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Complete Fractional Monotone Approximation

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here is developed the theory of complete fractional simultaneous monotone uniform polynomial approximation with rates using mixed fractional linear differential operators.

To achieve that, we establish first ordinary simultaneous polynomial approximation with respect to the highest order right and left fractional derivatives of the function under approximation using their moduli of continuity. Then we derive the complete right and left fractional simultaneous polynomial approximation with rates, as well we treat their affine combination. Based on the last and elegant analytical techniques, we derive preservation of monotonicity by mixed fractional linear differential operators. We study special cases.

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1 Introduction

The topic of monotone approximation started in [5] has become a major trend in approximation theory. A typical problem in this subject is: given a positive integer k, approximate a given function whose kth derivative is ≥ 0 by polynomials having this property.

In [2] the authors replaced the kth derivative with a linear differential operator of order k. We mention this motivating result.

Theorem 1 *Let h, k, p be integers, $0 \leq h \leq k \leq p$ and let f be a real function, $f^{(p)}$ continuous in $[-1, 1]$ with first modulus of continuity $\omega_1\left(f^{(p)}, x\right)$ there. Let $a_j(x)$, $j = h, h+1, ..., k$ be real functions, defined and bounded on $[-1, 1]$*

and assume $a_h(x)$ is either $\geq$ some number $\alpha > 0$ or $\leq$ some number $\beta < 0$ throughout $[-1,1]$. Consider the operator

$$L = \sum_{j=h}^{k} a_j(x) \left[\frac{d^j}{dx^j} \right] \tag{1}$$

and suppose, throughout $[-1,1]$,

$$L(f) \geq 0. \tag{2}$$

Then, for every integer $n \geq 1$, there is a real polynomial $Q_n(x)$ of degree $\leq n$ such that

$$L(Q_n) \geq 0 \text{ throughout } [-1,1] \tag{3}$$

and

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq C n^{k-p} \omega_1 \left(f^{(p)}, \frac{1}{n} \right), \tag{4}$$

where C is independent of n or f.

The purpose of this article is to extend completely Theorem 1 to the fractional level. All involved ordinary derivatives will become now fractional derivatives and even more we will have fractional simultaneous approximation.

We need and make

Definition 2 *([3], p. 50) Let $\alpha > 0$ and $\lceil \alpha \rceil = m$, ($\lceil \cdot \rceil$ ceiling of the number). Consider $f \in AC^m([0,1])$ (space of functions f with $f^{(m-1)} \in AC([0,1])$, absolutely continuous functions), $z \in [0,1]$. We define the left Caputo fractional derivative of f of order α as follows:*

$$\left(D_{*z}^{\alpha} f\right)(x) = \frac{1}{\Gamma(m-\alpha)} \int_z^x (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{5}$$

for any $x \in [z,1]$, where Γ is the gamma function.

We set

$$D_{*z}^{0} f(x) = f(x),$$
$$D_{*z}^{m} f(x) = f^{(m)}(x), \ \forall\, x \in [z,1]. \tag{6}$$

Definition 3 *([4]) Let $\alpha > 0$ and $\lceil \alpha \rceil = m$. Consider $f \in AC^m([0,1])$, $z \in [0,1]$. We define the right Caputo fractional derivative of f of order α as follows:*

$$\left(D_{z-}^{\alpha} f\right)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^z (t-x)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{7}$$

for any $x \in [0,z]$.

We set

$$D_{z-}^{0} f(x) = f(x),$$
$$D_{z-}^{m} f(x) = (-1)^m f^{(m)}(x), \ \forall\, x \in [0,z]. \tag{8}$$

Remark 4 *(to Definitions 2, 3) Let* $n \in \mathbb{N}$ *with* $f^{(n)} \in AC^m([0,1])$*, where* $\alpha > 0$*,* $\lceil \alpha \rceil = m$*, with* $\alpha \notin \mathbb{N}$*, here* $\lceil n+\alpha \rceil = n + \lceil \alpha \rceil = n+m$*, then*

$$\left(D_{*z}^{\alpha} f^{(n)}\right)(x) = \frac{1}{\Gamma(m-\alpha)} \int_z^x (x-t)^{m-\alpha-1} \left(f^{(n)}(t)\right)^{(m)} dt =$$

$$\frac{1}{\Gamma((n+m)-(n+\alpha))} \int_z^x (x-t)^{(n+m)-(n+\alpha)-1} f^{(n+m)}(t)\, dt = D_{*z}^{n+\alpha} f(x). \tag{9}$$

That is

$$\left(D_{*z}^{\alpha} f^{(n)}\right)(x) = D_{*z}^{n+\alpha} f(x), \quad \forall\, x \in [z,1]. \tag{10}$$

Similarly we get

$$\left(D_{z-}^{\alpha} f^{(n)}\right)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^z (t-x)^{m-\alpha-1} \left(f^{(n)}(t)\right)^{(m)} dt =$$

$$\frac{(-1)^{n+m}(-1)^n}{\Gamma((n+m)-(n+\alpha))} \int_x^z (t-x)^{(n+m)-(n+\alpha)-1} f^{(n+m)}(t)\, dt = (-1)^n D_{z-}^{n+\alpha} f(z). \tag{11}$$

That is

$$\left(D_{z-}^{\alpha} f^{(n)}\right)(x) = (-1)^n D_{z-}^{n+\alpha} f(z), \quad \forall\, x \in [0,z]. \tag{12}$$

We need the following:

Consider $f \in C([0,1])$ nad the Bernstein polynomials $(B_N f)(t) = \sum_{k=0}^{N} f\left(\frac{k}{N}\right) \binom{N}{k} t^k (1-t)^{N-k}$, $\forall\, t \in [0,1]$, $N \in \mathbb{N}$, of degree N.

We have $B_N 1 = 1$, and B_N are positive linear operators.

Theorem 5 *([1]) Let* $0 < \alpha < 1$*,* $r > 0$ *and* $f \in AC([0,1])$ *such that* $f' \in L_\infty([0,1])$.

Then we have

$$\|B_N f - f\|_\infty \leq \frac{1}{\Gamma(\alpha+1)} \left(1 + \frac{1}{(\alpha+1)r}\right)$$

$$\left[\sup_{x\in[0,1]} \omega_1 \left(D_{x-}^{\alpha} f, r \left\| B_n \left(|\cdot - x|^{\alpha+1} \chi_{[0,x]}(\cdot), x\right)\right\|_\infty^{\frac{1}{(\alpha+1)}}\right)_{[0,x]}\right.$$

$$\left\| B_n \left(|\cdot - x|^{\alpha+1} \chi_{[0,x]}(\cdot), x\right)\right\|_\infty^{\frac{\alpha}{(\alpha+1)}} +$$

$$\sup_{x\in[0,1]} \omega_1 \left(D_{*x}^{\alpha} f, r \left\| B_n \left(|\cdot - x|^{\alpha+1} \chi_{[x,1]}(\cdot), x\right)\right\|_\infty^{\frac{1}{(\alpha+1)}}\right)_{[x,1]}$$

$$\left\| B_n \left(\left| \cdot - x \right|^{\alpha+1} \chi_{[x,1]} (\cdot), x \right) \right\|_{\infty}^{\frac{\alpha}{(\alpha+1)}} \Bigg], \quad \forall\ N \in \mathbb{N}. \tag{13}$$

Above χ stands for the characteristic function, also the two first moduli of continuity are over the intervals $[0, x]$ and $[x, 1]$, respectively as indicated.

Remark 6 *(to Theorem 5) Next we choose $r = \frac{1}{\alpha+1}$, $p = \frac{2}{\alpha+1} > 1$, $q = \frac{2}{1-\alpha} > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.*

We observe that both

$$B_n \left(\left| \cdot - x \right|^{\alpha+1} \chi_{[0,x]} (\cdot), x \right), \ B_n \left(\left| \cdot - x \right|^{\alpha+1} \chi_{[x,1]} (\cdot), x \right) \leq$$

$$B_n \left(\left| \cdot - x \right|^{\alpha+1}, x \right) = \sum_{k=0}^{N} \left| x - \frac{k}{N} \right|^{\alpha+1} \binom{N}{k} x^k (1-x)^{N-k} \leq \tag{14}$$

(by discrete Hölder's inequality)

$$\left(\sum_{k=0}^{N} \left(x - \frac{k}{N} \right)^2 \binom{N}{k} x^k (1-x)^{N-k} \right)^{\frac{\alpha+1}{2}} = \left(\frac{x(1-x)}{N} \right)^{\frac{\alpha+1}{2}} \leq \tag{15}$$

$$\frac{1}{(4N)^{\frac{\alpha+1}{2}}} = \frac{1}{\left(2\sqrt{N} \right)^{\alpha+1}}, \quad \forall\ x \in [0,1].$$

We have proved the following important auxilliary result.

Theorem 7 *Let $0 < \alpha < 1$, $f \in AC([0,1])$, with $f' \in L_\infty([0,1])$, $N \in \mathbb{N}$. Then*

$$\| B_N f - f \|_\infty \leq \frac{2^{1-\alpha}}{\Gamma(\alpha+1) N^{\frac{\alpha}{2}}} \left[\sup_{x \in [0,1]} \omega_1 \left(D_{x-}^\alpha f, \frac{1}{2(\alpha+1) N^{\frac{1}{2}}} \right)_{[0,x]} + \right.$$

$$\left. \sup_{x \in [0,1]} \omega_1 \left(D_{*x}^\alpha f, \frac{1}{2(\alpha+1) N^{\frac{1}{2}}} \right)_{[x,1]} \right] =: T_N^\alpha (f) < \infty. \tag{16}$$

Proof. By (13) and (15).

By [1] we get that the quantity within the bracket of (16) is finite. ■

So as $N \to \infty$ we derive $B_N f \xrightarrow{u} f$ (uniformly) with rates.

2 Main Result

We give the following simultaneous approximation fractional result.

Theorem 8 *Let $\beta > 0$, $\beta \notin \mathbb{N}$, with integral part $[\beta] = n \in \mathbb{Z}_+$ such that $\beta = n + \alpha$, where $0 < \alpha < 1$. Let $f \in AC^{n+1}([0,1])$, and $f^{(n+1)} \in L_\infty([0,1])$, $N \in \mathbb{N}$. Set*

$$P_{N+n}(f)(x) := \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k +$$

$$\int_0^x \left(\int_0^{x_{n-1}} \ldots \left(\int_0^{x_1} B_N\left(f^{(n)}\right)(t_1)\, dt_1 \right) dx_1 \ldots \right) dx_{n-1} \tag{17}$$

$$= \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} B_N\left(f^{(n)}\right)(t)\, dt,$$

for all $0 \leq x \leq 1$, a polynomial of degree $(N+n)$. Here $B_N\left(f^{(n)}\right)$ is the Bernstein polynomial of degree N. If n=0 the sum in (17) collapses.

Set also

$$T_N^{\beta,\alpha}(f) := \frac{2^{1-\alpha}}{\Gamma(\alpha+1) N^{\frac{\alpha}{2}}} \left[\sup_{x\in[0,1]} \omega_1 \left(D_{x-}^{\beta} f, \frac{1}{2(\alpha+1) N^{\frac{1}{2}}} \right)_{[0,x]} + \right.$$

$$\left. \sup_{x\in[0,1]} \omega_1 \left(D_{*x}^{\beta} f, \frac{1}{2(\alpha+1) N^{\frac{1}{2}}} \right)_{[x,1]} \right] < \infty, \tag{18}$$

for every $N \in \mathbb{N}$.

Then $P_{N+n}^{(i)}(f) = P_{N+n-i}(f)$, and

$$\left\| P_{N+n}^{(i)}(f) - f^{(i)} \right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{(n-i)!}, \quad i = 0, 1, \ldots, n. \tag{19}$$

As $N \to \infty$ we derive with rates $P_{N+n}^{(i)}(f) \xrightarrow{u} f^{(i)}$.

Proof. Notice here $\lceil \beta \rceil = (n+1) \in \mathbb{N}$. We have by (16) that

$$\left\| B_N\left(f^{(n)}\right) - f^{(n)} \right\|_{\infty,[0,1]} \leq T_N^{\alpha}\left(f^{(n)}\right) < \infty. \tag{20}$$

that is

$$-T_N^{\alpha}\left(f^{(n)}\right) \leq B_N\left(f^{(n)}\right)(t_1) - f^{(n)}(t_1) \leq T_N^{\alpha}\left(f^{(n)}\right), \tag{21}$$

foe every $0 \leq t_1 \leq x_1 \leq 1$.

Set

$$P_N(f)(x) := B_N\left(f^{(n)}\right)(x). \tag{22}$$

Hence it holds

$$-T_N^{\alpha}\left(f^{(n)}\right) x_1 \leq \int_0^{x_1} B_N\left(f^{(n)}\right)(t_1)\, dt_1 - \int_0^{x_1} f^{(n)}(t_1)\, dt_1 \leq T_N^{\alpha}\left(f^{(n)}\right) x_1, \tag{23}$$

that is

$$-T_N^\alpha\left(f^{(n)}\right)x_1 \le f^{(n-1)}(0) + \int_0^{x_1} B_N\left(f^{(n)}\right)(t_1)\,dt_1$$

$$-f^{(n-1)}(x_1) \le T_N^\alpha\left(f^{(n)}\right)x_1. \tag{24}$$

Set

$$P_{N+1}(f)(x) := f^{(n-1)}(0) + \int_0^{x} B_N\left(f^{(n)}\right)(t_1)\,dt_1, \tag{25}$$

that is

$$P'_{N+1}(f)(x) = P_N(f)(x), \quad \text{all } x \in [0,1]. \tag{26}$$

Hence

$$-T_N^\alpha\left(f^{(n)}\right)x_1 \le P_{N+1}(f)(x_1) - f^{(n-1)}(x_1) \le T_N^\alpha\left(f^{(n)}\right)x_1, \tag{27}$$

for all $x_1 \in [0,1]$.

Continuing like this we get

$$-T_N^\alpha\left(f^{(n)}\right)\frac{x_2^2}{2} \le f^{(n-2)}(0) + \int_0^{x_2} P_{N+1}(f)(x_1)\,dx_1$$

$$-f^{(n-2)}(x_2) \le T_N^\alpha\left(f^{(n)}\right)\frac{x_2^2}{2}, \tag{28}$$

all $0 \le x_1 \le x_2 \le 1$.

Set

$$P_{N+2}(f)(x) := f^{(n-2)}(0) + \int_0^{x} P_{N+1}(f)(x_1)\,dx_1, \tag{29}$$

that is

$$P'_{N+2}(f)(x) = P_{N+1}(f)(x), \tag{30}$$

and

$$P''_{N+2}(f)(x) = P_N(f)(x), \quad \text{all } x \in [0,1]. \tag{31}$$

So far we have

$$-T_N^\alpha\left(f^{(n)}\right)\frac{x_2^2}{2} \le P_{N+2}(f)(x_2) - f^{(n-2)}(x_2) \le T_N^\alpha\left(f^{(n)}\right)\frac{x_2^2}{2}, \tag{32}$$

for all $x_2 \in [0,1]$.

Similarly we derive

$$-T_N^\alpha\left(f^{(n)}\right)\frac{x_3^3}{3!} \le f^{(n-3)}(0) + \int_0^{x_3} P_{N+2}(f)(x_2)\,dx_2$$

$$-f^{(n-3)}(x_3) \le T_N^\alpha\left(f^{(n)}\right)\frac{x_3^3}{3!}, \tag{33}$$

all $0 \le x_2 \le x_3 \le 1$.

Set

$$P_{N+3}(f)(x) := f^{(n-3)}(0) + \int_0^x P_{N+2}(f)(x_2)\,dx_2, \tag{34}$$

that is

$$P'_{N+3}(f)(x) = P_{N+2}(f)(x), \tag{35}$$

and

$$P'''_{N+3}(f)(x) = P_N(f)(x), \quad \text{all } x \in [0,1]. \tag{36}$$

Hence

$$-T_N^{\alpha}\left(f^{(n)}\right)\frac{x_3^3}{3!} \le P_{N+3}(f)(x_3) - f^{(n-3)}(x_3) \le T_N^{\alpha}\left(f^{(n)}\right)\frac{x_3^3}{3!}, \tag{37}$$

for all $x_3 \in [0,1]$.

Continuing as above, after n steps, we derive

$$-T_N^{\alpha}\left(f^{(n)}\right)\frac{x_n^n}{n!} \le P_{N+n}(f)(x_n) - f(x_n) \le T_N^{\alpha}\left(f^{(n)}\right)\frac{x_n^n}{n!}, \tag{38}$$

with $0 \le x_n \le 1$.

Above

$$P_{N+n}(f)(x) := f(0) + \int_0^x P_{N+n-1}(f)(x_{n-1})\,dx_{n-1}, \tag{39}$$

that is

$$P'_{N+n}(f)(x) = P_{N+n-1}(f)(x), \tag{40}$$

and

$$P_{N+n}^{(n)}(f)(x) = P_N(f)(x) = B_N\left(f^{(n)}\right)(x), \quad \text{all } x \in [0,1]. \tag{41}$$

So clearly here $P_{N+n}(f)$ has the representations (17), the second one comes by Taylor's theorem.

By (21) we get

$$\left\|P_N(f) - f^{(n)}\right\|_{\infty} \le T_N^{\alpha}\left(f^{(n)}\right), \tag{42}$$

by (27) we find

$$\left\|P_{N+1}(f) - f^{(n-1)}\right\|_{\infty} \le T_N^{\alpha}\left(f^{(n)}\right), \tag{43}$$

by (32) we derive

$$\left\|P_{N+2}(f) - f^{(n-2)}\right\|_{\infty} \le \frac{T_N^{\alpha}\left(f^{(n)}\right)}{2}, \tag{44}$$

by (37) we obtain

$$\left\|P_{N+3}(f) - f^{(n-3)}\right\|_{\infty} \le \frac{T_N^{\alpha}\left(f^{(n)}\right)}{3!}, \tag{45}$$

and by (38) we have

$$\left\|P_{N+n}(f)-f\right\|_{\infty} \leq \frac{T_N^{\alpha}\left(f^{(n)}\right)}{n!}. \tag{46}$$

So we have proved that

$$\left\|P_{N+n}^{(i)}(f)-f^{(i)}\right\|_{\infty} \leq \frac{T_N^{\alpha}\left(f^{(n)}\right)}{(n-i)!}, \quad i=0,1,...,n. \tag{47}$$

Based on (10) and (12) we derive that

$$T_n^{\alpha}\left(f^{(n)}\right)=T_N^{\beta,\alpha}(f). \tag{48}$$

The quantity within the bracket of (18), by [1], is finite.

The proof of the theorem now is complete. ■

We completely left fractionalize Theorem 8, to have

Theorem 9 *Here all terms and assumptions are as in Theorem 8. Consider* $\alpha_j>0$, $j=1,...,n\in\mathbb{N}$, *such that* $\alpha_0=0<\alpha_1\leq 1<\alpha_2\leq 2<\alpha_3\leq 3<...<...<\alpha_n\leq n$.

Then

$$\left\|D_{*0}^{\alpha_j}(f)-D_{*0}^{\alpha_j}\left(P_{N+n}(f)\right)\right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma\left(j-\alpha_j+1\right)(n-j)!}, \quad j=0,1,...,n. \tag{49}$$

Notice (49) generalizes (19).

Proof. Let $\alpha_j>0$, $j=1,...,n$, such that $0<\alpha_1\leq 1<\alpha_2\leq 2<\alpha_3\leq 3<...<...<\alpha_n\leq n$. That is $\lceil\alpha_j\rceil=j$, $j=1,...,n$.

We consider the left Caputo fractional derivatives

$$\left(D_*^{\alpha_j}f\right)(x)=\frac{1}{\Gamma\left(j-\alpha_j\right)}\int_0^x (x-t)^{j-\alpha_j-1} f^{(j)}(t)\,dt, \tag{50}$$

$$\left(D_{*0}^{j}f\right)(x)=f^{(j)}(x),$$

and

$$\left(D_{*0}^{\alpha_j}\left(P_{N+n}(f)\right)\right)(x)=\frac{1}{\Gamma\left(j-\alpha_j\right)}\int_0^x (x-t)^{j-\alpha_j-1}\left(P_{N+n}(f)\right)^{(j)}(t)\,dt,$$

$$\left(D_{*0}^{j}\left(P_{N+n}(f)\right)\right)(x)=\left(P_{N+n}(f)\right)^{(j)}. \tag{51}$$

We notice that

$$\left|\left(D_{*0}^{\alpha_j}f\right)(x)-\left(D_{*0}^{\alpha_j}\left(P_{N+n}(f)\right)\right)(x)\right|=$$

$$\frac{1}{\Gamma(j-\alpha_j)}\left|\int_0^x (x-t)^{j-\alpha_j-1} f^{(j)}(t)\,dt - \int_0^x (x-t)^{j-\alpha_j-1} (P_{N+n}(f))^{(j)}(t)\,dt\right|$$

$$= \frac{1}{\Gamma(j-\alpha_j)}\left|\int_0^x (x-t)^{j-\alpha_j-1}\left(f^{(j)}(t) - (P_{N+n}(f))^{(j)}(t)\right)dt\right| \leq$$

$$\frac{1}{\Gamma(j-\alpha_j)}\int_0^x (x-t)^{j-\alpha_j-1}\left|f^{(j)}(t) - (P_{N+n}(f))^{(j)}(t)\right|dt \overset{(19)}{\leq} \tag{52}$$

$$\frac{1}{\Gamma(j-\alpha_j)}\left(\int_0^x (x-t)^{j-\alpha_j-1}\,dt\right)\frac{T_N^{\beta,\alpha}(f)}{(n-j)!} =$$

$$\frac{x^{j-\alpha_j}}{\Gamma(j-\alpha_j)(j-\alpha_j)}\frac{T_N^{\beta,\alpha}(f)}{(n-j)!} = \frac{x^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{T_N^{\beta,\alpha}(f)}{(n-j)!}. \tag{53}$$

We have proved

$$\left|\left(D_{*0}^{\alpha_j} f\right)(x) - \left(D_{*0}^{\alpha_j}(P_{N+n}(f))\right)(x)\right| \leq \frac{x^{j-\alpha_j} T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!}$$

$$\leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!}, \tag{54}$$

for every $x \in [0,1]$, proving the claim. ■

We completely right fractionalize Theorem 8, to have

Theorem 10 *Here all terms and assumptions are as in Theorem 9. It holds*

$$\left\|D_{1-}^{\alpha_j}(f) - D_{1-}^{\alpha_j}(P_{N+n}(f))\right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!}, \tag{55}$$

$j = 0, 1, ..., n.$

Observe that (55) generalizes (19).

Proof. We notice that

$$\left|\left(D_{1-}^{\alpha_j} f\right)(x) - \left(D_{1-}^{\alpha_j}(P_{N+n}(f))\right)(x)\right| =$$

$$\frac{1}{\Gamma(j-\alpha_j)}\left|\int_x^1 (t-x)^{j-\alpha_j-1} f^{(j)}(t)\,dt - \int_x^1 (t-x)^{j-\alpha_j-1} (P_{N+n}(f))^{(j)}(t)\,dt\right|$$

$$= \frac{1}{\Gamma(j-\alpha_j)}\left|\int_x^1 (t-x)^{j-\alpha_j-1}\left(f^{(j)}(t) - (P_{N+n}(f))^{(j)}(t)\right)dt\right| \leq \tag{56}$$

$$\frac{1}{\Gamma(j-\alpha_j)}\int_x^1 (t-x)^{j-\alpha_j-1}\left|f^{(j)}(t) - (P_{N+n}(f))^{(j)}(t)\right|dt \overset{(19)}{\leq}$$

$$\frac{1}{\Gamma(j-\alpha_j)}\left(\int_x^1 (t-x)^{j-\alpha_j-1}\,dt\right)\frac{T_N^{\beta,\alpha}(f)}{(n-j)!} =$$

$$\frac{(1-x)^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{T_N^{\beta,\alpha}(f)}{(n-j)!}. \tag{57}$$

We have proved

$$\left|\left(D_{1-}^{\alpha_j}f\right)(x)-\left(D_{1-}^{\alpha_j}\left(P_{N+n}(f)\right)\right)(x)\right| \leq \frac{(1-x)^{j-\alpha_j}}{\Gamma(j-\alpha_j+1)}\frac{T_N^{\beta,\alpha}(f)}{(n-j)!}$$

$$\leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!}, \tag{58}$$

for all $x\in[0,1]$, proving the claim. ■

It follows to important

Corollary 11 *Here all terms and assumptions are as in Theorem 9. Let* $\lambda\in[0,1]$. *Then*

$$\left\|\left(\lambda D_{*0}^{\alpha_j}(f)+(1-\lambda)D_{1-}^{\alpha_j}(f)\right)-\right.$$

$$\left.\left(\lambda D_{*0}^{\alpha_j}\left(P_{N+n}(f)\right)+(1-\lambda)D_{1-}^{\alpha_j}\left(P_{N+n}(f)\right)\right)\right\|_{\infty,[0,1]}$$

$$\leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!},\quad j=0,1,...,n. \tag{59}$$

Proof. We see that

$$\left\|\left(\lambda D_{*0}^{\alpha_j}(f)+(1-\lambda)D_{1-}^{\alpha_j}(f)\right)-\right.$$

$$\left.\left(\lambda D_{*0}^{\alpha_j}\left(P_{N+n}(f)\right)+(1-\lambda)D_{1-}^{\alpha_j}\left(P_{N+n}(f)\right)\right)\right\|_{\infty,[0,1]}=$$

$$\left\|\lambda\left(D_{*0}^{\alpha_j}(f)-D_{*0}^{\alpha_j}\left(P_{N+n}(f)\right)\right)+(1-\lambda)\left(D_{1-}^{\alpha_j}(f)-D_{1-}^{\alpha_j}\left(P_{N+n}(f)\right)\right)\right\|_{\infty}\leq \tag{60}$$

$$\lambda\left\|D_{*0}^{\alpha_j}(f)-D_{*0}^{\alpha_j}\left(P_{N+n}(f)\right)\right\|_{\infty}+$$

$$(1-\lambda)\left\|D_{1-}^{\alpha_j}(f)-D_{1-}^{\alpha_j}\left(P_{N+n}(f)\right)\right\|_{\infty}\overset{((49),(55))}{\leq}$$

$$\lambda\frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!}+(1-\lambda)\frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!} \tag{61}$$

$$=\frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!},$$

proving the claim. ■

Next comes our main result: the complete fractional simultaneous monotone uniform approximation, using mixed fractional differential operators.

Theorem 12 *Let $\beta > 0$, $\beta \notin \mathbb{N}$, $n = [\beta] \in \mathbb{N} : \beta = n + \alpha$, $0 < \alpha < 1$; $f \in AC^{n+1}([0,1])$, and $f^{(n+1)} \in L_\infty([0,1])$. Let $\lambda \in [0,1]$, and $h, k \in \mathbb{Z}_+$ with $0 \leq h \leq k \leq n$, when $\lambda \neq 1$ we take h to be even. Consider the numbers $\alpha_0 = 0 < \alpha_1 \leq 1 < \alpha_2 \leq 2 < \alpha_3 \leq 3 < ... < ... < \alpha_n \leq n$. Let $\alpha_j(x)$, $j = h, h+1, ..., k$ be real functions, defined and bounded on $[0,1]$, and suppose $\alpha_h(x)$ is either $\geq \overline{\alpha} > 0$ or $\leq \overline{\beta} < 0$ on $[0,1]$. We set*

$$l_\tau :\equiv \sup_{x\in[0,1]} \left|\alpha_h^{-1}(x)\,\alpha_\tau(x)\right|, \quad \tau = h, ..., k. \tag{62}$$

Here $T_N^{\beta,\alpha}(f)$, $N \in \mathbb{N}$, as in (18), ($N \to \infty$). Consider the mixed fractional linear differential operator

$$L^* := \sum_{j=h}^{k} \alpha_j(x)\left[\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j}\right]. \tag{63}$$

Then, for any $N \in \mathbb{N}$, there exists a real polynomial Q_{N+n} of degree $(N+n)$ such that

1) for $j = h+1, ..., n$, it holds

$$\left\|\left(\lambda D_{*0}^{\alpha_j}(f(x)) + (1-\lambda) D_{1-}^{\alpha_j}(f(x))\right) - \left(\lambda D_{*0}^{\alpha_j}(Q_{N+n}(x)) + (1-\lambda) D_{1-}^{\alpha_j}(Q_{N+n}(x))\right)\right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!}, \tag{64}$$

2) for $j = 1, ..., h$, it holds

$$\left\|\left(\lambda D_{*0}^{\alpha_j}(f) + (1-\lambda) D_{1-}^{\alpha_j}(f)\right) - \left(\lambda D_{*0}^{\alpha_j}(Q_{N+n}) + (1-\lambda) D_{1-}^{\alpha_j}(Q_{N+n})\right)\right\|_{\infty,[0,1]} \tag{65}$$

$$\leq \frac{T_N^{\beta,\alpha}(f)}{(h-j)!}\left[\frac{1}{\Gamma(j-\alpha_j+1)} + \left(\sum_{\tau=h}^{k} \frac{l_\tau}{\Gamma(\tau-\alpha_\tau+1)(n-\tau)!}\right)\right.$$

$$\left.\left\{\lambda\frac{\Gamma(h-j+1)}{\Gamma(h-\alpha_j+1)} + (1-\lambda)\left[\sum_{\theta=0}^{h-j}\binom{h-j}{\theta}\frac{\Gamma(h-j-\theta+1)}{\Gamma(h-\alpha_j-\theta+1)}\right]\right\}\right],$$

and

3)

$$\|f - Q_{N+n}\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{h!}\left(\sum_{\tau=h}^{k}\frac{l_\tau}{\Gamma(\tau-\alpha_\tau+1)(n-\tau)!} + 1\right). \tag{66}$$

The set

$$\Lambda := \left\{(\lambda, x) \in (0,1)^2 : \lambda x^{h-\alpha_h} + (1-\lambda)(1-x)^{h-\alpha_h} \geq \Gamma(h-\alpha_h+1)\right\} \tag{67}$$

is not empty.

1) We assume that $L^{*}f(x)\geq 0$, *for every* $x\in\Lambda$, *then* $L^{*}(Q_{N+n})\geq 0$.

2) If $L^{*}f(0)\geq 0$ *and* $0\leq\lambda\leq 1-\Gamma(h-\alpha_h+1)$, *then* $L^{*}(Q_{N+n})(0)\geq 0$.

3) If $L^{*}f(1)\geq 0$ *and* $\Gamma(h-\alpha_h+1)\leq\lambda\leq 1$, *then* $L^{*}(Q_{N+n})(1)\geq 0$.

4) Given $L^{*}f(0)\geq 0$, $\lambda=0$, *we get* $L^{*}(Q_{N+n})(0)\geq 0$.

5) Given $L^{*}f(1)\geq 0$, $\lambda=1$, *we derive* $L^{*}(Q_{N+n})(1)\geq 0$.

6) Let $\lambda=0$, h *even,* $h>\alpha_h$, $L^{*}f(x)\geq 0$, *for* $x\in(0,1)$ *such that* $x\leq 1-\Gamma(h-\alpha_h+1)^{\frac{1}{h-\alpha_h}}$, *then* $L^{*}(Q_{N+n})(x)\geq 0$.

Finally:

7) Let $\lambda=1$, $h>\alpha_h$, *and* $x\in(0,1)$ *with* $x\geq(\Gamma(h-\alpha_h+1))^{\frac{1}{h-\alpha_h}}$. *Assume there* $L^{*}f(x)\geq 0$, *then* $L^{*}(Q_{N+n})(x)\geq 0$.

Proof. Here let h,k be integers $0\leq h\leq k\leq n$, and $\alpha_0=0<\alpha_1\leq 1<\alpha_2\leq 2<\alpha_3\leq 3<...<...<\alpha_n\leq n$, that is $\lceil\alpha_j\rceil=j$, $j=1,...,n$. We set

$$l_{j_*}:\equiv\sup_{x\in[0,1]}\left|\alpha_h^{-1}(x)\,\alpha_{j_*}(x)\right|<\infty,\quad h\leq j_*\leq k, \tag{68}$$

and

$$\rho_N:=T_N^{\beta,\alpha}(f)\left(\sum_{j_*=h}^{k}\frac{l_{j_*}}{\Gamma(j_*-\alpha_{j_*}+1)(n-j_*)!}\right). \tag{69}$$

I. Suppose, throughout $[0,1]$, $\alpha_h(x)\geq\overline{\alpha}>0$. Call

$$Q_{N+n}(x):=P_{N+n}(f)(x)+\rho_N\frac{x^h}{h!}, \tag{70}$$

where $P_{N+n}(f)(x)$ as in (17).

Then by (59) we obtain

$$\Big\|\left(\lambda D_{*0}^{\alpha_j}+(1-\lambda)D_{1-}^{\alpha_j}\right)\left(f(x)+\rho_N\frac{x^h}{h!}\right)-\left(\lambda D_{*0}^{\alpha_j}+(1-\lambda)D_{1-}^{\alpha_j}\right)(Q_{N+n}(x))\Big\|_{\infty,[0,1]}\leq\frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!}, \tag{71}$$

all $0\leq j\leq n$.

When $h+1\leq j\leq n$, immediately by (71) we obtain

$$\left\|\left(\lambda D_{*0}^{\alpha_j}+(1-\lambda)D_{1-}^{\alpha_j}\right)(f(x))-\left(\lambda D_{*0}^{\alpha_j}+(1-\lambda)D_{1-}^{\alpha_j}\right)(Q_{N+n}(x))\right\|_{\infty,[0,1]}\leq\frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!}, \tag{72}$$

proving (64).

Next we treat the case of $1 \leq j \leq h$. We have after calculations that

$$D_{*0}^{\alpha_j}\left(\frac{x^h}{h!}\right) = \frac{\Gamma(h-j+1)\, x^{h-\alpha_j}}{\Gamma(h-\alpha_j+1)(h-j)!}, \tag{73}$$

and

$$D_{1-}^{\alpha_j}\left(\frac{x^h}{h!}\right) = \frac{1}{(h-j)!} \cdot \tag{74}$$

$$\left[\sum_{\theta=0}^{h-j}\binom{h-j}{\theta}(-1)^{h+\theta}\left\{\frac{\Gamma(h-j-\theta+1)}{\Gamma(h-\alpha_j-\theta+1)}(1-x)^{h-\alpha_j-\theta}\right\}\right].$$

Hence by (71) we have

$$\Bigg\| \left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j}\right)(f(x)) + \rho_N \left\{\lambda \frac{\Gamma(h-j+1)\, x^{h-\alpha_j}}{\Gamma(h-\alpha_j+1)(h-j)!} + \right. \tag{75}$$

$$\left.\frac{(1-\lambda)}{(h-j)!}\left[\sum_{\theta=0}^{h-j}\binom{h-j}{\theta}(-1)^{h+\theta}\left\{\frac{\Gamma(h-j-\theta+1)}{\Gamma(h-\alpha_j-\theta+1)}(1-x)^{h-\alpha_j-\theta}\right\}\right]\right\}$$

$$-\left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j}\right)(Q_{N+n}(x))\Bigg\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!},$$

all $1 \leq j \leq h$.

By (75) and triangle inequality we get

$$\left\|\left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j}\right)(f) - \left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j}\right)(Q_{N+n})\right\|_{\infty,[0,1]}$$

$$\leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!} +$$

$$\frac{\rho_N}{(h-j)!}\left\{\lambda\frac{\Gamma(h-j+1)}{\Gamma(h-\alpha_j+1)} + (1-\lambda)\left[\sum_{\theta=0}^{h-j}\binom{h-j}{\theta}\frac{\Gamma(h-j-\theta+1)}{\Gamma(h-\alpha_j-\theta+1)}\right]\right\} = \tag{76}$$

$$\frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!} + \frac{T_N^{\beta,\alpha}(f)}{(h-j)!}\left(\sum_{j_*=h}^{k}\frac{l_{j_*}}{\Gamma(j_*-\alpha_{j_*}+1)(n-j_*)!}\right) \cdot$$

$$\left\{\lambda\frac{\Gamma(h-j+1)}{\Gamma(h-\alpha_j+1)} + (1-\lambda)\left[\sum_{\theta=0}^{h-j}\binom{h-j}{\theta}\frac{\Gamma(h-j-\theta+1)}{\Gamma(h-\alpha_j-\theta+1)}\right]\right\} \leq$$

$$\frac{T_N^{\beta,\alpha}(f)}{(h-j)!}\left[\frac{1}{\Gamma(j-\alpha_j+1)} + \left(\sum_{\tau=h}^{k}\frac{l_\tau}{\Gamma(\tau-\alpha_\tau+1)(n-\tau)!}\right)\right.$$

$$\left.\left\{\lambda\frac{\Gamma(h-j+1)}{\Gamma(h-\alpha_j+1)} + (1-\lambda)\left[\sum_{\theta=0}^{h-j}\binom{h-j}{\theta}\frac{\Gamma(h-j-\theta+1)}{\Gamma(h-\alpha_j-\theta+1)}\right]\right\}\right] =: K. \tag{77}$$

So we have derived

$$\left\| \left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j}\right)(f) - \left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j}\right)(Q_{N+n}) \right\|_{\infty,[0,1]} \leq K, \tag{78}$$

$j = 1, ..., h$, proving (65).

When $j = 0$ from (71) we obtain

$$\left\| f(x) + \rho_N \frac{x^h}{h!} - Q_{N+n}(x) \right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{n!}. \tag{79}$$

Hence

$$\|f - Q_{N+n}\|_{\infty,[0,1]} \leq \frac{\rho_N}{h!} + \frac{T_N^{\beta,\alpha}(f)}{n!} =$$

$$\frac{T_N^{\beta,\alpha}(f)}{h!} \left(\sum_{\tau=h}^{k} \frac{l_\tau}{\Gamma(\tau - \alpha_\tau + 1)(n-\tau)!} \right) + \frac{T_N^{\beta,\alpha}(f)}{n!} \leq \tag{80}$$

$$\frac{T_N^{\beta,\alpha}(f)}{h!} \left(\sum_{\tau=h}^{k} \frac{l_\tau}{\Gamma(\tau - \alpha_\tau + 1)(n-\tau)!} + 1 \right), \tag{81}$$

proving (66).

Furthermore, if (λ, x) is in the critical set Λ, see (67), and $L^* f(x) \geq 0$, we get

$$\alpha_h^{-1} L^*(Q_{N+n}) = \alpha_h^{-1}(x) L^*(f(x)) +$$

$$\rho_N \left\{ \lambda \frac{x^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} + (1-\lambda) \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} \right\} \tag{82}$$

(when $\lambda \in [0,1)$ we assumed that h is even)

$$+ \sum_{j=h}^{k} \alpha_h^{-1}(x) \alpha_j(x) \left\{ \left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j}\right) \left[Q_{N+n}(x) - f(x) - \rho_N \frac{x^h}{h!} \right] \right\} \overset{(71)}{\geq}$$

$$\rho_N \left\{ \lambda \frac{x^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} + (1-\lambda) \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} \right\} - \sum_{j=h}^{k} l_j \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!} =$$

$$\rho_N \left\{ \lambda \frac{x^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} + (1-\lambda) \frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)} - 1 \right\} =$$

$$\frac{\rho_N}{\Gamma(h-\alpha_h+1)} \left\{ \lambda x^{h-\alpha_h} + (1-\lambda)(1-x)^{h-\alpha_h} - \Gamma(h-\alpha_h+1) \right\} =: A(x,\lambda). \tag{83}$$

The set

$$\Lambda := \left\{ (\lambda, x) \in (0,1)^2 : \lambda x^{h-\alpha_h} + (1-\lambda)(1-x)^{h-\alpha_h} \geq \Gamma(h-\alpha_h+1) \right\} \tag{84}$$

is not empty.

If $h=\alpha_h$ then $\Lambda=(0,1)^2$. Assume $\alpha_h<h$:

Let us choose $\lambda=x=\delta\in(0,1)$, some want to find specific examples of

$$\delta^{1+h-\alpha_h}+(1-\delta)^{1+h-\alpha_h}\geq\Gamma(h-\alpha_h+1)\,. \tag{85}$$

The minimum value of Γ over $(0,\infty)$ is $\Gamma(1.46163)\simeq 0.885603$, we pick here $1+h-\alpha_h=1.46163$ and $\delta=0.99$.

Hence

$$(0.99)^{1.46163}+(0.01)^{1.46163}= \tag{86}$$

$$0.985417497+0.001193274=0.986610771>0.885603.$$

Similarly, we have that $\Gamma(1.4)=0.887264$, and we pick $\delta=0.95$ and $1+h-\alpha_h=1.4$. Then

$$(0.95)^{1.4}+(0.05)^{1.4}=$$

$$0.930707144+0.015085441=0.945792585>0.887264. \tag{87}$$

Hence $\Lambda\neq\varnothing$.

Hence over Λ we get that $A(x,\lambda)\geq 0$, thus there $L^*(Q_{N+n})\geq 0$.

We know that $\Gamma(1)=1$, $\Gamma(2)=1$, and Γ is convex and positive on $(0,\infty)$. Here in general $0\leq h-\alpha_h<1$, hence $1\leq h-\alpha_h+1<2$ and $0<\Gamma(h-\alpha_h+1)\leq 1$, that is $1-\Gamma(h-\alpha_h+1)\geq 0$.

Next we argue as in (82)-(83):

Hence we further have that

$$A(0,\lambda)=\frac{\rho_N}{\Gamma(h-\alpha_h+1)}\left\{(1-\lambda)-\Gamma(h-\alpha_h+1)\right\}\geq 0, \tag{88}$$

when $0\leq\lambda\leq 1-\Gamma(h-\alpha_h+1)$, proving in that case $L^*(Q_{N+n})(0)\geq 0$, given $L^*f(0)\geq 0$.

Similarly we observe that

$$A(1,\lambda)=\frac{\rho_N}{\Gamma(h-\alpha_h+1)}\left\{\lambda-\Gamma(h-\alpha_h+1)\right\}\geq 0, \tag{89}$$

when $\Gamma(h-\alpha_h+1)\leq\lambda\leq 1$, proving also $L^*(Q_{N+n})(1)\geq 0$ in this case, given that $L^*f(1)\geq 0$.

Clearly we have

$$A(0,0)=\frac{\rho_N}{\Gamma(h-\alpha_h+1)}\left\{1-\Gamma(h-\alpha_h+1)\right\}\geq 0, \tag{90}$$

proving $L^*(Q_{N+n})(0)\geq 0$, with $\lambda=0$, given $L^*f(0)\geq 0$, and

$$A(1,1)=\frac{\rho_N}{\Gamma(h-\alpha_h+1)}\left\{1-\Gamma(h-\alpha_h+1)\right\}\geq 0, \tag{91}$$

proving $L^*(Q_{N+n})(1)\geq 0$, given $L^*f(1)\geq 0$, with $\lambda=1$.

We see also that

$$A(x,0)=\frac{\rho_N}{\Gamma(h-\alpha_h+1)}\left\{(1-x)^{h-\alpha_h}-\Gamma(h-\alpha_h+1)\right\}\geq 0, \tag{92}$$

given $(1-x)\geq\Gamma(h-\alpha_h+1)^{\frac{1}{h-\alpha_h}}$, equivalently, given that $x\leq 1-\Gamma(h-\alpha_h+1)^{\frac{1}{h-\alpha_h}}$, with $h>\alpha_h$ and $x\in(0,1)$.

In that case $L^*(Q_{N+n})(x)\geq 0$, with $\lambda=0$, and h even, given there $L^*f(x)\geq 0$. At last we observe that

$$A(x,1)=\frac{\rho_N}{\Gamma(h-\alpha_h+1)}\left\{x^{h-\alpha_h}-\Gamma(h-\alpha_h+1)\right\}\geq 0, \tag{93}$$

given that $x\geq\Gamma(h-\alpha_h+1)^{\frac{1}{h-\alpha_h}}$, with $h>\alpha_h$ and $x\in(0,1)$.

In that case $L^*(Q_{N+n})(x)\geq 0$, with $\lambda=1$, given there $L^*f(x)\geq 0$.

II. Suppose, throughout $[0,1]$, $\alpha_h(x)\leq\overline{\beta}<0$. Call now

$$Q_{N+n}(x):=P_{N+n}(f)(x)-\rho_N\frac{x^h}{h!}. \tag{94}$$

Then by (59) we obtain

$$\Bigg\|\left(\lambda D_{*0}^{\alpha_j}+(1-\lambda)D_{1-}^{\alpha_j}\right)\left(f(x)-\rho_N\frac{x^h}{h!}\right)-$$
$$\left(\lambda D_{*0}^{\alpha_j}+(1-\lambda)D_{1-}^{\alpha_j}\right)(Q_{N+n}(x))\Big\|_{\infty,[0,1]}$$
$$\leq\frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!},\quad 0\leq j\leq n. \tag{95}$$

Again we obtain, as earlier, the inequalities (64), (65), (66). Furthermore, if $(\lambda,x)\in\Lambda$ and $L^*f(x)\geq 0$, we get

$$\alpha_h^{-1}(x)L^*(Q_{N+n})=\alpha_h^{-1}(x)L^*(f(x))-$$
$$\rho_N\left\{\lambda\frac{x^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}+(1-\lambda)\frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}\right\}$$

(when $\lambda\in[0,1)$ we assumed that h is even)

$$+\sum_{j=h}^{k}\alpha_h^{-1}(x)\alpha_j(x)\left\{\left(\lambda D_{*0}^{\alpha_j}+(1-\lambda)D_{1-}^{\alpha_j}\right)\left[Q_{N+n}(x)-f(x)+\rho_N\frac{x^h}{h!}\right]\right\}\overset{(95)}{\leq}$$

$$-\rho_N\left\{\lambda\frac{x^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}+(1-\lambda)\frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}\right\}+\sum_{j=h}^{k}l_j\frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!} \tag{96}$$

$$=\rho_N\left\{1-\left\{\lambda\frac{x^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}+(1-\lambda)\frac{(1-x)^{h-\alpha_h}}{\Gamma(h-\alpha_h+1)}\right\}\right\}=$$

$$\frac{\rho_N}{\Gamma(h-\alpha_h+1)}\left\{\Gamma(h-\alpha_h+1)-\left\{\lambda x^{h-\alpha_h}+(1-\lambda)(1-x)^{h-\alpha_h}\right\}\right\}=:B(x,\lambda). \tag{97}$$

Hence over Λ we get that $B(x,\lambda)\leq 0$, thus there $L^*(Q_{N+n})\geq 0$.

Next we argue as in (96)-(97):

Hence we further have that

$$B(0,\lambda)=\frac{\rho_N}{\Gamma(h-\alpha_h+1)}\{\Gamma(h-\alpha_h+1)-(1-\lambda)\}\leq 0, \tag{98}$$

when $0\leq\lambda\leq 1-\Gamma(h-\alpha_h+1)$, proving in that case $L^*(Q_{N+n})(0)\geq 0$, given $L^*f(0)\geq 0$.

Similarly we observe that

$$B(1,\lambda)=\frac{\rho_N}{\Gamma(h-\alpha_h+1)}\{\Gamma(h-\alpha_h+1)-\lambda\}\leq 0, \tag{99}$$

when $\Gamma(h-\alpha_h+1)\leq\lambda\leq 1$, proving also $L^*(Q_{N+n})(1)\geq 0$ in this case, given $L^*f(1)\geq 0$.

Clearly we have

$$B(0,0)=\frac{\rho_N}{\Gamma(h-\alpha_h+1)}\{\Gamma(h-\alpha_h+1)-1\}\leq 0, \tag{100}$$

proving $L^*(Q_{N+n})(0)\geq 0$, given $L^*f(0)\geq 0$, with $\lambda=0$.

Also it holds

$$B(1,1)=\frac{\rho_N}{\Gamma(h-\alpha_h+1)}\{\Gamma(h-\alpha_h+1)-1\}\leq 0, \tag{101}$$

proving $L^*(Q_{N+n})(1)\geq 0$, given $L^*f(1)\geq 0$, with $\lambda=1$.

We see also that

$$B(x,0)=\frac{\rho_N}{\Gamma(h-\alpha_h+1)}\left\{\Gamma(h-\alpha_h+1)-(1-x)^{h-\alpha_h}\right\}\leq 0, \tag{102}$$

given $(1-x)\geq\Gamma(h-\alpha_h+1)^{\frac{1}{h-\alpha_h}}$, with $h>\alpha_h$ and $x\in(0,1)$, h is even.

In that case $L^*(Q_{N+n})(x)\geq 0$, with $\lambda=0$, given there $L^*f(x)\geq 0$.

At last we observe that

$$B(x,1)=\frac{\rho_N}{\Gamma(h-\alpha_h+1)}\left\{\Gamma(h-\alpha_h+1)-x^{h-\alpha_h}\right\}\leq 0, \tag{103}$$

given that $x\geq\Gamma(h-\alpha_h+1)^{\frac{1}{h-\alpha_h}}$, with $h>\alpha_h$ and $x\in(0,1)$.

In that case again $L^*(Q_{N+n})(x)\geq 0$, with $\lambda=1$, given there $L^*f(x)\geq 0$.

■

Corollary 13 *Let $\beta>0$, $\beta\notin\mathbb{N}$, $n=[\beta]\in\mathbb{N}:\beta=n+\alpha$, $0<\alpha<1$; $f\in AC^{n+1}([0,1])$, and $f^{(n+1)}\in L_\infty([0,1])$. Let $h,k\in\mathbb{Z}_+$ with $0\leq h\leq k\leq n$.*

Consider the numbers $\alpha_0 = 0 < \alpha_1 \leq 1 < \alpha_2 \leq 2 < \alpha_3 \leq 3 < ... < ... < \alpha_n \leq n$. *Let* $\alpha_j(x)$, $j = h, h+1, ..., k$ *be real functions, defined and bounded on* $[0,1]$, *and suppose* $\alpha_h(x)$ *is either* $\geq \overline{\alpha} > 0$ *or* $\leq \overline{\beta} < 0$ *on* $[0,1]$. *We set*

$$l_\tau :\equiv \sup_{x \in [0,1]} \left| \alpha_h^{-1}(x) \alpha_\tau(x) \right|, \quad \tau = h, ..., k. \tag{104}$$

Consider the left fractional linear differential operator

$$L_1 := \sum_{j=h}^{k} \alpha_j(x) D_{*0}^{\alpha_j}. \tag{105}$$

Here $T_N^{\beta,\alpha}(f)$, $N \in \mathbb{N}$, *as in (18).*

Then, for any $N \in \mathbb{N}$, *there exists a real polynomial* Q_{N+n} *of degree* $(N+n)$ *such that*

1) for $j = h+1, ..., n$, *it holds*

$$\left\| D_{*0}^{\alpha_j} f - D_{*0}^{\alpha_j} Q_{N+n} \right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!}, \tag{106}$$

2) for $j = 1, ..., h$, *it holds*

$$\left\| D_{*0}^{\alpha_j}(f) - D_{*0}^{\alpha_j}(Q_{N+n}) \right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{(h-j)!} \left[\frac{1}{\Gamma(j-\alpha_j+1)} + \left(\sum_{\tau=h}^{k} \frac{l_\tau}{\Gamma(\tau-\alpha_\tau+1)(n-\tau)!} \right) \left(\frac{\Gamma(h-j+1)}{\Gamma(h-\alpha_j+1)} \right) \right], \tag{107}$$

and

3)

$$\|f - Q_{N+n}\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{h!} \left(\sum_{\tau=h}^{k} \frac{l_\tau}{\Gamma(\tau-\alpha_\tau+1)(n-\tau)!} + 1 \right). \tag{108}$$

We further have

1) Given $L_1 f(1) \geq 0$, *then* $L_1(Q_{N+n})(1) \geq 0$.

2) Let $h > \alpha_h$, *and* $x \in (0,1)$ *with* $x \geq (\Gamma(h-\alpha_h+1))^{\frac{1}{h-\alpha_h}}$. *Assume* $L_1 f(x) \geq 0$, *then* $L_1(Q_{N+n})(x) \geq 0$.

Proof. By Theorem 12 for $\lambda = 1$. ■

We finish with

Corollary 14 *Let* $\beta > 0$, $\beta \notin \mathbb{N}$, $n = [\beta] \in \mathbb{N} : \beta = n + \alpha$, $0 < \alpha < 1$; $f \in AC^{n+1}([0,1])$, *and* $f^{(n+1)} \in L_\infty([0,1])$. *Let* $h, k \in \mathbb{Z}_+$ *with* $0 \leq h \leq k \leq n$, *h is even. Consider the numbers* $\alpha_0 = 0 < \alpha_1 \leq 1 < \alpha_2 \leq 2 < \alpha_3 \leq 3 <$

... < ... < $\alpha_n \leq n$. *Let* $\alpha_j(x)$, $j = h, h+1, ..., k$ *be real functions, defined and bounded on* $[0,1]$, *and suppose* $\alpha_h(x)$ *is either* $\geq \overline{\alpha} > 0$ *or* $\leq \overline{\beta} < 0$ *on* $[0,1]$. *We set*

$$l_\tau :\equiv \sup_{x\in[0,1]} \left|\alpha_h^{-1}(x)\,\alpha_\tau(x)\right|, \quad \tau = h, ..., k. \tag{109}$$

Consider the right fractional linear differential operator

$$L_2 := \sum_{j=h}^{k} \alpha_j(x)\, D_{1-}^{\alpha_j}. \tag{110}$$

Here $T_N^{\beta,\alpha}(f)$, $N \in \mathbb{N}$, *as in (18).*

Then, for any $N \in \mathbb{N}$, *there exists a real polynomial* Q_{N+n} *of degree* $(N+n)$ *such that*

1) for $j = h+1, ..., n$, *it holds*

$$\left\|D_{1-}^{\alpha_j} f - D_{1-}^{\alpha_j} Q_{N+n}\right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j-\alpha_j+1)(n-j)!}, \tag{111}$$

2) for $j = 1, ..., h$, *it holds*

$$\left\|D_{1-}^{\alpha_j}(f) - D_{1-}^{\alpha_j}(Q_{N+n})\right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{(h-j)!}\left[\frac{1}{\Gamma(j-\alpha_j+1)} + \left(\sum_{\tau=h}^{k} \frac{l_\tau}{\Gamma(\tau-\alpha_\tau+1)(n-\tau)!}\right)\left[\sum_{\theta=0}^{h-j} \binom{h-j}{\theta} \frac{\Gamma(h-j-\theta+1)}{\Gamma(h-\alpha_j-\theta+1)}\right]\right], \tag{112}$$

and

3)

$$\|f - Q_{N+n}\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{h!}\left(\sum_{\tau=h}^{k} \frac{l_\tau}{\Gamma(\tau-\alpha_\tau+1)(n-\tau)!} + 1\right). \tag{113}$$

We further have

1) Given $L_2 f(0) \geq 0$, *then* $L_2(Q_{N+n})(0) \geq 0$.

2) Let even $h > \alpha_h$, $x \in (0,1)$ *such that* $x \leq 1 - (\Gamma(h-\alpha_h+1))^{\frac{1}{h-\alpha_h}}$, *and* $L_2 f(x) \geq 0$. *Then* $L_2(Q_{N+n})(x) \geq 0$.

Proof. By Theorem 12, $\lambda = 0$. ■

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