2\sum_{i=1}^n w_i \Phi'_+\left(|x_i|^2\right)} \Phi'_-\left(\frac{\sum_{i=1}^n w_i \Phi'_+\left(|x_i|^2\right)|x_i|^2}{\sum_{i=1}^n w_i \Phi'_+\left(|x_i|^2\right)}\right) \\
&\times \begin{cases} \left(M^2 - m^2\right)\left[\sum_{i=1}^n w_i \left[\Phi'_+\left(|x_i|^2\right)\right]^2 - \left(\sum_{i=1}^n w_i \left(|x_i|^2\right)\right)^2\right]^{1/2} \\ \\ \left[\Phi'_+\left(M^2\right) - \Phi'_+\left(m^2\right)\right]\left[\sum_{i=1}^n w_i |x_i|^4 - \left(\sum_{i=1}^n w_i |x_i|^2\right)^2\right]^{1/2} \end{cases} \\
&\leq \frac{1}{4\sum_{i=1}^n w_i \Phi'_+\left(|x_i|^2\right)} \Phi'_-\left(\frac{\sum_{i=1}^n w_i \Phi'_+\left(|x_i|^2\right)|x_i|^2}{\sum_{i=1}^n w_i \Phi'_+\left(|x_i|^2\right)}\right) \\
&\times \left(M^2 - m^2\right)\left[\Phi'_+\left(M^2\right) - \Phi'_+\left(m^2\right)\right].
\end{aligned}$$

We have the following particular case of interest.

If we take $\Phi(t) = t^r$, $r \geq 1$ and $t \in [0,\infty)$, then we can state the following power inequalities:

Corollary 5. *Let $r \geq 1$. If $f : \Omega \to \mathbb{C}$ is a μ-measurable function such that $|f|^2$, $|f|^{2(r-1)}$ and $|f|^{2r} \in L(\Phi, \mu)$, then*

$$0 \leq \left(\frac{\int_\Omega |f|^{2r} d\mu}{\int_\Omega |f|^{2(r-1)} d\mu} \right)^r - \int_\Omega |f|^{2r} d\mu \leq r \left(\frac{\int_\Omega |f|^{2r} d\mu}{\int_\Omega |f|^{2(r-1)} d\mu} \right)^{r-1} \left[\frac{\int_\Omega |f|^{2r} d\mu}{\int_\Omega |f|^{2(r-1)} d\mu} - \int_\Omega |f|^2 d\mu \right]. \tag{2.19}$$

If there exists the constants M, m such that (2.2) is valid, then

$$\begin{aligned} 0 &\leq \left(\frac{\int_\Omega |f|^{2r} d\mu}{\int_\Omega |f|^{2(r-1)} d\mu} \right)^r - \int_\Omega |f|^{2r} d\mu \\ &\leq \frac{r}{\int_\Omega |f|^{2(r-1)} d\mu} \left(\frac{\int_\Omega |f|^{2r} d\mu}{\int_\Omega |f|^{2(r-1)} d\mu} \right)^{r-1} \\ &\times \left[\int_\Omega |f|^{2r} d\mu - \int_\Omega |f|^{2(r-1)} d\mu \int_\Omega |f|^2 d\mu \right] \\ &\leq \frac{r}{2\int_\Omega |f|^{2(r-1)} d\mu} \left(\frac{\int_\Omega |f|^{2r} d\mu}{\int_\Omega |f|^{2(r-1)} d\mu} \right)^{r-1} \\ &\times \begin{cases} \left(M^2 - m^2\right) \int_\Omega \left| |f|^{2(r-1)} - \int_\Omega |f|^{2(r-1)} d\mu \right| d\mu \\ \\ \left[M^{2(r-1)} - m^{2(r-1)}\right] \int_\Omega \left| |f|^2 - \int_\Omega |f|^2 d\mu \right| d\mu \end{cases} \\ &\leq \frac{r}{2\int_\Omega |f|^{2(r-1)} d\mu} \left(\frac{\int_\Omega |f|^{2r} d\mu}{\int_\Omega |f|^{2(r-1)} d\mu} \right)^{r-1} \\ &\times \begin{cases} \left(M^2 - m^2\right) \left[\int_\Omega |f|^{4(r-1)} d\mu - \left(\int_\Omega |f|^{2(r-1)} d\mu \right)^2 \right]^{1/2} \\ \\ \left[M^{2(r-1)} - m^{2(r-1)}\right] \left[\int_\Omega |f|^4 d\mu - \left(\int_\Omega |f|^2 d\mu \right)^2 \right]^{1/2} \end{cases} \\ &\leq \frac{r}{4\int_\Omega |f|^{2(r-1)} d\mu} \left(\frac{\int_\Omega |f|^{2r} d\mu}{\int_\Omega |f|^{2(r-1)} d\mu} \right)^{r-1} \\ &\times \left(M^2 - m^2\right) \left[M^{2(r-1)} - m^{2(r-1)}\right]. \end{aligned} \tag{2.20}$$

If we take $\Phi(t) = -\ln t$, $t \in (0, \infty)$, then we can state the following logarithmic inequalities:

Corollary 6. *If $f : \Omega \to \mathbb{C}$ is a μ-measurable function such that $|f|^2$, $|f|^{-2}$, $\ln |f|^2 \in L(\Phi, \mu)$ and $\int_\Omega f d\mu \neq 0$, then*

$$0 \leq \int_\Omega \ln\left(|f|^2\right) d\mu - \ln\left(\frac{1}{\int_\Omega |f|^{-2} d\mu} \right) \leq \int_\Omega |f|^{-2} d\mu \int_\Omega |f|^2 d\mu - 1. \tag{2.21}$$

Finally, if we take $\Phi(t)=\exp(t)$, $t\in\mathbb{R}$, then we can state the following exponential inequalities:

Corollary 7. *If $f:\Omega\to\mathbb{C}$ is a μ-measurable function such that $|f|^2$, $\exp\left(|f|^2\right)$, $|f|^2\exp\left(|f|^2\right)\in L(\Phi,\mu)$, then*

$$0\le\exp\left(\frac{\int_\Omega|f|^2\exp\left(|f|^2\right)d\mu}{\int_\Omega\exp\left(|f|^2\right)d\mu}\right)-\int_\Omega\exp\left(|f|^2\right)d\mu \tag{2.22}$$
$$\le\exp\left(\frac{\int_\Omega|f|^2\exp\left(|f|^2\right)d\mu}{\int_\Omega\exp\left(|f|^2\right)d\mu}\right)\left[\frac{\int_\Omega\exp\left(|f|^2\right)|f|^2d\mu}{\int_\Omega\exp\left(|f|^2\right)d\mu}-\int_\Omega|f|^2d\mu\right].$$

If there exists the constants M, m such that (2.2) is valid, then

$$0\le\exp\left(\frac{\int_\Omega|f|^2\exp\left(|f|^2\right)d\mu}{\int_\Omega\exp\left(|f|^2\right)d\mu}\right)-\int_\Omega\exp\left(|f|^2\right)d\mu \tag{2.23}$$
$$\le\frac{1}{\int_\Omega\exp\left(|f|^2\right)d\mu}\exp\left(\frac{\int_\Omega|f|^2\exp\left(|f|^2\right)d\mu}{\int_\Omega\exp\left(|f|^2\right)d\mu}\right)$$
$$\times\left[\int_\Omega\exp\left(|f|^2\right)|f|^2d\mu-\int_\Omega\exp\left(|f|^2\right)d\mu\int_\Omega|f|^2d\mu\right]$$
$$\le\frac{1}{2\int_\Omega\exp\left(|f|^2\right)d\mu}\exp\left(\frac{\int_\Omega|f|^2\exp\left(|f|^2\right)d\mu}{\int_\Omega\exp\left(|f|^2\right)d\mu}\right)$$
$$\times\begin{cases}\left(M^2-m^2\right)\int_\Omega\left|\exp\left(|f|^2\right)-\int_\Omega\exp\left(|f|^2\right)d\mu\right|d\mu\\ \\ \left[\exp\left(M^2\right)-\exp\left(m^2\right)\right]\int_\Omega\left||f|^2-\int_\Omega|f|^2d\mu\right|d\mu\end{cases}$$

$$\le\frac{1}{2\int_\Omega\exp\left(|f|^2\right)d\mu}\exp\left(\frac{\int_\Omega|f|^2\exp\left(|f|^2\right)d\mu}{\int_\Omega\exp\left(|f|^2\right)d\mu}\right)$$
$$\times\begin{cases}\left(M^2-m^2\right)\left[\int_\Omega\exp\left(2|f|^2\right)d\mu-\left(\int_\Omega\exp\left(|f|^2\right)d\mu\right)^2\right]^{1/2}\\ \\ \left[\exp\left(M^2\right)-\exp\left(m^2\right)\right]\left[\int_\Omega|f|^4d\mu-\left(\int_\Omega|f|^2d\mu\right)^2\right]^{1/2}\end{cases}$$
$$\le\frac{1}{4\int_\Omega\exp\left(|f|^2\right)d\mu}\exp\left(\frac{\int_\Omega|f|^2\exp\left(|f|^2\right)d\mu}{\int_\Omega\exp\left(|f|^2\right)d\mu}\right)$$
$$\times\left(M^2-m^2\right)\left[\exp\left(M^2\right)-\exp\left(m^2\right)\right].$$

3. Applications for Selfadjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$. Let $A\in\mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda\in\mathbb{R}$ as follows

$$\varphi_\lambda(s):=\begin{cases}1, \text{ for } -\infty<s\leq\lambda,\\ \\ 0, \text{ for } \lambda<s<+\infty.\end{cases}$$

Then for every $\lambda\in\mathbb{R}$ the operator

$$E_\lambda:=\varphi_\lambda(A) \tag{3.1}$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [35, p. 256]:

Theorem 5 (Spectral Representation Theorem)**.** *Let A be a bounded selfadjoint operator on the Hilbert space H and let $m=\min\{\lambda\,|\lambda\in Sp(A)\}=:\min Sp(A)$ and $M=\max\{\lambda\,|\lambda\in Sp(A)\}=:\max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, called the spectral family of A, with the following properties*

a) $E_\lambda\leq E_{\lambda'}$ *for* $\lambda\leq\lambda'$;
b) $E_{m-0}=0, E_M=I$ *and* $E_{\lambda+0}=E_\lambda$ *for all* $\lambda\in\mathbb{R}$*;*
c) *We have the representation*

$$A=\int_{m-0}^{M}\lambda dE_\lambda.$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ and for every $\varepsilon>0$ there exists a $\delta>0$ such that

$$\left\|\varphi(A)-\sum_{k=1}^{n}\varphi(\lambda_k')\left[E_{\lambda_k}-E_{\lambda_{k-1}}\right]\right\|\leq\varepsilon$$

whenever

$$\begin{cases}\lambda_0<m=\lambda_1<...<\lambda_{n-1}<\lambda_n=M,\\ \\ \lambda_k-\lambda_{k-1}\leq\delta \text{ for } 1\leq k\leq n,\\ \\ \lambda_k'\in[\lambda_{k-1},\lambda_k] \text{ for } 1\leq k\leq n\end{cases}$$

this means that

$$\varphi(A)=\int_{m-0}^{M}\varphi(\lambda)\,dE_\lambda, \tag{3.2}$$

where the integral is of Riemann-Stieltjes type.

Corollary 8. *With the assumptions of Theorem 5 for A, E_λ and φ we have the representations*

$$\varphi(A)x=\int_{m-0}^{M}\varphi(\lambda)\,dE_\lambda x \quad \text{for all } x\in H$$

and

$$\langle\varphi(A)x,y\rangle=\int_{m-0}^{M}\varphi(\lambda)\,d\langle E_\lambda x,y\rangle \quad \text{for all } x,y\in H. \tag{3.3}$$

In particular,

$$\langle \varphi(A)x,x\rangle = \int_{m-0}^{M} \varphi(\lambda)\, d\langle E_\lambda x,x\rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(A)x\|^2 = \int_{m-0}^{M} |\varphi(\lambda)|^2\, d\|E_\lambda x\|^2 \quad \text{for all } x \in H.$$

For a bounded linear operator B on H we denote $|B| := \sqrt{B^*B}$.

We can state the following result for functions of selfadjoint operators:

Theorem 6. *Let A be a bounded selfadjoint operator on the Hilbert space H and let $m = \min\{\lambda \,|\lambda \in Sp(A)\} =: \min Sp(A)$ and $M = \max\{\lambda \,|\lambda \in Sp(A)\} =: \max Sp(A)$. Assume that $\Phi : I \to \mathbb{R}$ is a differentiable convex function on the interior $\mathring{I}$ of I with $[0,\infty) \subset \mathring{I}$, and the derivative Φ' is continuous on $\mathring{I}$. If $f : J \to \mathbb{C}$ is a continuous function on J with $[m,M] \subset \mathring{J}$, then we have the inequalities*

$$\begin{aligned} &\left(\|f(A)x\|^2 - |\langle f(A)x,x\rangle|^2\right)\Phi'\left(|\langle f(A)x,x\rangle|^2\right) \\ &\le \left\langle \Phi\left(|f(A)|^2\right)x,x\right\rangle - \Phi\left(|\langle f(A)x,x\rangle|^2\right) \\ &\le \left\langle \Phi'\left(|f(A)|^2\right)|f(A)|^2 x,x\right\rangle - |\langle f(A)x,x\rangle|^2 \left\langle \Phi'\left(|f(A)|^2\right)x,x\right\rangle \end{aligned} \tag{3.4}$$

for any $x \in H$, $\|x\| = 1$.

If $\gamma := \min_{t\in[m,M]} |f(t)|$ and $\Gamma = \max_{t\in[m,M]} |f(t)|$, then

$$\begin{aligned} &\left\langle \Phi'\left(|f(A)|^2\right)|f(A)|^2 x,x\right\rangle - |\langle f(A)x,x\rangle|^2 \left\langle \Phi'\left(|f(A)|^2\right)x,x\right\rangle \\ &\le \left(\|f(A)x\|^2 - |\langle f(A)x,x\rangle|^2\right)\left\langle \Phi'\left(|f(A)|^2\right)x,x\right\rangle \\ &+ \frac{1}{4}\left(\Gamma^2-\gamma^2\right)\left[\Phi'\left(\Gamma^2\right) - \Phi'\left(\gamma^2\right)\right] \end{aligned} \tag{3.5}$$

for any $x \in H$, $\|x\| = 1$.

In particular, for $f(t) = t$, we have

$$\begin{aligned} &\left(\|Ax\|^2 - |\langle Ax,x\rangle|^2\right)\Phi'\left(|\langle Ax,x\rangle|^2\right) \\ &\le \left\langle \Phi\left(A^2\right)x,x\right\rangle - \Phi\left(|\langle Ax,x\rangle|^2\right) \\ &\le \left\langle \Phi'\left(A^2\right)A^2x,x\right\rangle - |\langle Ax,x\rangle|^2\left\langle \Phi'\left(A^2\right)x,x\right\rangle \end{aligned} \tag{3.6}$$

for any $x \in H$, $\|x\| = 1$.

If we denote $n := \min_{t\in[m,M]} |t|$ and $N := \max_{t\in[m,M]} |t|$, then

$$\begin{aligned} &\left\langle \Phi'\left(A^2\right)A^2x,x\right\rangle - |\langle Ax,x\rangle|^2\left\langle \Phi'\left(A^2\right)x,x\right\rangle \\ &\le \left(\|Ax\|^2 - |\langle Ax,x\rangle|^2\right)\left\langle \Phi'\left(A^2\right)x,x\right\rangle \\ &+ \frac{1}{4}\left(N^2-n^2\right)\left[\Phi'\left(N^2\right) - \Phi'\left(n^2\right)\right] \end{aligned} \tag{3.7}$$

for any $x \in H$, $\|x\| = 1$.

Proof. Let $x\in H$, $\|x\|=1$, $\varepsilon>0$ and $f:[m-\varepsilon,M]\subset J\to\mathbb{C}$, which is continuous on $[m-\varepsilon,M]$. If we use the inequality (2.1) for the measure $d\mu=dg$, where $g:[m-\varepsilon,M]\to\mathbb{R}$ is the monotonic nondecreasing function $g(t):=\langle E_t x,x\rangle$, and $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, is the spectral family of A, then we have the inequality

$$\begin{aligned}(3.8)\qquad &\left(\int_{m-\varepsilon}^{M}|f(t)|^2\,d\langle E_t x,x\rangle-\left|\int_{m-\varepsilon}^{M}f(t)\,d\langle E_t x,x\rangle\right|^2\right)\\ &\times\Phi'\left(\left|\int_{m-\varepsilon}^{M}f(t)\,d\langle E_t x,x\rangle\right|^2\right)\\ &\le\int_{m-\varepsilon}^{M}\Phi\left(|f(t)|^2\right)d\langle E_t x,x\rangle-\Phi\left(\left|\int_{m-\varepsilon}^{M}f(t)\,d\langle E_t x,x\rangle\right|^2\right)\\ &\le\int_{m-\varepsilon}^{M}\Phi'\left(|f(t)|^2\right)|f(t)|^2\,d\langle E_t x,x\rangle\\ &-\left|\int_{m-\varepsilon}^{M}f(t)\,d\langle E_t x,x\rangle\right|^2\int_{m-\varepsilon}^{M}\Phi'\left(|f(t)|^2\right)d\langle E_t x,x\rangle.\end{aligned}$$

Taking the limit over $\varepsilon\to 0+$ in (3.8) and utilizing the Spectral Representation Theorem for selfadjoint operators we get the desired inequality (3.4).

The inequality (3.5) follows in a similar manner by making use of (2.3). The details are omitted.

For $f(t)=t$ we have $|f(A)|^2=|A|^2=A^*A=A^2$ and the inequalities (3.6) and (3.7) follow from (3.4) and (3.5). □

We have the following power inequalities:

Corollary 9. *Let A be a bounded selfadjoint operator on the Hilbert space H and $0\le mI\le A\le MI$. Then for $r\ge 1$ we have*

$$\begin{aligned}(3.9)\qquad & r\left(\|Ax\|^2-|\langle Ax,x\rangle|^2\right)|\langle Ax,x\rangle|^{2(r-1)}\\ &\le\left\langle A^{2r}x,x\right\rangle-|\langle Ax,x\rangle|^{2r}\\ &\le r\left[\left\langle A^{2r}x,x\right\rangle-|\langle Ax,x\rangle|^2\left\langle A^{2(r-1)}x,x\right\rangle\right]\\ &\le r\left(\|Ax\|^2-|\langle Ax,x\rangle|^2\right)\left\langle A^{2(r-1)}x,x\right\rangle\\ &+\frac{1}{4}r\left(M^2-m^2\right)\left[M^{2(r-1)}-m^{2(r-1)}\right]\end{aligned}$$

for any $x\in H$, $\|x\|=1$.

We have the following logarithmic inequalities:

Corollary 10. *Let A be a positive definite operator on the Hilbert space H. Then*

$$\begin{aligned}(3.10)\qquad \frac{\|Ax\|^2}{|\langle Ax,x\rangle|^2}-1&\ge\left\langle\ln\left(A^2\right)x,x\right\rangle-\ln\left(|\langle Ax,x\rangle|^2\right)\\ &\ge 1-|\langle Ax,x\rangle|^2\left\langle A^{-2}x,x\right\rangle\end{aligned}$$

for any $x\in H$, $\|x\|=1$.

We have the following exponential inequalities:

Corollary 11. *Let A be a bounded selfadjoint operator on the Hilbert space H and $mI \leq A \leq MI$. Then we have*

$$\left(\|Ax\|^2 - |\langle Ax, x\rangle|^2\right) \exp\left(|\langle Ax, x\rangle|^2\right) \tag{3.11}$$
$$\leq \left\langle \exp\left(A^2\right) x, x\right\rangle - \exp\left(|\langle Ax, x\rangle|^2\right)$$
$$\leq \left\langle \exp\left(A^2\right) A^2 x, x\right\rangle - |\langle Ax, x\rangle|^2 \left\langle \exp\left(A^2\right) x, x\right\rangle$$

for any $x \in H$, $\|x\| = 1$.

If we denote $n := \min_{t\in[m,M]} |t|$ and $N := \max_{t\in[m,M]} |t|$, then

$$\left\langle \exp\left(A^2\right) A^2 x, x\right\rangle - |\langle Ax, x\rangle|^2 \left\langle \exp\left(A^2\right) x, x\right\rangle \tag{3.12}$$
$$\leq \left(\|Ax\|^2 - |\langle Ax, x\rangle|^2\right) \left\langle \exp\left(A^2\right) x, x\right\rangle$$
$$+ \frac{1}{4}\left(N^2 - n^2\right)\left[\exp\left(N^2\right) - \exp\left(n^2\right)\right]$$

for any $x \in H$, $\|x\| = 1$.

For recent inequalities for functions of selfadjoint operators on Hilbert spaces, see [5]-[20] and the monographs [21] and [22].

4. Applications for Unitary Operators

A *unitary operator* is a bounded linear operator $U : H \to H$ on a Hilbert space H satisfying

$$U^*U = UU^* = 1_H$$

where U^* is the adjoint of U, and $1_H : H \to H$ is the identity operator. This property is equivalent to the following:

(i) U preserves the inner product $\langle\cdot,\cdot\rangle$ of the Hilbert space, i.e., for all vectors x and y in the Hilbert space, $\langle Ux, Uy\rangle = \langle x, y\rangle$ and
(ii) U is surjective.

The following result is well known [35, p. 275 - p. 276]:

Theorem 7 (Spectral Representation Theorem)**.** *Let U be a unitary operator on the Hilbert space H. Then there exists a family of projections $\{P_\lambda\}_{\lambda\in[0,2\pi]}$, called the spectral family of U, with the following properties*

a) $P_\lambda \leq P_{\lambda'}$ *for* $\lambda \leq \lambda'$;
b) $P_0 = 0, P_{2\pi} = I$ *and* $P_{\lambda+0} = P_\lambda$ *for all* $\lambda \in [0, 2\pi)$;
c) *We have the representation*

$$U = \int_0^{2\pi} \exp(i\lambda)\, dP_\lambda.$$

More generally, for every continuous complex-valued function f defined on the unit circle $\mathcal{C}(0,1)$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left\| f(U) - \sum_{k=1}^{n} f\left(\exp\left(i\lambda_k'\right)\right)\left[P_{\lambda_k} - P_{\lambda_{k-1}}\right] \right\| \leq \varepsilon$$

whenever

$$\begin{cases} 0=\lambda_1<...<\lambda_{n-1}<\lambda_n=2\pi, \\ \lambda_k-\lambda_{k-1}\le\delta \text{ for } 1\le k\le n, \\ \lambda_k'\in[\lambda_{k-1},\lambda_k] \text{ for } 1\le k\le n \end{cases}$$

this means that

$$f(U)=\int_0^{2\pi} f(\exp(i\lambda))\,dP_\lambda, \tag{4.1}$$

where the integral is of Riemann-Stieltjes type.

Corollary 12. *With the assumptions of Theorem 7 for* U, P_λ *and* f *we have the representations*

$$f(U)x=\int_0^{2\pi} f(\exp(i\lambda))\,dP_\lambda x \quad \text{for all } x\in H$$

and

$$\langle f(U)x,y\rangle=\int_0^{2\pi} f(\exp(i\lambda))\,d\langle P_\lambda x,y\rangle \quad \text{for all } x,y\in H. \tag{4.2}$$

In particular,

$$\langle f(U)x,x\rangle=\int_0^{2\pi} f(\exp(i\lambda))\,d\langle P_\lambda x,x\rangle \quad \text{for all } x\in H.$$

Moreover, we have the equality

$$\|f(U)x\|^2=\int_0^{2\pi} |f(\exp(i\lambda))|^2\,d\|P_\lambda x\|^2 \quad \text{for all } x\in H.$$

The following result holds:

Theorem 8. *Let* U *be a unitary operator on the Hilbert space* H. *Assume that* $\Phi:[0,2\pi]\to\mathbb{R}$ *is a differentiable convex function on* $(0,2\pi)$ *and the derivative* Φ' *is continuous on* $(0,2\pi)$. *If* $f:\mathcal{C}(0,1)\to\mathbb{C}$ *is a continuous function on the unit circle* $\mathcal{C}(0,1)$, *then we have the inequalities*

$$\begin{aligned} &\left(\|f(U)x\|^2-|\langle f(U)x,x\rangle|^2\right)\Phi'\left(|\langle f(U)x,x\rangle|^2\right) \\ &\le\left\langle \Phi\left(|f(U)|^2\right)x,x\right\rangle-\Phi\left(|\langle f(U)x,x\rangle|^2\right) \\ &\le\left\langle \Phi'\left(|f(U)|^2\right)|f(U)|^2x,x\right\rangle-|\langle f(U)x,x\rangle|^2\left\langle \Phi'\left(|f(U)|^2\right)x,x\right\rangle \end{aligned} \tag{4.3}$$

for any $x\in H$, $\|x\|=1$.

If $k=\min_{z\in\mathcal{C}(0,1)}|f(z)|$ *and* $K=\max_{z\in\mathcal{C}(0,1)}|f(z)|$, *then*

$$\begin{aligned} &\left\langle \Phi'\left(|f(U)|^2\right)|f(U)|^2x,x\right\rangle-|\langle f(U)x,x\rangle|^2\left\langle \Phi'\left(|f(U)|^2\right)x,x\right\rangle \\ &\le\left(\|f(U)x\|^2-|\langle f(U)x,x\rangle|^2\right)\left\langle \Phi'\left(|f(U)|^2\right)x,x\right\rangle \\ &+\frac{1}{4}\left(K^2-k^2\right)\left[\Phi'\left(K^2\right)-\Phi'\left(k^2\right)\right] \end{aligned} \tag{4.4}$$

for any $x\in H$, $\|x\|=1$.

Proof. Let $x \in H$ with $\|x\| = 1$ and $\{P_\lambda\}_{\lambda \in [0,2\pi]}$, the spectral family of U. Utilising the inequality (2.1) we have

$$\begin{aligned} &\left(\int_0^{2\pi} |f(\exp(i\lambda))|^2 d\langle P_\lambda x, x\rangle - \left| \int_0^{2\pi} f(\exp(i\lambda))\, d\langle P_\lambda x, x\rangle \right|^2 \right) \\ &\times \Phi'\left(\left| \int_0^{2\pi} f(\exp(i\lambda))\, d\langle P_\lambda x, x\rangle \right|^2 \right) \\ &\le \int_0^{2\pi} \Phi\left(|f(\exp(i\lambda))|^2 \right) d\langle P_\lambda x, x\rangle - \Phi\left(\left| \int_0^{2\pi} f(\exp(i\lambda))\, d\langle P_\lambda x, x\rangle \right|^2 \right) \\ &\le \int_0^{2\pi} \Phi'\left(|f(\exp(i\lambda))|^2 \right) |f(\exp(i\lambda))|^2 d\langle P_\lambda x, x\rangle \\ &- \left| \int_0^{2\pi} f(\exp(i\lambda))\, d\langle P_\lambda x, x\rangle \right|^2 \int_0^{2\pi} \Phi'\left(|f(\exp(i\lambda))|^2 \right) d\langle P_\lambda x, x\rangle . \end{aligned} \tag{4.5}$$

By making use of the Spectral Representation Theorem for unitary operators we get from (4.5) the desired result (4.3). □

Corollary 13. *Let U be a unitary operator on the Hilbert space H. Assume that $\Phi : [0, 2\pi] \to \mathbb{R}$ is a differentiable convex function on $(0, 2\pi)$ and the derivative Φ' is continuous on $(0, 2\pi)$. Then we have the inequalities*

$$\begin{aligned} &\left(\langle \exp[2\operatorname{Re}(U)]\, x, x\rangle - |\langle \exp(U)\, x, x\rangle|^2 \right) \Phi'\left(|\langle \exp(U)\, x, x\rangle|^2 \right) \\ &\le \langle \Phi(\exp[2\operatorname{Re}(U)])\, x, x\rangle - \Phi\left(|\langle \exp(U)\, x, x\rangle|^2 \right) \\ &\le \langle \Phi'(\exp[2\operatorname{Re}(U)]) \exp[2\operatorname{Re}(U)]\, x, x\rangle \\ &- |\langle \exp(U)\, x, x\rangle|^2 \langle \Phi'(\exp[2\operatorname{Re}(U)])\, x, x\rangle \end{aligned} \tag{4.6}$$

for any $x \in H$, $\|x\| = 1$, where

$$\operatorname{Re}(U) = \frac{1}{2}(U^* + U).$$

Proof. If we take in (4.3) $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \exp(z)$, then

$$\begin{aligned} |f(U)|^2 &= |\exp(U)|^2 = [\exp(U)]^* \exp(U) \\ &= \exp(U^*) \exp(U) = \exp(U^* + U) = \exp[2\operatorname{Re}(U)] \end{aligned}$$

since $U^*U = UU^* = I$.

This proves the inequality (4.6). □

Remark 4. *Assume that $f : \mathcal{C}(0,1) \to \mathbb{C}$ is a continuous function on the unit circle $\mathcal{C}(0,1)$. If we take $\Phi(t) = t^r$, $r \ge 1$, then we have from (4.3) and (4.4) that*

$$\begin{aligned} &r\left(\|f(U)\, x\|^2 - |\langle f(U)\, x, x\rangle|^2 \right) |\langle f(U)\, x, x\rangle|^{2(r-1)} \\ &\le \left\langle |f(U)|^{2r} x, x \right\rangle - |\langle f(U)\, x, x\rangle|^{2r} \\ &\le r\left\langle |f(U)|^{2r} x, x \right\rangle - |\langle f(U)\, x, x\rangle|^2 \left\langle |f(U)|^{2(r-1)} x, x \right\rangle \end{aligned} \tag{4.7}$$

for any $x \in H$, $\|x\| = 1$.

If $k=\min_{z\in\mathcal{C}(0,1)}|f(z)|$ *and* $K=\max_{z\in\mathcal{C}(0,1)}|f(z)|$, *then*

$$\begin{aligned}\left\langle |f(U)|^{2r}x,x\right\rangle - |\langle f(U)x,x\rangle|^{2}\left\langle |f(U)|^{2(r-1)}x,x\right\rangle \\ \leq \left(\|f(U)x\|^{2}-|\langle f(U)x,x\rangle|^{2}\right)\left\langle |f(U)|^{2(r-1)}x,x\right\rangle \\ +\frac{1}{4}\left(K^{2}-k^{2}\right)\left[K^{2(r-1)}-k^{2(r-1)}\right]\end{aligned} \tag{4.8}$$

for any $x\in H$, $\|x\|=1$.

Assume that $f(z)\neq 0$ *for any* $z\in\mathcal{C}(0,1)$. *By taking* $\Phi(t)=-\ln t$ *in (4.3) and (4.4), we get*

$$\begin{aligned}\frac{\|f(U)x\|^{2}}{|\langle f(U)x,x\rangle|^{2}}-1 \geq \left\langle \ln\left(|f(U)|^{2}\right)x,x\right\rangle - \ln\left(|\langle f(U)x,x\rangle|^{2}\right) \\ \geq 1-|\langle f(U)x,x\rangle|^{2}\left\langle |f(U)|^{-2}x,x\right\rangle\end{aligned} \tag{4.9}$$

for any $x\in H$, $\|x\|=1$.

For recent inequalities for functions of unitary operators, see [26]-[33].

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa