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Lower order Fractional Monotone Approximation

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Abstract

Here is presented the theory of lower order fractional simultaneous monotone uniform polynomial approximation with rates using mixed lower order fractional linear differential operators.

To obtain that, we use first ordinary simultaneous polynomial approximation with respect to the highest lower order right and left fractional derivatives of the function under approximation using their moduli of continuity. Then we use the total right and left fractional simultaneous polynomial approximation with rates, as well their convex combination. Based on the last and elegant analytical techniques, we derive preservation of monotonicity by mixed lower order fractional linear differential operators.

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1 Introduction

The topic of monotone approximation started in [6] has become a major trend in approximation theory. A typical problem in this subject is: given a positive integer k, approximate a given function whose kth derivative is ≥ 0 by polynomials having this property.

In [3] the authors replaced the kth derivative with a linear differential operator of order k. We mention this motivating result.

Theorem 1 *Let h, k, p be integers, $0 \leq h \leq k \leq p$ and let f be a real function, $f^{(p)}$ continuous in $[-1, 1]$ with first modulus of continuity $\omega_1 \left(f^{(p)}, x\right)$ there. Let $a_j(x)$, $j = h, h+1, ..., k$ be real functions, defined and bounded on $[-1, 1]$ and assume $a_h(x)$ is either $\geq$ some number $\alpha > 0$ or $\leq$ some number $\beta < 0$ throughout $[-1, 1]$. Consider the operator*

$$L = \sum_{j=h}^{k} a_j(x) \left[\frac{d^j}{dx^j}\right] \tag{1}$$

and suppose, throughout $[-1, 1]$,

$$L(f) \geq 0. \tag{2}$$

Then, for every integer $n \geq 1$, there is a real polynomial $Q_n(x)$ of degree $\leq n$ such that

$$L(Q_n) \geq 0 \text{ throughout } [-1, 1] \tag{3}$$

and

$$\max_{-1 \leq x \leq 1} |f(x) - Q_n(x)| \leq C n^{k-p} \omega_1 \left(f^{(p)}, \frac{1}{n}\right), \tag{4}$$

where C is independent of n or f.

The purpose of this article is to extend completely Theorem 1 to the lower order fractional level. All involved ordinary derivatives will become now fractional derivatives of lower order and even more we will have fractional simultaneous approximation.

We need

Definition 2 *([4], p. 50) Let $\alpha > 0$ and $\lceil \alpha \rceil = m$, ($\lceil \cdot \rceil$ ceiling of the number). Consider $f \in AC^m([0, 1])$ (space of functions f with $f^{(m-1)} \in AC([0, 1])$, absolutely continuous functions), $z \in [0, 1]$. We define the left Caputo fractional derivative of f of order α as follows:*

$$\left(D_{*z}^{\alpha} f\right)(x) = \frac{1}{\Gamma(m-\alpha)} \int_z^x (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{5}$$

for any $x \in [z, 1]$, where Γ is the gamma function.

We set

$$D_{*z}^{0} f(x) = f(x),$$
$$D_{*z}^{m} f(x) = f^{(m)}(x), \quad \forall\, x \in [z, 1]. \tag{6}$$

Definition 3 *([5]) Let $\alpha > 0$ and $\lceil \alpha \rceil = m$. Consider $f \in AC^m([0, 1])$, $z \in [0, 1]$. We define the right Caputo fractional derivative of f of order α as follows:*

$$\left(D_{z-}^{\alpha} f\right)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^z (t-x)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{7}$$

for any $x \in [0, z]$.

We set

$$D_{z-}^{0} f(x) = f(x),$$

$$D_{z-}^{m} f(x) = (-1)^m f^{(m)}(x), \ \ \forall\ x \in [0, z]. \tag{8}$$

In particular we give

Definition 4 *Let* $0 < \alpha < 1$ *and* $f \in AC([0,1])$ *(absolutely continuous functions),* $z \in [0,1]$*. We define the left Caputo fractional derivative of* f *of order* α *as follows:*

$$\left(D_{*z}^{\alpha} f\right)(x) = \frac{1}{\Gamma(1-\alpha)} \int_{z}^{x} (x-t)^{-\alpha} f'(t)\, dt, \tag{9}$$

for any $x \in [z, 1]$.

We set

$$D_{*z}^{0} f(x) = f(x),$$

$$D_{*z}^{1} f(x) = f'(x), \ \ \forall\ x \in [z, 1]. \tag{10}$$

Definition 5 *Let* $0 < \alpha < 1$ *and* $f \in AC([0,1])$*,* $z \in [0,1]$*. We define the right Caputo fractional derivative of* f *of order* α *as follows:*

$$\left(D_{z-}^{\alpha} f\right)(x) = \frac{-1}{\Gamma(1-\alpha)} \int_{x}^{z} (t-x)^{-\alpha} f'(t)\, dt, \tag{11}$$

for any $x \in [0, z]$.

We set

$$D_{z-}^{0} f(x) = f(x),$$

$$D_{z-}^{1} f(x) = -f'(x), \ \ \forall\ x \in [0, z]. \tag{12}$$

Definition 6 *Let* $f \in C([0,1])$*, we define the Bernstein polynomials*

$$\left(B_N f\right)(t) = \sum_{k=0}^{N} f\left(\frac{k}{N}\right) \binom{N}{k} t^k (1-t)^{N-k}, \tag{13}$$

$\forall\ t \in [0,1]$*,* $N \in \mathbb{N}$*, of degree* N*.*

Our article [1] was the base to develop the general article [2]. We rely alot on [2].

We need the following special result from [2], here is the $n = 1$ case.

Theorem 7 *Let* $1 < \beta < 2$ *and* $0 < \alpha < 1$ *such that* $\beta = 1 + \alpha$*. Let* $f \in AC^2([0,1])$*, and* $f'' \in L_\infty([0,1])$*,* $N \in \mathbb{N}$*. Set*

$$P_{N+1}(f)(x) := f(0) + \int_0^x B_N(f')(t)\, dt, \tag{14}$$

for all $0 \leq x \leq 1$, a polynomial of degree $(N+1)$. Set also

$$T_N^{\beta,\alpha}(f) := \frac{2^{1-\alpha}}{\Gamma(\alpha+1) N^{\frac{\alpha}{2}}} \left[\sup_{x \in [0,1]} \omega_1 \left(D_{x-}^{\beta} f, \frac{1}{2(\alpha+1) N^{\frac{1}{2}}} \right)_{[0,x]} + \right.$$

$$\left. \sup_{x \in [0,1]} \omega_1 \left(D_{*x}^{\beta} f, \frac{1}{2(\alpha+1) N^{\frac{1}{2}}} \right)_{[x,1]} \right] < \infty. \quad (15)$$

for every $N \in \mathbb{N}$.

Then

1) the quantity within the bracket of (15) is finite,

2) $P'_{N+1}(f) = B_N(f')$,

3)

$$\left\| P_{N+1}^{(i)}(f) - f^{(i)} \right\|_{\infty,[0,1]} \leq T_N^{\beta,\alpha}(f), \quad i = 0, 1. \quad (16)$$

As $N \to \infty$ we derive with rates $P_{N+1}^{(i)}(f) \xrightarrow{u} f^{(i)}$ (uniformly), $i = 0, 1$.

We completely left fractionalize Theorem 7, to have

Theorem 8 *Here all terms and assumptions as in Theorem 7 and $\alpha_j \in [0,1]$, $j \in \mathbb{Z}_+$. Then*

$$\left\| D_{*0}^{\alpha_j}(f) - D_{*}^{\alpha_j}(P_{N+1}(f)) \right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j^* - \alpha_j + 1)}, \quad (17)$$

where $j^* = \lceil \alpha_j \rceil = 0$ or 1.

Observe that (17) generalizes (16).

Proof. By [2], see there Theorem 9. ■

We completely right fractionalize Theorem 7, to have

Theorem 9 *Here all terms and assumptions as in Theorem 7, and $\alpha_j \in [0,1]$, $j \in \mathbb{Z}_+$. Then*

$$\left\| D_{1-}^{\alpha_j}(f) - D_{1-}^{\alpha_j}(P_{N+1}(f)) \right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j^* - \alpha_j + 1)}, \quad (18)$$

where $j^* = \lceil \alpha_j \rceil = 0$ or 1.

Observe that (18) generalizes (16).

Proof. By [2], see there Theorem 10. ■

It follows the important

Corollary 10 *Here all as in Theorem 7, $\lambda \in [0,1]$. Then*

$$\left\| \left(\lambda D_{*0}^{\alpha_j} (f) + (1-\lambda) D_{1-}^{\alpha_j} (f) \right) - \right.$$

$$\left. \left(\lambda D_{*0}^{\alpha_j} (P_{N+1}(f)) + (1-\lambda) D_{1-}^{\alpha_j} (P_{N+1}(f)) \right) \right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(j^* - \alpha_j + 1)}, \quad (19)$$

where $\alpha_j \in [0,1]$, $j \in \mathbb{Z}_+$; $j^* = \lceil \alpha_j \rceil = 0$ or 1.

Proof. By [2], see there Corollary 11. ■

2 Main Results

Next comes our main result: the totally lower order fractional simultaneous monotone uniform approximation, using mixed fractional differential operators.

Theorem 11 *Let $1 < \beta < 2$ and $0 < \alpha < 1 : \beta = 1 + \alpha$. Let $f \in AC^2([0,1])$ with $f'' \in L_\infty([0,1])$, $N \in \mathbb{N}$. Here let $k, \rho \in \mathbb{N} : 0 < k \leq \rho$, and $\alpha_0 = 0 < \alpha_1 < \alpha_2 < ... < \alpha_k < ... < \alpha_\rho \leq 1$. Let $\lambda \in [0,1]$, and $\alpha_j(x)$, $j = 0,1,...,k$ be real functions, defined and bounded on $[0,1]$, and suppose $\alpha_0(x)$ is either $\geq \overline{\alpha} > 0$ or $\leq \overline{\beta} < 0$ on $[0,1]$. We set*

$$l_\tau :\equiv \sup_{x\in[0,1]} \left| \alpha_0^{-1}(x) \alpha_\tau(x) \right|, \quad \tau = 0,1,...,k. \quad (20)$$

Here $T_N^{\beta,\alpha}(f)$, $N \in \mathbb{N}$, as in (15), ($N \to \infty$). Consider the mixed fractional linear differential operator

$$L_\lambda^* := \sum_{j=0}^{k} \alpha_j(x) \left[\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j} \right]. \quad (21)$$

Then, for any $N \in \mathbb{N}$, there exists a real polynomial Q_{N+1} of degree $(N+1)$ such that

1)

$$\left\| \left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j} \right)(f) - \right.$$

$$\left. \left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j} \right)(Q_{N+1}(f)) \right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(2-\alpha_j)}, \ j = 1,...,\rho, \quad (22)$$

and

2)

$$\|f - Q_{N+1}(f)\|_{\infty,[0,1]} \leq T_N^{\beta,\alpha}(f) \left[\sum_{\tau=1}^{k} \frac{l_\tau}{\Gamma(2-\alpha_\tau)} + 2 \right]. \quad (23)$$

Assuming $L_\lambda^* f(x) \geq 0$, for all $x \in [0,1]$ we get $(L_\lambda^*(Q_{N+1}(f)))(x) \geq 0$ for all $x \in [0,1]$.

Proof. Here let $k, \rho \in \mathbb{N} : 0 < k \leq \rho$, and $\alpha_0 = 0 < \alpha_1 < \alpha_2 < ... < \alpha_k < ... < \alpha_\rho \leq 1$, that is $\lceil \alpha_0 \rceil = 0$; $\lceil \alpha_j \rceil = 1$, $j = 1, ..., \rho$. We set

$$l_\tau := \sup_{x \in [0,1]} \left| \alpha_0^{-1}(x) \alpha_\tau(x) \right| < \infty, \quad 0 \leq \tau \leq k, \tag{24}$$

with $l_0 = 1$, and

$$\rho_N := T_N^{\beta,\alpha}(f) \left(\sum_{\tau=1}^{k} \frac{l_\tau}{\Gamma(2-\alpha_\tau)} + 1 \right). \tag{25}$$

I. Suppose, throughout $[0,1]$, $\alpha_0(x) \geq \overline{\alpha} > 0$. Call

$$Q_{N+1}(f)(x) := P_{N+1}(f)(x) + \rho_N, \tag{26}$$

where $P_{N+1}(f)(x)$ as in (14).

Then by (19) we obtain

$$\left\| \left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j} \right) (f(x) + \rho_N) - \right.$$

$$\left. \left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j} \right) (Q_{N+1}(f)(x)) \right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(2-\alpha_j)}, \quad j = 1, ..., \rho. \tag{27}$$

And of course it holds

$$\| (f(x) + \rho_N) - Q_{N+1}(f)(x) \|_{\infty,[0,1]} \overset{(16)}{\leq} T_N^{\beta,\alpha}(f). \tag{28}$$

So that we find

$$\| f - Q_{N+1}(f) \|_{\infty,[0,1]} \overset{(28)}{\leq} \rho_N + T_N^{\beta,\alpha}(f) = \tag{29}$$

$$T_N^{\beta,\alpha}(f) \left(\sum_{\tau=1}^{k} \frac{l_\tau}{\Gamma(2-\alpha_j)} + 1 \right) + T_N^{\beta,\alpha}(f) = T_N^{\beta,\alpha}(f) \left[\sum_{\tau=1}^{k} \frac{l_\tau}{\Gamma(2-\alpha_j)} + 2 \right],$$

proving (23).

From (27) and (9), (11), we get

$$\left\| \left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j} \right) (f(x)) - \right.$$

$$\left. \left(\lambda D_{*0}^{\alpha_j} + (1-\lambda) D_{1-}^{\alpha_j} \right) (Q_{N+1}(f))(x) \right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(2-\alpha_j)}, \quad j = 1, ..., \rho, \tag{30}$$

proving (22).

Next we use the assumption $L_\lambda^* f(x) \geq 0$, all $x \in [0,1]$, to get

$$\alpha_0^{-1}(x) L_\lambda^* (Q_{N+1}(f))(x) = \alpha_0^{-1}(x) L_\lambda^* f(x) + \rho_N +$$

$$\sum_{j=0}^{k}\alpha_0^{-1}(x)\,\alpha_j(x)\left\{\left(\lambda D_{*0}^{\alpha_j}+(1-\lambda)\,D_{1-}^{\alpha_j}\right)[Q_{N+1}(f)(x)-f(x)-\rho_N]\right\}\overset{((27),(28))}{\geq}$$

$$\rho_N-\left(\sum_{\tau=1}^{k}\frac{l_\tau}{\Gamma(2-\alpha_j)}+1\right)T_N^{\beta,\alpha}(f)=\rho_N-\rho_N=0. \quad (31)$$

Hence $L_\lambda^*(Q_{N+1}(f))(x)\geq 0$, all $x\in[0,1]$.

II. Suppose, throughout $[0,1]$, $\alpha_0(x)\leq\overline{\beta}<0$. Call

$$Q_{N+1}(f)(x):=P_{N+1}(f)(x)-\rho_N, \quad (32)$$

where $P_{N+1}(f)(x)$ as in (14).

Then by (19) we obtain

$$\left\|\left(\lambda D_{*0}^{\alpha_j}+(1-\lambda)\,D_{1-}^{\alpha_j}\right)(f(x)-\rho_N)-\right.$$

$$\left.\left(\lambda D_{*0}^{\alpha_j}+(1-\lambda)\,D_{1-}^{\alpha_j}\right)(Q_{N+1}(f)(x))\right\|_{\infty,[0,1]}\leq\frac{T_N^{\beta,\alpha}(f)}{\Gamma(2-\alpha_j)},\quad j=1,...,\rho. \quad (33)$$

And of course it holds

$$\|(f(x)-\rho_N)-Q_{N+1}(f)(x)\|_{\infty,[0,1]}\overset{(16)}{\leq}T_N^{\beta,\alpha}(f). \quad (34)$$

Similarly we obtain again (22) and (23).

Next we use the assumption $L_\lambda^*f(x)\geq 0$, all $x\in[0,1]$, to get

$$\alpha_0^{-1}(x)\,L_\lambda^*(Q_{N+1}(f))(x)=\alpha_0^{-1}(x)\,L_\lambda^*f(x)-\rho_N+$$

$$\sum_{j=0}^{k}\alpha_0^{-1}(x)\,\alpha_j(x)\left\{\left(\lambda D_{*0}^{\alpha_j}+(1-\lambda)\,D_{1-}^{\alpha_j}\right)[Q_{N+1}(f)(x)-f(x)+\rho_N]\right\}\overset{((33),(34))}{\leq}$$

$$-\rho_N+\left(\sum_{\tau=1}^{k}\frac{l_\tau}{\Gamma(2-\alpha_j)}+1\right)T_N^{\beta,\alpha}(f)=-\rho_N+\rho_N=0. \quad (35)$$

Hence $L_\lambda^*(Q_{N+1}(f))(x)\geq 0$, for any $x\in[0,1]$. ■

Corollary 12 *(to Theorem 11, $\lambda=1$ case) Let* $L_1^*:=\sum_{j=0}^{k}\alpha_j(x)\,D_{*0}^{\alpha_j}$.

Then, for any $N\in\mathbb{N}$, there exists a real polynomial Q_{N+1} of degree $(N+1)$ such that

1)

$$\left\|D_{*0}^{\alpha_j}(f)-D_{*0}^{\alpha_j}(Q_{N+1}(f))\right\|_{\infty,[0,1]}\leq\frac{T_N^{\beta,\alpha}(f)}{\Gamma(2-\alpha_j)},\ j=1,...,\rho, \quad (36)$$

2) inequality (23) is again valid.

Assuming $L_1^*f(x)\geq 0$, for all $x\in[0,1]$, we get $(L_1^*(Q_{N+1}(f)))(x)\geq 0$ for all $x\in[0,1]$.

Corollary 13 *(to Theorem 11, $\lambda = 0$ case) Let* $L_0^* := \sum_{j=0}^{k} \alpha_j(x) D_{1-}^{\alpha_j}$.

Then, for any $N \in \mathbb{N}$, there exists a real polynomial Q_{N+1} of degree $(N+1)$ such that

1)

$$\left\| D_{1-}^{\alpha_j} f - D_{1-}^{\alpha_j} (Q_{N+1}(f)) \right\|_{\infty,[0,1]} \leq \frac{T_N^{\beta,\alpha}(f)}{\Gamma(2-\alpha_j)}, \ j = 1, ..., \rho, \tag{37}$$

2) inequality (23) is again valid.

Assuming $L_0^* f(x) \geq 0$, we get $L_0^*(Q_{N+1}(f))(x) \geq 0$ for any $x \in [0,1]$.

Finally we give

Corollary 14 *(to Theorem 11, $\lambda = \frac{1}{2}$ case) Let* $L_{\frac{1}{2}}^* := \sum_{j=0}^{k} \alpha_j(x) \left[\frac{D_{*0}^{\alpha_j} + D_{1-}^{\alpha_j}}{2} \right]$.

Then, for any $N \in \mathbb{N}$, there exists a real polynomial Q_{N+1} of degree $(N+1)$ such that

1)

$$\left\| \left(D_{*0}^{\alpha_j} + D_{1-}^{\alpha_j} \right)(f) - \left(D_{*0}^{\alpha_j} + D_{1-}^{\alpha_j} \right)(Q_{N+1}(f)) \right\|_{\infty,[0,1]} \leq \frac{2T_N^{\beta,\alpha}(f)}{\Gamma(2-\alpha_j)}, \ j = 1, ..., \rho, \tag{38}$$

2) inequality (23) is again valid.

Assuming $L_{\frac{1}{2}}^* f(x) \geq 0$, we get $L_{\frac{1}{2}}^*(Q_{N+1}(f))(x) \geq 0$ for any $x \in [0,1]$.

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