Received 26/03/14

JENSEN INTEGRAL INEQUALITY FOR POWER SERIES WITH NONNEGATIVE COEFFICIENTS AND APPLICATIONS

S.S. DRAGOMIR[1,2]

Abstract. We establish in this paper some Jensen's type inequalities for functions defined by power series with nonnegative coefficients. Applications for functions of selfadjoint operators on complex Hilbert spaces are provided as well.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a $\sigma-$ algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$. Assume, for simplicity, that $\int_\Omega d\mu = 1$. Consider the *Lebesgue space*

$$L(\Omega,\mu) := \{f : \Omega \to \mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(t)|\, d\mu(t) < \infty\}.$$

For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(t)\, d\mu(t)$.

Assume that $f, g \in L(\Phi, \mu)$ with $fg \in L(\Phi,\mu)$ and consider the *Čebyšev functional*

$$C(f,g) := \int_\Omega fg d\mu - \int_\Omega f d\mu \int_\Omega g d\mu.$$

It is known that if the function f, g are *synchronous*, i.e.

$$(f(t) - f(s))(g(t) - g(s)) \geq 0$$

for μ-almost every $t, s \in \Omega$, then we have the *Čebyšev inequality*

$$C(f,g) \geq 0. \tag{1.1}$$

If there exists the constants γ, Γ such that $\infty < \gamma \leq f \leq \Gamma < \infty$ μ-almost everywhere on Ω, then we have the following refinement of Grüss' inequality due to Cerone & Dragomir [2], which was obtained for univariate functions of real variable by Cheng & Sun in [3]:

$$\begin{aligned} |C(f,g)| &\leq \frac{1}{2}(\Gamma - \gamma)\int_\Omega \left| g - \int_\Omega g d\mu \right| d\mu \\ &\leq \frac{1}{2}(\Gamma-\gamma)\left[\int_\Omega g^2 d\mu - \left(\int_\Omega g d\mu\right)^2\right]^{1/2}. \end{aligned} \tag{1.2}$$

The constant $\frac{1}{2}$ is sharp [2].

1991 *Mathematics Subject Classification.* Primary 26D15, 26D20; Secondary 47A63.

Key words and phrases. Jensen's inequality, Measurable functions, Lebesgue integral, Selfadjoint operators, Functions of selfadjoint operators.

If there exists the constants $\infty < \lambda \leq g \leq \Lambda < \infty$ μ-almost everywhere on Ω, then we have the sequence of inequalities

$$|C(f,g)| \leq \frac{1}{2}(\Gamma-\gamma)\int_{\Omega}\left|g-\int_{\Omega}gd\mu\right|d\mu \leq \frac{1}{2}(\Gamma-\gamma)\left[\int_{\Omega}g^2d\mu-\left(\int_{\Omega}gd\mu\right)^2\right]^{1/2} \leq \frac{1}{4}(\Gamma-\gamma)(\Lambda-\lambda). \tag{1.3}$$

The inequality between the first and last term in (1.3) is known in the literature as *Grüss' inequality.*

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, the author obtained in 2002 [4] the following result:

Theorem 1. *Let $\Phi : [m,M] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable convex function on (m,M) and $f : \Omega \to [m,M]$ so that $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L(\Omega,\mu)$. Then we have the inequality:*

$$0 \leq \int_{\Omega}\Phi\circ f d\mu - \Phi\left(\int_{\Omega}fd\mu\right) \leq \int_{\Omega}(\Phi'\circ f)fd\mu - \int_{\Omega}\Phi'\circ fd\mu\int_{\Omega}fd\mu \leq \frac{1}{2}[\Phi'(M)-\Phi'(m)]\int_{\Omega}\left|f-\int_{\Omega}fd\mu\right|d\mu. \tag{1.4}$$

In the case of discrete measure, we have:

Corollary 1. *Let $\Phi : [m,M] \to \mathbb{R}$ be a differentiable convex function on (m,M). If $x_i \in [m,M]$ and $w_i \geq 0$ $(i = 1,\dots,n)$ with $W_n := \sum_{i=1}^n w_i = 1$, then one has the reverse of Jensen's weighted discrete inequality:*

$$0 \leq \sum_{i=1}^n w_i\Phi(x_i) - \Phi\left(\sum_{i=1}^n w_ix_i\right) \leq \sum_{i=1}^n w_i\Phi'(x_i)x_i - \sum_{i=1}^n w_i\Phi'(x_i)\sum_{i=1}^n w_ix_i \leq \frac{1}{2}[\Phi'(M)-\Phi'(m)]\sum_{i=1}^n w_i\left|x_i-\sum_{j=1}^n w_jx_j\right|. \tag{1.5}$$

Remark 1. *We notice that the inequality between the first and the second term in (1.5) was proved in 1994 by Dragomir & Ionescu, see* [9].

On making use of the results (1.4) and (1.3), we can state the following string of reverse inequalities

$$
\begin{aligned}
(1.6)\qquad 0 &\leq \int_\Omega \Phi\circ f d\mu - \Phi\left(\int_\Omega f d\mu\right)\\
&\leq \int_\Omega f\cdot(\Phi'\circ f)\,d\mu - \int_\Omega \Phi'\circ f d\mu \int_\Omega f d\mu\\
&\leq \frac{1}{2}\left[\Phi'(M)-\Phi'(m)\right]\int_\Omega\left|f-\int_\Omega f d\mu\right| d\mu\\
&\leq \frac{1}{2}\left[\Phi'(M)-\Phi'(m)\right]\left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right]^{\frac{1}{2}}\\
&\leq \frac{1}{4}\left[\Phi'(M)-\Phi'(m)\right](M-m),
\end{aligned}
$$

provided that $\Phi : [m,M]\subset\mathbb{R}\to\mathbb{R}$ is a differentiable convex function on (m,M) and $f:\Omega\to[m,M]$ so that $\Phi\circ f,\ f,\ \Phi'\circ f,\ (\Phi'\circ f)\,f\in L(\Omega,\mu)$, with $\int_\Omega d\mu = 1$.

For other reverses of Jensen inequality and applications to divergence measures see [6], [7] and [8].

Motivated by the above results we establish in this paper some Jensen's type inequalities for functions defined by power series with nonnegative coefficients. Applications for functions of selfadjoint operators on complex Hilbert spaces are provided as well.

2. The Results

The most important power series with nonnegative coefficients are:

$$
\begin{aligned}
(2.1)\qquad \exp(z) &= \sum_{n=0}^{\infty}\frac{1}{n!}z^n,\ z\in\mathbb{C},\ \frac{1}{1-z}=\sum_{n=0}^{\infty}z^n,\ z\in D(0,1),\\
\ln\frac{1}{1-z} &= \sum_{n=1}^{\infty}\frac{1}{n}z^n,\ z\in D(0,1),\ \cosh z=\sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n},\ z\in\mathbb{C},\\
\sinh z &= \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1},\ z\in\mathbb{C}.
\end{aligned}
$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$
\begin{aligned}
(2.2)\qquad \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) &= \sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\ z\in D(0,1),\\
\sin^{-1}(z) &= \sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,(2n+1)\,n!}z^{2n+1},\ z\in D(0,1),\\
\tanh^{-1}(z) &= \sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\ z\in D(0,1),\\
{}_2F_1(\alpha,\beta,\gamma,z) &:= \sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}z^n,\alpha,\beta,\gamma>0\\
&\quad z\in D(0,1),
\end{aligned}
$$

where Γ is *Gamma function*.

The following results that improves Jensen inequality as well as provides some reverse inequalities can be stated:

Theorem 2. *Let $\Phi(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$. Assume that $f:\Omega\to\mathbb{R}$ is μ-measurable and with $0<f(u)<R$ for μ-almost every u in Ω and such that $\Phi\circ f$, $(\Phi'\circ f)f$, $(\Phi'\circ f)f^{-1}\in L(\Omega,\mu)$. Then we have the inequalities*

$$
\begin{aligned}
(2.3)\quad 0 &\leq \frac{1}{2}\left[\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2\right]\Phi''(0)\\
&\leq \frac{1}{2}\left[\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2\right]\frac{\Phi'\left(\int_\Omega fd\mu\right)-\Phi'(0)}{\int_\Omega fd\mu}\\
&\leq \int_\Omega \Phi\circ fd\mu-\Phi\left(\int_\Omega fd\mu\right)\\
&\leq \frac{1}{2}\left[\int_\Omega\left[\Phi'\circ f-\Phi'(0)\right]fd\mu-\int_\Omega\left[\Phi'\circ f-\Phi'(0)\right]f^{-1}d\mu\left(\int_\Omega fd\mu\right)^2\right]\\
&\leq \frac{1}{2}\left[\int_\Omega(\Phi''\circ f)f^2d\mu-\Phi''(0)\left(\int_\Omega fd\mu\right)^2\right].
\end{aligned}
$$

Proof. If $g:I\to\mathbb{R}$ is a differentiable convex function on the interior $\mathring{I}$ of the interval I then we have the *gradient inequality*

$$g'(t)(t-s)\geq g(t)-g(s)\geq g'(s)(t-s) \tag{2.4}$$

for any $t,s\in\mathring{I}$.

If we write the inequality (2.4) for the power function $g(t)=t^r$, $r\geq 1$ on the interval $(0,\infty)$, then we have

$$rt^{r-1}(t-s)\geq g(t)-g(s)\geq rs^{r-1}(t-s) \tag{2.5}$$

for any $s,t>0$.

Let $n\geq 2$ be a natural number, then $g(t)=t^{n/2}$ is convex on $(0,\infty)$ and by taking $t=x^2$ and $s=y^2$ then we get from (2.5) that

$$\frac{n}{2}x^{n-2}\left(x^2-y^2\right)\geq x^n-y^n\geq\frac{n}{2}y^{n-2}\left(x^2-y^2\right) \tag{2.6}$$

for any $n\geq 2$ and any $x,y\geq 0$.

From (2.6) we have

$$
\begin{aligned}
\frac{n}{2}[f(u)]^{n-2}\left[[f(u)]^2-\left(\int_\Omega fd\mu\right)^2\right]&\geq[f(u)]^n-\left(\int_\Omega fd\mu\right)^n\\
&\geq\frac{n}{2}\left(\int_\Omega fd\mu\right)^{n-2}\left[[f(u)]^2-\left(\int_\Omega fd\mu\right)^2\right]
\end{aligned}
$$

for any $n \geq 2$, or, equivalently

$$\frac{n}{2}\left[f\left(u\right)\right]^{n}-\frac{n}{2}\left[f\left(u\right)\right]^{n-2}\left(\int_{\Omega}fd\mu\right)^{2}\geq\left[f\left(u\right)\right]^{n}-\left(\int_{\Omega}fd\mu\right)^{n} \geq\frac{n}{2}\left(\int_{\Omega}fd\mu\right)^{n-2}\left[\left[f\left(u\right)\right]^{2}-\left(\int_{\Omega}fd\mu\right)^{2}\right] \tag{2.7}$$

for any $n \geq 2$.

Integrating the inequality over u on Ω we get

$$\frac{n}{2}\int_{\Omega}f^{n}d\mu-\frac{n}{2}\int_{\Omega}f^{n-2}d\mu\left(\int_{\Omega}fd\mu\right)^{2}\geq\int_{\Omega}f^{n}d\mu-\left(\int_{\Omega}fd\mu\right)^{n} \geq\frac{n}{2}\left(\int_{\Omega}fd\mu\right)^{n-2}\left[\int_{\Omega}f^{2}d\mu-\left(\int_{\Omega}fd\mu\right)^{2}\right] \tag{2.8}$$

for any $n \geq 2$, which is an inequality of interest in itself.

Let $m \geq 2$. If we multiply (2.8) by $a_n \geq 0$ and sum over n from 2 to m we get

$$\begin{aligned}&\frac{1}{2}\int_{\Omega}\left(\sum_{n=2}^{m}na_{n}f^{n}\right)d\mu-\frac{1}{2}\int_{\Omega}\left(\sum_{n=2}^{m}na_{n}f^{n-2}\right)d\mu\left(\int_{\Omega}fd\mu\right)^{2}\\&\geq\int_{\Omega}\left(\sum_{n=2}^{m}a_{n}f^{n}\right)d\mu-\sum_{n=2}^{m}a_{n}\left(\int_{\Omega}fd\mu\right)^{n}\\&\geq\frac{1}{2}\sum_{n=2}^{m}na_{n}\left(\int_{\Omega}fd\mu\right)^{n-2}\left[\int_{\Omega}f^{2}d\mu-\left(\int_{\Omega}fd\mu\right)^{2}\right].\end{aligned} \tag{2.9}$$

Observe that

$$\begin{aligned}&\int_{\Omega}\left(\sum_{n=0}^{m}a_{n}f^{n}\right)d\mu-\sum_{n=0}^{m}a_{n}\left(\int_{\Omega}fd\mu\right)^{n}\\&=\int_{\Omega}a_{0}d\mu-a_{0}\left(\int_{\Omega}fd\mu\right)^{0}+\int_{\Omega}\left(a_{1}f\right)d\mu-a_{1}\left(\int_{\Omega}fd\mu\right)^{1}\\&+\int_{\Omega}\left(\sum_{n=2}^{m}a_{n}f^{n}\right)d\mu-\sum_{n=2}^{m}a_{n}\left(\int_{\Omega}fd\mu\right)^{n}\\&=\int_{\Omega}\left(\sum_{n=2}^{m}a_{n}f^{n}\right)d\mu-\sum_{n=2}^{m}a_{n}\left(\int_{\Omega}fd\mu\right)^{n}\end{aligned}$$

for any $m \geq 2$.

From (2.9) we get

$$\begin{aligned}&\frac{1}{2}\int_{\Omega}\left(\sum_{n=2}^{m}na_{n}f^{n}\right)d\mu-\frac{1}{2}\int_{\Omega}\left(\sum_{n=2}^{m}na_{n}f^{n-2}\right)d\mu\left(\int_{\Omega}fd\mu\right)^{2}\\&\geq\int_{\Omega}\left(\sum_{n=0}^{m}a_{n}f^{n}\right)d\mu-\sum_{n=0}^{m}a_{n}\left(\int_{\Omega}fd\mu\right)^{n}\\&\geq\frac{1}{2}\sum_{n=2}^{m}na_{n}\left(\int_{\Omega}fd\mu\right)^{n-2}\left[\int_{\Omega}f^{2}d\mu-\left(\int_{\Omega}fd\mu\right)^{2}\right],\end{aligned} \tag{2.10}$$

for any $m \geq 2$.

Observe that the power series $\sum_{n=2}^{\infty} na_n z^n$ and $\sum_{n=2}^{m} na_n z^{n-2}$ are convergent on $D(0,R)$ and

$$\sum_{n=2}^{\infty} na_n z^n = z\sum_{n=2}^{\infty} na_n z^{n-1} = z\left(\sum_{n=1}^{\infty} na_n z^{n-1} - a_1\right)$$
$$= z\left(\Phi'(z) - \Phi'(0)\right),\ z \in D(0,R)$$

while

$$\sum_{n=2}^{\infty} na_n z^{n-2} = \frac{1}{z}\sum_{n=2}^{\infty} na_n z^{n-1} = \frac{\Phi'(z) - \Phi'(0)}{z},\ z \in D(0,R) \setminus \{0\}.$$

Since $0 < f(u) < R$ for μ-almost every u in Ω, then $0 < \int_{\Omega} f d\mu < R$, the series $\sum_{n=2}^{m} na_n [f(u)]^n$, $\sum_{n=2}^{m} na_n [f(u)]^{n-2}$, $\sum_{n=0}^{m} a_n [f(u)]^n$ are convergent for μ-almost every u in Ω, $\sum_{n=0}^{\infty} a_n \left(\int_{\Omega} f d\mu\right)^n$ and $\sum_{n=2}^{\infty} na_n \left(\int_{\Omega} f d\mu\right)^{n-2}$ are convergent and

$$\sum_{n=2}^{m} na_n [f(u)]^n = f(u)\left(\Phi'(f(u)) - \Phi'(0)\right),$$

$$\sum_{n=2}^{m} na_n [f(u)]^{n-2} = \frac{\Phi'(f(u)) - \Phi'(0)}{f(u)},\ \sum_{n=0}^{m} a_n [f(u)]^n = \Phi(f(u))$$

for μ-almost every u in Ω.

We also have

$$\sum_{n=0}^{\infty} a_n \left(\int_{\Omega} f d\mu\right)^n = \Phi\left(\int_{\Omega} f d\mu\right)$$

and

$$\sum_{n=2}^{\infty} na_n \left(\int_{\Omega} f d\mu\right)^{n-2} = \frac{\Phi'\left(\int_{\Omega} f d\mu\right) - \Phi'(0)}{\int_{\Omega} f d\mu}.$$

By taking the limit in (2.10) over $m \to \infty$, swapping the limit with the integral, we get the third and fourth inequalities in (2.3).

Since Φ' is also a convex function on $(0,R)$ then we have by (2.4)

$$\Phi'\left(\int_{\Omega} f d\mu\right) - \Phi'(0) \geq \Phi''(0)\int_{\Omega} f d\mu$$

and since $\int_{\Omega} f d\mu > 0$, we obtain the second inequality in (2.3). The first inequality is obvious.

By the inequality (2.4) applied for Φ' we also have

$$\Phi''(0) \leq \frac{\Phi'(f(u)) - \Phi'(0)}{f(u)} \leq \Phi''(f(u))$$

for μ-almost every u in Ω.

This implies that

$$\int_{\Omega} \left(\Phi' \circ f - \Phi'(0)\right) f^{-1} d\mu \geq \Phi''(0)$$

and

$$\int_{\Omega} \left(\Phi' \circ f - \Phi'(0)\right) f d\mu \leq \int_{\Omega} \left(\Phi'' \circ f\right) f^2 d\mu,$$

which prove the fifth inequality in (2.3). □

Remark 2. *Let* $\Phi(z)=\sum_{n=0}^{\infty}a_nz^n$ *be a power series with nonnegative coefficients and convergent on the open disk* $D(0,R)$ *with* $R>0$ *or* $R=\infty$. *If* $x_i\in(0,R)$ *and* $w_i\geq 0\ \ (i=1,\dots,n)$ *with* $W_n:=\sum_{i=1}^{n}w_i=1$, *then we have the inequalities*

$$\begin{aligned}(2.11)\quad 0&\leq\frac{1}{2}\left[\sum_{i=1}^{n}w_ix_i^2-\left(\sum_{i=1}^{n}w_ix_i\right)^2\right]\Phi''(0)\\&\leq\frac{1}{2}\left[\sum_{i=1}^{n}w_ix_i^2-\left(\sum_{i=1}^{n}w_ix_i\right)^2\right]\frac{\Phi'\left(\sum_{i=1}^{n}w_ix_i\right)-\Phi'(0)}{\sum_{i=1}^{n}w_ix_i}\\&\leq\sum_{i=1}^{n}w_i\Phi(x_i)-\Phi\left(\sum_{i=1}^{n}w_ix_i\right)\\&\leq\frac{1}{2}\left[\sum_{i=1}^{n}w_ix_i\left[\Phi'(x_i)-\Phi'(0)\right]-\sum_{i=1}^{n}\frac{w_i}{x_i}\left[\Phi'(x_i)-\Phi'(0)\right]\left(\sum_{i=1}^{n}w_ix_i\right)^2\right]\\&\leq\frac{1}{2}\left[\sum_{i=1}^{n}w_ix_i^2\Phi''(x_i)-\Phi''(0)\left(\sum_{i=1}^{n}w_ix_i\right)^2\right].\end{aligned}$$

We have the following particular inequalities of interest:

Corollary 2. *Assume that* $f:\Omega\to\mathbb{R}$ *is* μ*-measurable and with* $0<f(u)$ *for* μ*-almost every* u *in* Ω *and such that* $\exp\circ f,\ (\exp\circ f)f,\ (\exp\circ f)f^{-1}\in L(\Omega,\mu)$. *Then we have the inequalities*

$$\begin{aligned}(2.12)\qquad 0&\leq\frac{1}{2}\left[\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2\right]\\&\leq\frac{1}{2}\left[\int_\Omega f^2d\mu-\left(\int_\Omega fd\mu\right)^2\right]\frac{\exp\left(\int_\Omega fd\mu\right)-1}{\int_\Omega fd\mu}\\&\leq\int_\Omega\exp\circ fd\mu-\exp\left(\int_\Omega fd\mu\right)\\&\leq\frac{1}{2}\left[\int_\Omega(\exp\circ f-1)fd\mu-\int_\Omega(\exp\circ f-1)f^{-1}d\mu\left(\int_\Omega fd\mu\right)^2\right]\\&\leq\frac{1}{2}\left[\int_\Omega f^2\exp\circ fd\mu-\left(\int_\Omega fd\mu\right)^2\right].\end{aligned}$$

The inequality (2.12) follows by (2.3) for $\Phi(z)=\exp(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^n,\ z\in\mathbb{C}$.

If we use the inequality (2.3) for $\Phi(z)=\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n,\ z\in D(0,1)$, then we can state:

Corollary 3. *Assume that* $f:\Omega\to\mathbb{R}$ *is* μ*-measurable and with* $0<f(u)<1$ *for* μ*-almost every* u *in* Ω *and such that* $(1-f)^{-1},\ (1-f)^{-2}f,\ (1-f)^{-2}f^{-1}\in L(\Omega,\mu)$.

Then we have the inequalities

$$0 \leq \int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2 \tag{2.13}$$
$$\leq \frac{1}{2}\left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right] \frac{2-\int_\Omega f d\mu}{\left(1-\int_\Omega f d\mu\right)^2}$$
$$\leq \int_\Omega (1-f)^{-1} d\mu - \left(1-\int_\Omega f d\mu\right)^{-1}$$
$$\leq \frac{1}{2}\left[\int_\Omega \frac{(2-f) f^2}{(1-f)^2} d\mu - \int_\Omega \frac{2-f}{(1-f)^2} d\mu \left(\int_\Omega f d\mu\right)^2\right]$$
$$\leq \int_\Omega \frac{f^2}{(1-f)^3} d\mu - \left(\int_\Omega f d\mu\right)^2.$$

We also have:

Corollary 4. *Assume that $f : \Omega \to \mathbb{R}$ is μ-measurable and with $0 < f(u) < 1$ for μ-almost every u in Ω and such that $\ln(1-f)^{-1}$, $(1-f)^{-1} f$, $(1-f)^{-1} f^{-1} \in L(\Omega,\mu)$. Then we have the inequalities*

$$0 \leq \frac{1}{2}\left[\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right] \leq \frac{1}{2}\frac{\int_\Omega f^2 d\mu - \left(\int_\Omega f d\mu\right)^2}{\left(1-\int_\Omega f d\mu\right)\int_\Omega f d\mu} \tag{2.14}$$
$$\leq \int_\Omega \ln(1-f)^{-1} d\mu - \ln\left(1-\int_\Omega f d\mu\right)^{-1}$$
$$\leq \frac{1}{2}\left[\int_\Omega \frac{f^2}{1-f} d\mu - \int_\Omega \frac{1}{1-f} d\mu \left(\int_\Omega f d\mu\right)^2\right]$$
$$\leq \frac{1}{2}\left[\int_\Omega \left(\frac{f}{1-f}\right)^2 d\mu - \left(\int_\Omega f d\mu\right)^2\right].$$

The proof follows by (2.3) for $\Phi(z) = \ln\frac{1}{1-z} = \sum_{n=1}^{\infty}\frac{1}{n}z^n$, $z \in D(0,1)$.

3. Applications for Functions of Selfadjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$. Let $A \in \mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda \in \mathbb{R}$ as follows

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \leq \lambda, \\ \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

Then for every $\lambda \in \mathbb{R}$ the operator $E_\lambda := \varphi_\lambda(A)$ is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [12, p. 256]:

Theorem 3 (Spectral Representation Theorem)**.** *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m = \min\{\lambda \,|\lambda \in Sp(A)\} =: \min Sp(A)$ and $M = \max\{\lambda\,|\lambda \in Sp(A)\} =: \max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, called the spectral family of A, with the following properties*

a) $E_\lambda \le E_{\lambda'}$ *for* $\lambda \le \lambda'$;
b) $E_{m-0} = 0, E_M = I$ *and* $E_{\lambda+0} = E_\lambda$ *for all* $\lambda \in \mathbb{R}$;
c) *We have the representation* $A = \int_{m-0}^{M} \lambda dE_\lambda$.

More generally, for every continuous complex-valued function φ *defined on* $\mathbb{R}$ *and for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi(\lambda_k') \left[E_{\lambda_k} - E_{\lambda_{k-1}} \right] \right\| \le \varepsilon \tag{3.1}$$

whenever

$$\begin{cases} \lambda_0 < m = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = M, \\ \lambda_k - \lambda_{k-1} \le \delta \text{ for } 1 \le k \le n, \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \le k \le n \end{cases} \tag{3.2}$$

this means that

$$\varphi(A) = \int_{m-0}^{M} \varphi(\lambda)\, dE_\lambda, \tag{3.3}$$

where the integral is of Riemann-Stieltjes type.

Corollary 5. *With the assumptions of Theorem 3 for* A, E_λ *and* φ *we have the representations*

$$\varphi(A)\, x = \int_{m-0}^{M} \varphi(\lambda)\, dE_\lambda x \quad \text{for all } x \in H \tag{3.4}$$

and

$$\langle \varphi(A)\, x, y \rangle = \int_{m-0}^{M} \varphi(\lambda)\, d \langle E_\lambda x, y \rangle \quad \text{for all } x, y \in H. \tag{3.5}$$

In particular,

$$\langle \varphi(A)\, x, x \rangle = \int_{m-0}^{M} \varphi(\lambda)\, d \langle E_\lambda x, x \rangle \quad \text{for all } x \in H. \tag{3.6}$$

Moreover, we have the equality

$$\|\varphi(A)\, x\|^2 = \int_{m-0}^{M} |\varphi(\lambda)|^2\, d \|E_\lambda x\|^2 \quad \text{for all } x \in H. \tag{3.7}$$

The next result shows that it is legitimate to talk about "*the*" spectral family of the bounded selfadjoint operator A since it is uniquely determined by the requirements a), b) and c) in Theorem 3, see for instance [12, p. 258]:

Theorem 4. *Let* A *be a bonded selfadjoint operator on the Hilbert space* H *and let* $m = \min Sp(A)$ *and* $M = \max Sp(A)$. *If* $\{F_\lambda\}_{\lambda \in \mathbb{R}}$ *is a family of projections satisfying the requirements a), b) and c) in Theorem 3, then* $F_\lambda = E_\lambda$ *for all* $\lambda \in \mathbb{R}$ *where* E_λ *is defined above.*

By the above two theorems, the spectral family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ uniquely determines and in turn is uniquely determined by the bounded selfadjoint operator A.

Theorem 5. *Let $\Phi(z)=\sum_{n=0}^{\infty}a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$ and A a bonded selfadjoint operator on the Hilbert space H with $m=\min Sp(A)$ and $M=\max Sp(A)$. Assume that $f:I\to\mathbb{R}$ is continuous on I with $[m,M]\subset\mathring{I}$ and $0<f(u)<R$ for any $u\in\mathring{I}$. Then we have the inequalities*

$$
\begin{aligned}
(3.8)\qquad 0 &\le \frac{1}{2}\left[\left\langle f^2(A)x,x\right\rangle-\langle f(A)x,x\rangle^2\right]\Phi''(0)\\
&\le \frac{1}{2}\left[\left\langle f^2(A)x,x\right\rangle-\langle f(A)x,x\rangle^2\right]\frac{\Phi'(\langle f(A)x,x\rangle)-\Phi'(0)}{\langle f(A)x,x\rangle}\\
&\le \langle(\Phi\circ f)(A)x,x\rangle-\Phi(\langle f(A)x,x\rangle)\\
&\le \frac{1}{2}[\langle((\Phi'\circ f)(A)-\Phi'(0)I)f(A)x,x\rangle\\
&-\left\langle((\Phi'\circ f)(A)-\Phi'(0)I)f^{-1}(A)x,x\right\rangle\langle f(A)x,x\rangle^2\Big]\\
&\le \frac{1}{2}\left[\left\langle(\Phi''\circ f)(A)f^2(A)x,x\right\rangle-\Phi''(0)(\langle f(A)x,x\rangle)^2\right]
\end{aligned}
$$

for any $x\in H$, $\|x\|=1$.

In particular, if $0<m\le M<R$, then

$$
\begin{aligned}
(3.9)\quad 0 &\le \frac{1}{2}\left[\left\langle A^2x,x\right\rangle-\langle Ax,x\rangle^2\right]\Phi''(0)\\
&\le \frac{1}{2}\left[\left\langle A^2x,x\right\rangle-\langle Ax,x\rangle^2\right]\frac{\Phi'(\langle Ax,x\rangle)-\Phi'(0)}{\langle Ax,x\rangle}\\
&\le \langle\Phi(A)x,x\rangle-\Phi(\langle Ax,x\rangle)\\
&\le \frac{1}{2}\left[\langle(\Phi'(A)-\Phi'(0)I)Ax,x\rangle-\left\langle(\Phi'(A)-\Phi'(0)I)A^{-1}x,x\right\rangle\langle Ax,x\rangle^2\right]\\
&\le \frac{1}{2}\left[\left\langle\Phi''(A)A^2x,x\right\rangle-\Phi''(0)\langle Ax,x\rangle^2\right]
\end{aligned}
$$

for any $x\in H$, $\|x\|=1$.

Proof. Let $x\in H$, $\|x\|=1$. For small $\varepsilon>0$, consider $f:[m-\varepsilon,M]\to\mathbb{R}$ continuous and $g(\lambda)=\langle E_\lambda x,x\rangle$ monotonic nondecreasing on $[m-\varepsilon,M]$. Utilising the inequality (2.3) for the positive measure $d\mu=dg$ we have

$$
\begin{aligned}
0 &\le \frac{1}{2}\left[\int_{m-\varepsilon}^{M}f^2(\lambda)\,d\langle E_\lambda x,x\rangle-\left(\int_{m-\varepsilon}^{M}f(\lambda)\,d\langle E_\lambda x,x\rangle\right)^2\right]\Phi''(0)\\
&\le \frac{1}{2}\left[\int_{m-\varepsilon}^{M}f^2(\lambda)\,d\langle E_\lambda x,x\rangle-\left(\int_{m-\varepsilon}^{M}f(\lambda)\,d\langle E_\lambda x,x\rangle\right)^2\right]\\
&\times\frac{\Phi'\left(\int_{m-\varepsilon}^{M}f(\lambda)\,d\langle E_\lambda x,x\rangle\right)-\Phi'(0)}{\int_{m-\varepsilon}^{M}f(\lambda)\,d\langle E_\lambda x,x\rangle}
\end{aligned}
$$

$$\leq \int_{m-\varepsilon}^{M} \Phi(f(\lambda))\, d\langle E_\lambda x, x\rangle - \Phi\left(\int_{m-\varepsilon}^{M} f(\lambda)\, d\langle E_\lambda x, x\rangle\right)$$
$$\leq \frac{1}{2}\left[\int_{m-\varepsilon}^{M} \left(\Phi'(f(\lambda)) - \Phi'(0)\right) f(\lambda)\, d\langle E_\lambda x, x\rangle\right.$$
$$\left. - \int_{m-\varepsilon}^{M} \left(\Phi'(f(\lambda)) - \Phi'(0)\right) f^{-1}(\lambda)\, d\langle E_\lambda x, x\rangle \left(\int_{m-\varepsilon}^{M} f(\lambda)\, d\langle E_\lambda x, x\rangle\right)^2\right]$$
$$\leq \frac{1}{2}\left[\int_{m-\varepsilon}^{M} \Phi''(f(\lambda)) f^2(\lambda)\, d\langle E_\lambda x, x\rangle - \Phi''(0)\left(\int_{m-\varepsilon}^{M} f(\lambda)\, d\langle E_\lambda x, x\rangle\right)^2\right].$$

Taking the limit over $\varepsilon \to 0+$ we deduce the desired result (3.8). □

We can give some examples as follows:

Example 1. *If $A > 0$ (is a positive definite operator) on H, then we have the exponential inequalities*

$$\begin{aligned}
0 &\leq \frac{1}{2}\left[\left\langle A^2 x, x\right\rangle - \langle Ax, x\rangle^2\right] \\
&\leq \frac{1}{2}\left[\left\langle A^2 x, x\right\rangle - \langle Ax, x\rangle^2\right] \frac{\exp(\langle Ax, x\rangle) - 1}{\langle Ax, x\rangle} \\
&\leq \langle \exp(A) x, x\rangle - \exp(\langle Ax, x\rangle) \\
&\leq \frac{1}{2}\left[\langle (\exp(A) - I) Ax, x\rangle - \left\langle (\exp(A) - I) A^{-1} x, x\right\rangle \langle Ax, x\rangle^2\right] \\
&\leq \frac{1}{2}\left[\left\langle A^2 \exp(A) x, x\right\rangle - \langle Ax, x\rangle^2\right]
\end{aligned} \tag{3.10}$$

for any $x \in H$, $\|x\| = 1$.

Example 2. *If $0 < A < I$, then we have*

$$\begin{aligned}
0 &\leq \left\langle A^2 x, x\right\rangle - \langle Ax, x\rangle^2 \\
&\leq \frac{1}{2}\left[\left\langle A^2 x, x\right\rangle - \langle Ax, x\rangle^2\right] \frac{\langle (2I - A) x, x\rangle}{\langle (I - A) x, x\rangle^2} \\
&\leq \left\langle (I - A)^{-1} x, x\right\rangle - \langle (I - A) x, x\rangle^{-1} \\
&\leq \frac{1}{2}\left[\left\langle (2I - A) A^2 (I - A)^{-2} x, x\right\rangle - \left\langle (2I - A)(I - A)^{-2} x, x\right\rangle \langle Ax, x\rangle^2\right] \\
&\leq \left\langle A^2 (I - A)^{-3} x, x\right\rangle - \langle Ax, x\rangle^2
\end{aligned} \tag{3.11}$$

for any $x \in H$, $\|x\| = 1$.

Example 3. *If* $0 < A < I$, *then we have the logarithmic inequalities*

$$\begin{aligned} 0 &\leq \frac{1}{2}\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right] \\ &\leq \frac{1}{2}\frac{\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2}{\langle (I-A)\,x,x\rangle\,\langle Ax,x\rangle} \\ &\leq \left\langle \ln (I-A)^{-1}x,x\right\rangle - \ln\langle (I-A)\,x,x\rangle^{-1} \\ &\leq \frac{1}{2}\left[\left\langle A^2 (I-A)^{-1}x,x\right\rangle - \left\langle (I-A)^{-1}x,x\right\rangle \langle Ax,x\rangle^2\right] \\ &\leq \frac{1}{2}\left[\left\langle A^2 (I-A)^{-2}x,x\right\rangle - \langle Ax,x\rangle^2\right] \end{aligned} \tag{3.12}$$

for any $x \in H$, $\|x\| = 1$.

For recent inequalities for continuous functions of selfadjoint operators see the papers [1], [5], the monographs [10], [11] and the references therein.

References

[1] R. P. Agarwal and S. S. Dragomir, A survey of Jensen type inequalities for functions of selfadjoint operators in Hilbert spaces. *Comput. Math. Appl.* **59** (2010), no. 12, 3785–3812.

[2] P. Cerone and S. S. Dragomir, A refinement of the Grüss inequality and applications, *Tamkang J. Math.* **38**(2007), No. 1, 37-49. Preprint *RGMIA Res. Rep. Coll.,* **5**(2) (2002), Art. 14.

[3] X.-L. Cheng and J. Sun, Note on the perturbed trapezoid inequality, *J. Inequal. Pure & Appl. Math.,* **3**(2) (2002), Art. 21.

[4] S. S. Dragomir, A Grüss type inequality for isotonic linear functionals and applications. *Demonstratio Math.* **36** (2003), no. 3, 551–562. Preprint RGMIA *Res. Rep. Coll.* **5**(2002), Suplement, Art. 12. [Online http://rgmia.org/v5(E).php].

[5] S. S. Dragomir, Some reverses of the Jensen inequality for functions of selfadjoint operators in Hilbert spaces. *J. Inequal. Appl.* **2010**, Art. ID 496821, 15 pp.

[6] S. S. Dragomir, Reverses of the Jensen inequality in terms of the first derivative and applications, *Acta Math. Vietnam.* **38** (2013), no. 3, 429–446. Preprint *RGMIA Res. Rep. Coll.* **14** (2011), Art. 71. [http://rgmia.org/papers/v14/v14a71.pdf].

[7] S. S. Dragomir, Some reverses of the Jensen inequality with applications, *Bull. Aust. Math. Soc.* **87** (2013), no. 2, 177–194. Preprint *RGMIA Res. Rep. Coll.* **14** (2011), Art. 72. [http://rgmia.org/papers/v14/v14a72.pdf].

[8] S. S. Dragomir, A refinement and a divided difference reverse of Jensen's inequality with applications, Preprint *RGMIA Res. Rep. Coll.* **14** (2011), Art. 74. [http://rgmia.org/papers/v14/v14a74.pdf].

[9] S. S. Dragomir and N. M. Ionescu, Some converse of Jensen's inequality and applications. *Rev. Anal. Numér. Théor. Approx.* **23** (1994), no. 1, 71–78.

[10] S. S. Dragomir, *Operator Inequalities of the Jensen, Čebyšev and Grüss Type.* Springer Briefs in Mathematics. Springer, New York, 2012. xii+121 pp. ISBN: 978-1-4614-1520-6

[11] S. S. Dragomir, *Operator Inequalities of Ostrowski and Trapezoidal Type.* Springer Briefs in Mathematics. Springer, New York, 2012. x+112 pp. ISBN: 978-1-4614-1778-1

[12] G. Helmberg, *Introduction to Spectral Theory in Hilbert Space*, John Wiley & Sons, Inc. -New York, 1969.

[1]Mathematics, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa.