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NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR $n-$ TIME DIFFERENTIABLE FUNCTIONS WHICH ARE QUASI-CONVEX

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Abstract. In this paper, by using an integral identity and Hölder integral inequality we establish several new inequalities for $n-$ time differentiable mappings that are connected with the celebrated Hermite-Hadamard integral inequality. Some applications for special means of real numbers are also provided.

1. INRODUCTION

On November 22, 1881, Hermite (1822-1901) sent a letter to the Journal Mathesis. This letter was published in Mathesis 3 (1883, p: 82) and in this letter an inequality presented which is well-known in the literature as Hermite-Hadamard integral inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I of a real numbers and $a, b \in I$ with $a < b$. If the function f is concave, the inequality in (1.1) is reversed.

The inequalities (1.1) have become an important cornerstone in mathematical anlysis and optimization. Many uses of these inequalities have been discovered in a variety of settings. Moreover , many inequalities of special means can be obtained for a particular choice of the function f. Due to the rich geometrical significance of Hermite-Hadamard's inequlity, there is growing literature providing its new proofs, extensions, refinements and generalizations, see for example ([1], [5], [9]-[14], [16]-[19]) and the references therein.

Definition 1. *A function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if whenever $x, y \in [a, b]$ and $t \in [0, 1]$, the following inequality holds:*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

We say that f is concave if $(-f)$ is convex. This definition has its origins in Jensen's results from [8] and has opened up the most extended, useful and multidisciplinary domain of mathematics, namely, convex analysis. Convex curves and convex bodies have appeared in mathematical literature since antiquity and there are many important results related to them.

We recall that the notion of quasi-convex functions generalizes the notion of convex functions.

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Definition 2. *A function $f:[a,b]\to\mathbb{R}$ is said to be quasi-convex on $[a,b]$ if*

$$f(tx+(1-t)y)\leq\max\{f(x),f(y)\}, \text{ for all } x,\ y\in[a,b].$$

Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see [7], [15]).

For example, consider the following:

Let $f:\mathbb{R}^{+}\to\mathbb{R}$,

$$f(x)=\ln x,\quad x\in\mathbb{R}^{+}.$$

This function is quasi-convex. However f is not convex functions.

In [1], Alomari, Darus and Dragomir proved the following theorem for quasi-convex functions:

Theorem 1. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be twice differentiable mapping on I°, $a,b\in I$ with $a<b$ and f'' is integrable on $[a,b]$. If $|f''|^{q}$ is an quasi-convex on $[a,b]$, $q\geq 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|\leq\frac{(b-a)^{2}}{12}\left[\max\left\{\left|f''(a)\right|^{q},\left|f''(b)\right|^{q}\right\}\right]^{\frac{1}{q}}. \tag{1.2}$$

In [18], Wang *et al.* proved the following lemma:

Lemma 1. *For $n\in\mathbb{N}$, let $f:I\subset\mathbb{R}\to\mathbb{R}$ be $n-$time differentiable. If $a,b\in I$ with $a<b$ and $f^{(n)}(x)\in L[a,b]$, then*

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_{a}^{b}f(x)dx-\sum_{k=1}^{n-1}\frac{k-1}{2(k+1)!}(a-b)^{k}f^{(f)}(b)\\ =\;&\frac{(-1)^{n}(b-a)^{n}}{2n!}\int_{0}^{1}(1-t)^{n-1}(2t+n-2)f^{(n)}(ta+(1-t)b)dt,\end{aligned}$$

where an empty sum is understood to be nil.

For other recent results concerning the $n-$time differentiable functions see [2]-[4], [6], [9], [12], [18] where further references are given.

The main purpose of the present paper is to establish several new inequalities for $n-$ time differantiable mappings that are connected with the celebrated Hermite-Hadamard integral inequality. Also, some applications for special means of real numbers are provided.

2. MAIN RESULTS

Theorem 2. *For $n\geq 2$, let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be $n-$time differentiable and $0\leq a<b$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^{q}$ is quasi-convex on $[a,b]$, for $q>1$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_{a}^{b}f(x)dx-\sum_{k=1}^{n-1}\frac{k-1}{2(k+1)!}(a-b)^{k}f^{(f)}(b)\right| \qquad (2.1)\\ \leq\;&\frac{(b-a)^{n}}{2n!}\left(\frac{q-1}{nq-1}\right)^{1-\frac{1}{q}}\\ &\times\left(\frac{n^{q+1}-(n-2)^{q+1}}{2q+2}\right)^{\frac{1}{q}}\left[\max\left\{\left|f^{(n)}(a)\right|^{q},\left|f^{(n)}(b)\right|^{q}\right\}\right]^{\frac{1}{q}}.\end{aligned}$$

Proof. Since $\left|f^{(n)}\right|^q$ is quasi-convex on $[a,b]$, for $q>1$, from Lemma 1 and Hölder integral inequality, it follows that

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\sum_{k=1}^{n-1}\frac{k-1}{2(k+1)!}(a-b)^k f^{(f)}(b)\right| \\
\leq\ & \frac{(b-a)^n}{2n!}\int_0^1 (1-t)^{n-1}(2t+n-2)\left|f^{(n)}(ta+(1-t)b)\right|dt
\end{aligned}$$

$$\begin{aligned}
\leq\ & \frac{(b-a)^n}{2n!}\left(\int_0^1 (1-t)^{\frac{(n-1)q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (2t+n-2)^q\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{(b-a)^n}{2n!}\left(\int_0^1 (1-t)^{\frac{(n-1)q}{q-1}}dt\right)^{1-\frac{1}{q}} \\
& \times\left(\int_0^1 (2t+n-2)^q dt\right)^{\frac{1}{q}}\left[\max\left\{\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right\}\right]^{\frac{1}{q}} \\
=\ & \frac{(b-a)^n}{2n!}\left(\frac{q-1}{nq-1}\right)^{1-\frac{1}{q}} \\
& \times\left(\frac{n^{q+1}-(n-2)^{q+1}}{2q+2}\right)^{\frac{1}{q}}\left[\max\left\{\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right\}\right]^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

Corollary 1. *Under conditions of Theorem 2, if we choose $n=2$, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{(b-a)^2}{2(q+1)^{\frac{1}{q}}}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left[\max\left\{\left|f''(a)\right|^q,\left|f''(b)\right|^q\right\}\right]^{\frac{1}{q}} \\
\leq\ & \frac{(b-a)^2}{2}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left[\max\left\{\left|f''(a)\right|^q,\left|f''(b)\right|^q\right\}\right]^{\frac{1}{q}}.
\end{aligned}$$

Theorem 3. *For $n\geq 2$, let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be $n-$time differentiable and $0\leq a<b$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^q$ is quasi-convex on $[a,b]$, then we have*

$$\begin{aligned}
(2.2)\quad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\sum_{k=1}^{n-1}\frac{k-1}{2(k+1)!}(a-b)^k f^{(f)}(b)\right| \\
\leq\ & \frac{(b-a)^n}{2n!}\left(\frac{1}{q(n-1)+1}\right)^{\frac{1}{q}} \\
& \times\left(\frac{(q-1)\left[n^{\frac{2q-1}{q-1}}-(n-2)^{\frac{2q-1}{q-1}}\right]}{2(2q-1)}\right)^{1-\frac{1}{q}}\left[\max\left\{\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right\}\right]^{\frac{1}{q}}
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1 and Hölder integral inequality, we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\sum_{k=1}^{n-1}\frac{k-1}{2(k+1)!}(a-b)^k f^{(f)}(b)\right| \\
\leq\ & \frac{(b-a)^n}{2n!}\int_0^1 (1-t)^{n-1}(2t+n-2)\left|f^{(n)}(ta+(1-t)b)\right|dt \\
\leq\ & \frac{(b-a)^n}{2n!}\left(\int_0^1 (2t+n-2)^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)^{q(n-1)}\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\left|f^{(n)}\right|^q$ is quasi-convex on $[a,b]$, for $q>1$, then

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\sum_{k=1}^{n-1}\frac{k-1}{2(k+1)!}(a-b)^k f^{(f)}(b)\right| \\
\leq\ & \frac{(b-a)^n}{2n!}\left(\int_0^1 (2t+n-2)^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}} \\
& \times\left(\int_0^1 (1-t)^{q(n-1)}dt\right)^{\frac{1}{q}}\left[\max\left\{\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right\}\right]^{\frac{1}{q}} \\
=\ & \frac{(b-a)^n}{2n!}\left(\frac{1}{q(n-1)+1}\right)^{\frac{1}{q}} \\
& \times\left(\frac{(q-1)\left[n^{\frac{2q-1}{q-1}}-(n-2)^{\frac{2q-1}{q-1}}\right]}{2(2q-1)}\right)^{1-\frac{1}{q}}\left[\max\left\{\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right\}\right]^{\frac{1}{q}}
\end{aligned}$$

which completes the proof. □

Corollary 2. *If we choose $n=2$ in the inequality (2.2), then we have*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{(b-a)^2}{2}\left(\frac{(q-1)^{q-1}}{(3q+1)(2q-1)^{q-1}}\right)^{\frac{1}{q}}\left[\max\left\{\left|f''(a)\right|^q,\left|f''(b)\right|^q\right\}\right]^{\frac{1}{q}}.
\end{aligned}$$

Theorem 4. *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be $n-$time differentiable for $n>1$ and $0\leq a<b<\infty$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^q$ is quasi-convex on $[a,b]$, for $q>1$, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\sum_{k=1}^{n-1}\frac{k-1}{2(k+1)!}(a-b)^k f^{(f)}(b)\right| \qquad (2.3)\\
\leq\ & \frac{(b-a)^n}{2n(n-2)!} \\
& \times\left(\frac{p(n-2)+2}{[p(n-1)+1]\,[p(n-1)+2]}\right)^{\frac{1}{p}}\left[\max\left\{\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right\}\right]^{\frac{1}{q}}.
\end{aligned}$$

Proof. Since $\left|f^{(n)}\right|^q$ is quasi-convex on $[a,b]$, for $q>1$, from Lemma 1 and Hölder integral inequality, it follows that

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\sum_{k=1}^{n-1}\frac{k-1}{2(k+1)!}(a-b)^k f^{(f)}(b)\right| \\
\leq\ & \frac{(b-a)^n}{2n!}\int_0^1 (1-t)^{n-1}(2t+n-2)\left|f^{(n)}(ta+(1-t)b)\right|dt \\
\leq\ & \frac{(b-a)^n}{2n!}\left(\int_0^1 (1-t)^{p(n-1)}(2t+n-2)dt\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1 (2t+n-2)\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{(b-a)^n}{2n!}\left(\int_0^1 (1-t)^{p(n-1)}(2t+n-2)dt\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1 (2t+n-2)dt\right)^{\frac{1}{q}}\left[\max\left\{\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right\}\right]^{\frac{1}{q}} \\
=\ & \frac{(b-a)^n}{2n(n-2)!} \\
&\times\left(\frac{p(n-2)+2}{[p(n-1)+1]\,[p(n-1)+2]}\right)^{\frac{1}{p}}\left[\max\left\{\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right\}\right]^{\frac{1}{q}}.
\end{aligned}
$$

On the other hand, we have

$$\int_0^1 (1-t)^{p(n-1)}(2t+n-2)dt=\frac{(n-1)\,[p(n-2)+2]}{[p(n-1)+1]\,[p(n-1)+2]}.$$

Thsi completes the proof. □

Corollary 3. *In Theorem 4, if $n=2$, we have*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{(b-a)^2}{4}\left(\frac{2}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left[\max\left\{\left|f''(a)\right|^q,\left|f''(b)\right|^q\right\}\right]^{\frac{1}{q}}.
\end{aligned}
$$

Theorem 5. *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be $n-$time differentiable for $n>1$ and $0\leq a<b<\infty$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^q$ is quasi-convex on $[a,b]$, for $q>1$, then*

$$
\begin{aligned}
(2.4)\quad &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\sum_{k=1}^{n-1}\frac{k-1}{2(k+1)!}(a-b)^k f^{(f)}(b)\right| \\
\leq\ & \frac{(b-a)^n}{2n!}\left(\frac{1}{q(n-2)+2}\right)^{\frac{1}{q}} \\
&\times\left(\frac{n^{p+2}-(n+2p+2)(n-2)^{p+1}}{4(p+1)(p+2)}\right)^{\frac{1}{p}}\left[\max\left\{\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right\}\right]^{\frac{1}{q}}.
\end{aligned}
$$

Proof. From Lemma 1 and Hölder integral inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\sum_{k=1}^{n-1}\frac{k-1}{2(k+1)!}(a-b)^k f^{(f)}(b)\right| \\
\leq\ & \frac{(b-a)^n}{2n!}\int_0^1 (1-t)^{n-1}(2t+n-2)\left|f^{(n)}(ta+(1-t)b)\right|dt \\
\leq\ & \frac{(b-a)^n}{2n!}\left(\int_0^1 (1-t)(2t+n-2)^p dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)^{nq-2q+1}\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\left|f^{(n)}\right|^q$ is quasi-convex on $[a,b]$, for $q>1$, then we get

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\sum_{k=1}^{n-1}\frac{k-1}{2(k+1)!}(a-b)^k f^{(f)}(b)\right| \\
\leq\ & \frac{(b-a)^n}{2n!}\left(\int_0^1 (1-t)(2t+n-2)^p dt\right)^{\frac{1}{p}} \\
& \times\left(\int_0^1 (1-t)^{nq-2q+1}dt\right)^{\frac{1}{q}}\left[\max\left\{\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right\}\right]^{\frac{1}{q}} \\
=\ & \frac{(b-a)^n}{2n!}\left(\frac{1}{q(n-2)+2}\right)^{\frac{1}{q}} \\
& \times\left(\frac{n^{p+2}-(n+2p+2)(n-2)^{p+1}}{4(p+1)(p+2)}\right)^{\frac{1}{q}}\left[\max\left\{\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right\}\right]^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

Theorem 6. *For $n\geq 2$, let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be $n-$time differentiable and $0\leq a<b$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^q$ is quasi-convex on $[a,b]$, for $q\geq 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\sum_{k=1}^{n-1}\frac{k-1}{2(k+1)!}(a-b)^k f^{(f)}(b)\right| \\
\leq\ & \frac{(b-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)\left[\max\left\{\left|f^{(n)}(a)\right|^q,\left|f^{(n)}(b)\right|^q\right\}\right]^{\frac{1}{q}}.
\end{aligned}\tag{2.5}$$

Proof. From Lemma 1 and using the well known power-mean integral inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\sum_{k=1}^{n-1}\frac{k-1}{2(k+1)!}(a-b)^k f^{(f)}(b)\right| \\
\leq\ & \frac{(b-a)^n}{2n!}\int_0^1 (1-t)^{n-1}(2t+n-2)\left|f^{(n)}(ta+(1-t)b)\right|dt \\
\leq\ & \frac{(b-a)^n}{2n!}\left(\int_0^1 (1-t)^{n-1}(2t+n-2)dt\right)^{1-\frac{1}{q}} \\
& \times\left(\int_0^1 (1-t)^{n-1}(2t+n-2)\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\left|f^{(n)}\right|^q$ is quasi-convex on $[a,b]$, for $q \geq 1$, then we obtain

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \sum_{k=1}^{n-1} \frac{k-1}{2(k+1)!}(a-b)^k f^{(f)}(b) \right| \\ \leq & \frac{(b-a)^n}{2n!}\left(\int_0^1 (1-t)^{n-1}(2t+n-2)dt\right)^{1-\frac{1}{q}} \\ & \times \left(\int_0^1 (1-t)^{n-1}(2t+n-2)dt\right)^{\frac{1}{q}} \left[\max\left\{\left|f^{(n)}(a)\right|^q, \left|f^{(n)}(b)\right|^q\right\}\right]^{\frac{1}{q}} \\ = & \leq \frac{(b-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)\left[\max\left\{\left|f^{(n)}(a)\right|^q, \left|f^{(n)}(b)\right|^q\right\}\right]^{\frac{1}{q}} \end{aligned}$$

whisch completes the proof. □

Corollary 4. *Under conditions of Theorem 6, if we choose* $q = 1$, *then we obtain*

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx - \sum_{k=1}^{n-1} \frac{k-1}{2(k+1)!}(a-b)^k f^{(f)}(b) \right| \\ \leq & \frac{(b-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)\left[\max\left\{\left|f^{(n)}(a)\right|, \left|f^{(n)}(b)\right|\right\}\right]. \end{aligned}$$

Remark 1. *Under conditions of Theorem 6, if we choose* $n = 2$, *then we obtain the inequality (1.2).*

3. APPLICATIONS TO SPECIAL MEANS

We now consider the means for arbitrary real numbers α, β $(\alpha \neq \beta)$. We take

(1) *Arithmetic mean* :

$$A(\alpha,\beta) = \frac{\alpha+\beta}{2}, \quad \alpha,\beta \in \mathbb{R}^+.$$

(2) *Logarithmic mean*:

$$L(\alpha,\beta) = \frac{\alpha-\beta}{\ln|\alpha| - \ln|\beta|}, \quad |\alpha| \neq |\beta|,\ \alpha,\beta \neq 0,\ \alpha,\beta \in \mathbb{R}^+.$$

(3) *Generalized Logarithmic − mean*:

$$L_n(\alpha,\beta) = \left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{\frac{1}{n}}, \quad n \in \mathbb{Z}\backslash\{-1,0\},\ \alpha,\beta \in \mathbb{R}^+.$$

Now using the results of Section 2, we give some applications for special means of real numbers.

Proposition 1. *Let* $a, b \in \mathbb{R}$, $0 < a < b$ *and* $n \in \mathbb{Z}$, $n > 1$, *then, for all* $q > 1$, *the following inequality holds:*

$$|A(a^n,b^n) - L_n^n(a,b)| \leq \frac{n(n-1)(b-a)^2}{2(q+1)^{\frac{1}{q}}}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left[\max\left\{(a^{n-2})^q, (b^{n-2})^q\right\}\right]^{\frac{1}{q}}.$$

Proof. The proof is obvious from Corollary 1 applied to the quasi-convex mapping $f(x) = x^n$, $x \in [a,b]$. □

Proposition 2. *Let $a, b \in \mathbb{R}$, $0 < a < b$ and $n \in \mathbb{Z}$, $n > 1$, then, for all $q > 1$, the following inequality holds:*

$$\begin{aligned} |A(a^n, b^n) - L_n^n(a,b)| &\leq \frac{n(n-1)(b-a)^2}{2} \\ &\times \left(\frac{(q-1)^{q-1}}{(3q+1)(2q-1)^{q-1}} \right)^{\frac{1}{q}} \left[\max\left\{ (a^{n-2})^q, (b^{n-2})^q \right\} \right]^{\frac{1}{q}}. \end{aligned}$$

Proof. The proof is obvious from Corollary 2 applied to the quasi-convex mapping $f(x) = x^n$, $x \in [a, b]$. □

Proposition 3. *Let $a, b \in \mathbb{R}$, $0 < a < b$ and $n \in \mathbb{Z}$, $n > 1$, then, for all $q > 1$, the following inequality holds:*

$$|A(a^n, b^n) - L_n^n(a,b)| \leq \frac{n(n-1)(b-a)^2}{4} \left(\frac{2}{(p+1)(p+2)} \right)^{\frac{1}{p}} \left[\max\left\{ (a^{n-2})^q, (b^{n-2})^q \right\} \right]^{\frac{1}{q}}.$$

Proof. The proof is obvious from Corollary 3 applied to the quasi-convex mapping $f(x) = x^n$, $x \in [a, b]$. □

References

[1] M. Alomari, M. Darus and S.S. Dragomir, New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are quasi-convex. Tamkang Journal of Mathematics, 41(4), 2010, 353-359.

[2] S.-P. Bai, S.-H. Wang and F. Qi, Some Hermite-Hadamard type inequalities for n-time differentiable (α, m)-convex functions, Jour. of Ineq. and Appl., 2012, 2012:267.

[3] P. Cerone, S.S. Dragomir and J. Roumeliotis, Some Ostrowski type inequalities for n-time differentiable mappings and applications, Demonstratio Math., 32 (4) (1999), 697-712.

[4] P. Cerone, S.S. Dragomir and J. Roumeliotis and J. Šunde, A new generalization of the trapezoid formula for n-time differentiable mappings and applications, Demonstratio Math., 33 (4) (2000), 719-736.

[5] S.S. Dragomir and C.E.M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000. Online:[http://www.staxo.vu.edu.au/RGMIA/monographs/hermite hadamard.html].

[6] D.-Y. Hwang, Some Inequalities for n-time Differentiable Mappings and Applications, Kyung. Math. Jour., 43 (2003), 335-343.

[7] D.A. Ion, Some estimates on the Hermite-Hadamard inequalities through quasi-convex functions, Annals of University of Craiova, Math. Comp. Sci. Ser., 34 (2007), 82-87.

[8] J. L. W. V. Jensen, On konvexe funktioner og uligheder mellem middlvaerdier, Nyt. Tidsskr. Math. B., 16, 49-69, 1905.

[9] W.-D. Jiang, D.-W. Niu, Y. Hua and F. Qi, Generalizations of Hermite-Hadamard inequality to n-time differentiable function which are s-convex in the second sense, Analysis (Munich), 32 (2012), 209-220

[10] U.S. Kırmacı, M.K. Bakula, M.E. Özdemir and J. Pečarić, Hadamard-type inequalities for s-convex functions, Appl. Math. and Comp., 193(2007), 26-35.

[11] M.E. Özdemir and U.S. Kırmacı, Two new theorem on mappings uniformly continuous and convex with applications to quadrature rules and means, Appl. Math. and Comp., 143(2003), 269-274.

[12] M.E. Özdemir, Ç. Yıldız, New Inequalities for n-time differentiable functions, Arxiv:1402.4959v1.

[13] M.E. Özdemir, Ç. Yıldız, New Inequalities for Hermite-Hadamard and Simpson Type with Applications, Tamkang J. of Math., 44, 2, 209-216, 2013.

[14] M.E. Özdemir, Ç. Yıldız, A.O. Akdemir and E. Set, New Inequalities of Hadamard Type for Quasi-Convex Functions, AIP Conference Proceedings, 1470, 99-101 (2012), doi: 10.1063/1.4747649.

[15] J.E. Pečarić, F. Porschan and Y.L. Tong, Convex Functions, Partial Orderings, and Statistical Applications, Academic Press Inc., 1992.
[16] A. Saglam, M.Z Sarıkaya and H. Yıldırım, Some new inequalities of Hermite-Hadamard's type, Kyung. Math. Jour., 50 (2010), 399-410.
[17] E. Set, M.E. Özdemir and S.S. Dragomir, On Hadamard-Type Inequalities Involving Several Kinds of Convexity, Jour. of Ineq. and Appl., 2010, 286845.
[18] S.H. Wang, B.-Y. Xi and F. Qi, Some new inequalities of Hermite-Hadamard type for n-time differentiable functions which are m-convex, Analysis (Munich), 32 (2012), 247-262.
[19] B.-Y. Xi and F. Qi, Some integral inequalities of Hermite-Hadamard type for convex functions with applications to means, J. Funct. Spaces Appl., 2012 (2012), http://dx.doi.org/10.1155/2012/980438.

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