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NEW INEQUALITIES OF CBS-TYPE FOR POWER SERIES OF COMPLEX NUMBERS

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Abstract. Let $f(\lambda)=\sum_{n=0}^{\infty}a_n\lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. In this paper we show amongst other that, if $\alpha,z\in\mathbb{C}$ are such that $|\alpha|$, $|\alpha||z|^2<R$, then

$$\left|f(\alpha)f\left(\alpha z^2\right)-f^2(\alpha z)\right|\leq f_A(|\alpha|)f_A\left(|\alpha||z|^2\right)-|f_A(|\alpha|z)|^2.$$

where $f_A(z)=\sum_{n=0}^{\infty}|\alpha_n|z^n$.

Applications for some fundamental functions defined by power series are also provided.

1. Introduction

If we consider an analytic function $f(z)$ defined by the power series $\sum_{n=0}^{\infty}a_nz^n$ with complex coefficients a_n and apply the well-known Cauchy-Bunyakovsky-Schwarz (CBS) inequality

$$\left|\sum_{j=1}^{n}a_jb_j\right|^2\leq\sum_{j=1}^{n}|a_j|^2\sum_{j=1}^{n}|b_j|^2, \tag{1.1}$$

holding for the complex numbers a_j, b_j, $j\in\{1,\dots,n\}$, then we can deduce that

$$|f(z)|^2=\left|\sum_{n=0}^{\infty}a_nz^n\right|^2\leq\sum_{n=0}^{\infty}|a_n|^2\sum_{n=0}^{\infty}|z|^{2n}=\frac{1}{1-|z|^2}\cdot\sum_{n=0}^{\infty}|a_n|^2 \tag{1.2}$$

for any $z\in D(0,R)\cap D(0,1)$, where R is the radius of convergence of f.

The above inequality gives some information about the magnitude of the function f provided that numerical series $\sum_{n=0}^{\infty}|a_n|^2$ is convergent and z is not too close to the boundary of the open disk $D(0,1)$.

If we restrict ourselves more and assume that the coefficients in the representation $f(z)=\sum_{n=0}^{\infty}a_nz^n$ are nonnegative, and the assumption incorporates various examples of complex functions that will be indicated in the sequel, on utilizing the weighted version of the CBS-inequality, namely

$$\left|\sum_{j=1}^{n}w_ja_jb_j\right|^2\leq\sum_{j=1}^{n}w_j|a_j|^2\sum_{j=1}^{n}w_j|b_j|^2, \tag{1.3}$$

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where $w_j \geq 0$, while $a_j, b_j \in \mathbb{C}$, $j \in \{1, \dots, n\}$, we can state that

$$|f(zw)|^2 = \left| \sum_{n=0}^{\infty} a_n z^n w^n \right|^2 \leq \sum_{n=0}^{\infty} a_n |z|^{2n} \sum_{n=0}^{\infty} a_n |w|^{2n} = f\left(|z|^2\right) f\left(|w|^2\right) \tag{1.4}$$

for any $z, w \in \mathbb{C}$ with $|z|^2, |w|^2 \in D(0, R)$.

In an effort to provide a refinement for the celebrated Cauchy-Bunyakovsky-Schwarz inequality for complex numbers (1.1) de Bruijn established in 1960, [2] (see also [8, p. 89] or [3, p. 48]) the following result:

Lemma 1 (de Bruijn, 1960)**.** *If* $\mathbf{b} = (b_1, \dots, b_n)$ *is an* n*-tuple of real numbers and* $z = (z_1, \dots, z_n)$ *an* n*-tuple of complex numbers, then*

$$\left| \sum_{k=1}^{n} b_k z_k \right|^2 \leq \frac{1}{2} \sum_{k=1}^{n} b_k^2 \left[\sum_{k=1}^{n} |z_k|^2 + \left| \sum_{k=1}^{n} z_k^2 \right| \right]. \tag{1.5}$$

Equality holds in (1.5) if and only if for $k \in \{1, \dots, n\}$, $b_k = \operatorname{Re}(\lambda z_k)$, *where* λ *is a complex number such that the quantity* $\lambda^2 \sum_{k=1}^{n} z_k^2$ *is a nonnegative real number.*

On utilizing this result, Cerone & Dragomir established in [1] some inequalities for power series with nonnegative coefficients as follows:

Theorem 1 (Cerone & Dragomir, 2007 [1])**.** *Let* $f(z) := \sum_{n=0}^{\infty} a_n z^n$ *be an analytic function defined by a power series with nonnegative coefficients* a_n, $n \in \mathbb{N}$ *and convergent on the open disk* $D(0, R) \subset \mathbb{C}$, $R > 0$. *If* a *is a real number and* z *a complex number such that* $a^2, |z|^2 \in D(0, R)$, *then:*

$$|f(az)|^2 \leq \frac{1}{2} f\left(a^2\right) \left[f\left(|z|^2\right) + \left| f\left(z^2\right) \right| \right]. \tag{1.6}$$

For other similar results and applications for special functions see the research papers [1], [4]-[6] and the survey [7].

2. The Results

Denote by:

$$D(0,R) = \begin{cases} \{z \in \mathbb{C} : |z| < R\}, & \text{if } R < \infty \\ \mathbb{C}, & \text{if } R = \infty, \end{cases}$$

and consider the functions:

$$\lambda \mapsto f(\lambda) : D(0,R) \to \mathbb{C},\ f(\lambda) := \sum_{n=0}^{\infty} \alpha_n \lambda^n$$

and

$$\lambda \mapsto f_A(\lambda) : D(0,R) \to \mathbb{C},\ f_A(\lambda) := \sum_{n=0}^{\infty} |\alpha_n| \lambda^n.$$

As some natural examples that are useful for applications, we can point out that, if

$$f(\lambda)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\lambda^n=\ln\frac{1}{1+\lambda},\ \lambda\in D(0,1);\tag{2.1}$$
$$g(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\lambda^{2n}=\cos\lambda,\ \lambda\in\mathbb{C};$$
$$h(\lambda)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\lambda^{2n+1}=\sin\lambda,\ \lambda\in\mathbb{C};$$
$$l(\lambda)=\sum_{n=0}^{\infty}(-1)^n\lambda^n=\frac{1}{1+\lambda},\ \lambda\in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_A(\lambda)=\sum_{n=1}^{\infty}\frac{1}{n}\lambda^n=\ln\frac{1}{1-\lambda},\ \lambda\in D(0,1);\tag{2.2}$$
$$g_A(\lambda)=\sum_{n=0}^{\infty}\frac{1}{(2n)!}\lambda^{2n}=\cosh\lambda,\ \lambda\in\mathbb{C};$$
$$h_A(\lambda)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\lambda^{2n+1}=\sinh\lambda,\ \lambda\in\mathbb{C};$$
$$l_A(\lambda)=\sum_{n=0}^{\infty}\lambda^n=\frac{1}{1-\lambda},\ \lambda\in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(\lambda)=\sum_{n=0}^{\infty}\frac{1}{n!}\lambda^n\qquad\lambda\in\mathbb{C},\tag{2.3}$$
$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1},\qquad\lambda\in D(0,1);$$
$$\sin^{-1}(\lambda)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}\lambda^{2n+1},\qquad\lambda\in D(0,1);$$
$$\tanh^{-1}(\lambda)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1},\qquad\lambda\in D(0,1)$$
$${}_2F_1(\alpha,\beta,\gamma,\lambda)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^n,\alpha,\beta,\gamma>0,$$
$$\lambda\in D(0,1);$$

where Γ is *Gamma function.*

The following result holds:

Theorem 2. *Let $f(\lambda)=\sum_{n=0}^{\infty}a_n\lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. If*

$\alpha, z \in \mathbb{C}$ are such that $|\alpha|$, $|\alpha| |z|^2 < R$, then

$$\left|f(\alpha) f\left(\alpha z^2\right) - f^2(\alpha z)\right| \le f_A(|\alpha|) f_A\left(|\alpha| |z|^2\right) - |f_A(|\alpha| z)|^2 . \tag{2.4}$$

Proof. Let $n \ge 1$. Observe that

$$\begin{aligned}\sum_{j=0}^{n}\sum_{k=0}^{n} a_j a_k \alpha^j \alpha^k \left(z^j - z^k\right)^2 &= \sum_{j=0}^{n}\sum_{k=0}^{n} a_j a_k \alpha^j \alpha^k \left(z^{2j} - 2z^j z^k + z^{2k}\right) \\ &= \sum_{j=0}^{n} a_j \alpha^j z^{2j} \sum_{k=0}^{n} a_k \alpha^k + \sum_{j=0}^{n} a_j \alpha^j \sum_{k=0}^{n} a_k \alpha^k z^{2k} \\ &- 2\sum_{j=0}^{n} a_j \alpha^j z^j \sum_{k=0}^{n} a_k \alpha^k z^k \\ &= 2\left[\sum_{j=0}^{n} a_j \alpha^j \sum_{j=0}^{n} a_j \alpha^j z^{2j} - \left(\sum_{j=0}^{n} a_j \alpha^j z^j\right)^2\right],\end{aligned}$$

which gives us the useful identity

$$\sum_{j=0}^{n} a_j \alpha^j \sum_{j=0}^{n} a_j \alpha^j z^{2j} - \left(\sum_{j=0}^{n} a_j \alpha^j z^j\right)^2 = \frac{1}{2}\sum_{j=0}^{n}\sum_{k=0}^{n} a_j a_k \alpha^j \alpha^k \left(z^j - z^k\right)^2 \tag{2.5}$$

for any $\alpha, z \in \mathbb{C}$ and $n \ge 1$.

Taking the modulus in (2.5) and utilizing the generalized triangle inequality we have

$$\begin{aligned}&\left|\sum_{j=0}^{n} a_j \alpha^j \sum_{j=0}^{n} a_j \alpha^j z^{2j} - \left(\sum_{j=0}^{n} a_j \alpha^j z^j\right)^2\right| \\ &\le \frac{1}{2}\sum_{j=0}^{n}\sum_{k=0}^{n} |a_j| |a_k| |\alpha|^j |\alpha|^k \left|z^j - z^k\right|^2 \\ &= \frac{1}{2}\sum_{j=0}^{n}\sum_{k=0}^{n} |a_j| |a_k| |\alpha|^j |\alpha|^k \left[|z|^{2j} - 2\operatorname{Re}\left(z^j \bar{z}^k\right) + |z|^{2k}\right] \\ &= \frac{1}{2}\sum_{j=0}^{n}\sum_{k=0}^{n} |a_j| |a_k| |\alpha|^j |\alpha|^k |z|^{2j} + \frac{1}{2}\sum_{j=0}^{n}\sum_{k=0}^{n} |a_j| |a_k| |\alpha|^j |\alpha|^k |z|^{2k} \\ &- \sum_{j=0}^{n}\sum_{k=0}^{n} |a_j| |a_k| |\alpha|^j |\alpha|^k \operatorname{Re}\left(z^j \bar{z}^k\right).\end{aligned} \tag{2.6}$$

Observe that

$$\begin{aligned}\sum_{j=0}^{n}\sum_{k=0}^{n} |a_j| |a_k| |\alpha|^j |\alpha|^k |z|^{2j} &= \sum_{j=0}^{n}\sum_{k=0}^{n} |a_j| |a_k| |\alpha|^j |\alpha|^k |z|^{2k} \\ &= \sum_{j=0}^{n} |a_j| |\alpha|^j \sum_{j=0}^{n} |a_j| |\alpha|^j |z|^{2j}\end{aligned} \tag{2.7}$$

and

$$\begin{aligned} (2.8) \quad & \sum_{j=0}^{n}\sum_{k=0}^{n}|a_j|\,|a_k|\,|\alpha|^j\,|\alpha|^k \operatorname{Re}\left(z^j\bar{z}^k\right) \\ &= \operatorname{Re}\left(\sum_{j=0}^{n}|a_j|\,|\alpha|^j\,z^j\sum_{k=0}^{n}|a_k|\,|\alpha|^k\,\bar{z}^k\right) \\ &= \operatorname{Re}\left(\sum_{j=0}^{n}|a_j|\,|\alpha|^j\,z^j\overline{\sum_{j=0}^{n}|a_j|\,|\alpha|^j\,z^j}\right) = \left|\sum_{j=0}^{n}|a_j|\,|\alpha|^j\,z^j\right|^2 . \end{aligned}$$

Making use of (2.6)-(2.8) we get

$$\begin{aligned} (2.9) \quad & \left|\sum_{j=0}^{n}a_j\alpha^j\sum_{j=0}^{n}a_j\alpha^j z^{2j}-\left(\sum_{j=0}^{n}a_j\alpha^j z^j\right)^2\right| \\ &\le \sum_{j=0}^{n}|a_j|\,|\alpha|^j\sum_{j=0}^{n}|a_j|\,|\alpha|^j\,|z|^{2j}-\left|\sum_{j=0}^{n}|a_j|\,|\alpha|^j\,z^j\right|^2 \end{aligned}$$

for any $\alpha, z \in \mathbb{C}$ and $n \ge 1$.

Since all series whose partial sums involved in the inequality (2.9) are convergent, then by letting $n \to \infty$ in (2.9) we deduce the desired result (2.4). □

Remark 1. *If $f(\lambda)=\sum_{n=0}^{\infty}a_n\lambda^n$ is a function defined by power series with non-negative coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$, then*

$$(2.10) \quad \left|f(\alpha)\,f\left(\alpha z^2\right)-f^2(\alpha z)\right| \le f(|\alpha|)\,f\left(|\alpha|\,|z|^2\right)-|f(|\alpha|\,z)|^2$$

for $\alpha, z\in\mathbb{C}$ with $|\alpha|,\ |\alpha|\,|z|^2<R$.

Corollary 1. *Let $f(\lambda)=\sum_{n=0}^{\infty}a_n\lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$. If $x,y\in\mathbb{C}$ are such that $|x|^2,\ |y|^2<R$, then*

$$(2.11) \quad \left|f\left(x^2\right)f\left(y^2\right)-f^2(xy)\right| \le f_A\left(|x|^2\right)f_A\left(|y|^2\right)-|f_A(\bar{x}y)|^2$$

and

$$(2.12) \quad \left|f\left(x^2\right)f\left(\sigma^2(x)\,y^2\right)-f^2(\sigma(x)\,xy)\right| \le f_A\left(|x|^2\right)f_A\left(|y|^2\right)-|f_A(xy)|^2$$

where $\sigma(x):=\frac{x}{\bar{x}}$ is the "sign" of the complex number $x\neq 0$.

Proof. If we take in (2.4) $\alpha=x^2$ and $z=\frac{y}{x}$, then we have

$$(2.13) \quad \left|f\left(x^2\right)f\left(y^2\right)-f^2(xy)\right| \le f_A\left(|x|^2\right)f_A\left(|y|^2\right)-\left|f_A\left(|x|^2\,\frac{y}{x}\right)\right|^2,$$

which is equivalent to (2.11).

If we take in (2.4) $\alpha=x^2$ and $z=\frac{y}{\bar{x}}$, then we have

$$\left|f\left(x^2\right)f\left(x^2\left(\frac{y}{\bar{x}}\right)^2\right)-f^2\left(x^2\frac{y}{\bar{x}}\right)\right| \le f_A\left(|x|^2\right)f_A\left(|y|^2\right)-|f_A(xy)|^2,$$

which is equivalent to (2.12). □

Remark 2. *If $a \in \mathbb{R}$ and $y \in \mathbb{C}$ are such that $a^2, |y|^2 < R$, then*

$$\left| f\left(a^2\right) f\left(y^2\right) - f^2(ay) \right| \le f_A\left(a^2\right) f_A\left(|y|^2\right) - |f_A(ay)|^2 . \tag{2.14}$$

In particular, if the power series is with nonnegative coefficients, then

$$\left| f\left(a^2\right) f\left(y^2\right) - f^2(ay) \right| \le f\left(a^2\right) f\left(|y|^2\right) - |f(ay)|^2 \tag{2.15}$$

for $a \in \mathbb{R}$ and $y \in \mathbb{C}$ such that $a^2, |y|^2 < R$.

We also remark that, since

$$|f(ay)|^2 - f\left(a^2\right) f\left(y^2\right) \le \left| f\left(a^2\right) f\left(y^2\right) - f^2(ay) \right| ,$$

then by (2.15) we get

$$|f(ay)|^2 - f\left(a^2\right) f\left(y^2\right) \le f\left(a^2\right) f\left(|y|^2\right) - |f(ay)|^2 ,$$

which is equivalent to Cerone-Dragomir's result

$$|f(ay)|^2 \le \frac{1}{2} f\left(a^2\right) \left[f\left(|y|^2\right) + \left| f\left(y^2\right) \right| \right] ,$$

where $a \in \mathbb{R}$ and $y \in \mathbb{C}$ such that $a^2, |y|^2 < R$.

The following result also holds:

Theorem 3. *Let $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If $\alpha, z \in \mathbb{C}$ are such that $|\alpha|, |\alpha| |z|^2 < R$, then*

$$\begin{aligned} &\left| f\left(\alpha |z|^2\right) f(\alpha) - f(\alpha z) f(\alpha \bar{z}) \right|^2 \\ &\le f_A\left(|\alpha| |z|^2\right) \left[f_A\left(|\alpha| |z|^2\right) |f(\alpha)|^2 + |f(\alpha z)|^2 f_A(|\alpha|) \right. \\ &\left. -2 |f(\alpha)|^2 \operatorname{Re}\left(f_A(|\alpha| z) \overline{(f(\alpha z) / f(\alpha))} \right) \right] . \end{aligned} \tag{2.16}$$

Proof. Let $n \ge 1$. Observe that

$$\sum_{j=0}^{n} a_j \alpha^j \left(z^j - \frac{f(\alpha z)}{f(\alpha)} \right) \bar{z}^j = \sum_{j=0}^{n} a_j \alpha^j |z|^{2j} - \frac{f(\alpha x)}{f(\alpha)} \sum_{j=0}^{n} a_j \alpha^j \bar{z}^j \tag{2.17}$$

for any $\alpha, z \in \mathbb{C}$.

Taking the modulus in (2.17) and utilizing the generalized triangle inequality we get

$$\left| \sum_{j=0}^{n} a_j \alpha^j |z|^{2j} - \frac{f(\alpha z)}{f(\alpha)} \sum_{j=0}^{n} a_j \alpha^j \bar{z}^j \right| \le \sum_{j=0}^{n} |a_j| |\alpha|^j \left(\left| z^j - \frac{f(\alpha z)}{f(\alpha)} \right| \right) |z|^j \tag{2.18}$$

for any $\alpha, z \in \mathbb{C}$.

Making use of the weighted discrete Cauchy-Bunyakovsky-Schwarz inequality we have

$$\begin{aligned} &\sum_{j=0}^{n} |a_j| |\alpha|^j \left(\left| z^j - \frac{f(\alpha z)}{f(\alpha)} \right| \right) |z|^j \\ &\le \left(\sum_{j=0}^{n} |a_j| |\alpha|^j |z|^{2j} \right)^{1/2} \left[\sum_{j=0}^{n} |a_j| |\alpha|^j \left| z^j - \frac{f(\alpha z)}{f(\alpha)} \right|^2 \right]^{1/2} \end{aligned} \tag{2.19}$$

for any $\alpha, z \in \mathbb{C}$.

We also have

$$\begin{aligned}
&\sum_{j=0}^{n} |a_j| |\alpha|^j \left| z^j - \frac{f(\alpha z)}{f(\alpha)} \right|^2 \\
&= \sum_{j=0}^{n} |a_j| |\alpha|^j \left[|z|^{2j} - 2\operatorname{Re}\left(z^j \overline{\left(\frac{f(\alpha z)}{f(\alpha)} \right)} \right) + \left| \frac{f(\alpha z)}{f(\alpha)} \right|^2 \right] \\
&= \sum_{j=0}^{n} |a_j| |\alpha|^j |z|^{2j} - 2\operatorname{Re}\left(\sum_{j=0}^{n} |a_j| |\alpha|^j z^j \overline{\left(\frac{f(\alpha z)}{f(\alpha)} \right)} \right) \\
&+ \left| \frac{f(\alpha z)}{f(\alpha)} \right|^2 \sum_{j=0}^{n} |a_j| |\alpha|^j
\end{aligned} \tag{2.20}$$

for any $\alpha, z \in \mathbb{C}$.

By (2.18)-(2.20) we get

$$\begin{aligned}
&\left| \sum_{j=0}^{n} a_j \alpha^j |z|^{2j} - \frac{f(\alpha z)}{f(\alpha)} \sum_{j=0}^{n} a_j \alpha^j \overline{z}^j \right| \\
&\le \left(\sum_{j=0}^{n} |a_j| |\alpha|^j |z|^{2j} \right)^{1/2} \left[\sum_{j=0}^{n} |a_j| |\alpha|^j |z|^{2j} \right. \\
&\left. -2\operatorname{Re}\left(\overline{\left(\frac{f(\alpha z)}{f(\alpha)} \right)} \sum_{j=0}^{n} |a_j| |\alpha|^j z^j \right) + \left| \frac{f(\alpha z)}{f(\alpha)} \right|^2 \sum_{j=0}^{n} |a_j| |\alpha|^j \right]^{1/2}
\end{aligned} \tag{2.21}$$

for any $\alpha, z \in \mathbb{C}$.

Since all series whose partial sums involved in the inequality (2.21) are convergent, then by letting $n \to \infty$ in (2.21) we deduce

$$\begin{aligned}
&\left| f\left(\alpha |z|^2\right) - \frac{f(\alpha z) f(\alpha \overline{z})}{f(\alpha)} \right| \\
&\le \left[f_A\left(|\alpha| |z|^2\right) \right]^{1/2} \\
&\times \left[f_A\left(|\alpha| |z|^2\right) - 2\operatorname{Re}\left(f_A(|\alpha| z) \overline{\left(\frac{f(\alpha z)}{f(\alpha)} \right)} \right) + \left| \frac{f(\alpha z)}{f(\alpha)} \right|^2 f_A(|\alpha|) \right]^{1/2},
\end{aligned}$$

which is equivalent to the desired result (2.16). □

Corollary 2. *Let $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If $x, y \in \mathbb{C}$ are such that $|x|^2, |y|^2 < R$, then*

$$\begin{aligned}
&\left| f\left(\sigma(x) |y|^2\right) f\left(x^2\right) - f(xy) f\left(\sigma(x) x\overline{y}\right) \right|^2 \\
&\le f_A\left(|y|^2\right) \left[f_A\left(|y|^2\right) \left| f\left(x^2\right) \right|^2 + |f(xy)|^2 f_A\left(\left|x^2\right|\right) \right. \\
&\left. -2 \left| f\left(x^2\right) \right|^2 \operatorname{Re}\left(f_A(\overline{x} y) \overline{\left(f(xy) / f\left(x^2\right) \right)} \right) \right].
\end{aligned} \tag{2.22}$$

Proof. If we take in (2.16) $\alpha = x^2$ and $z = \frac{y}{x}$, then we have

$$\begin{aligned}
&\left| f\left(x^2\left|\frac{y}{x}\right|^2\right) f\left(x^2\right) - f\left(x^2\frac{y}{x}\right) f\left(x^2\overline{\frac{y}{x}}\right)\right|^2 \\
&\leq f_A\left(\left|x^2\right|\left|\frac{y}{x}\right|^2\right) \\
&\times \left[f_A\left(\left|x^2\right|\left|\frac{y}{x}\right|^2\right)\left|f\left(x^2\right)\right|^2 - 2\left|f\left(x^2\right)\right| \operatorname{Re}\left(f_A\left(\left|x^2\right|\frac{y}{x}\right)\overline{\left(\frac{f\left(x^2\frac{y}{x}\right)}{f\left(x^2\right)}\right)}\right)\right. \\
&\left. + \left|f\left(x^2\frac{y}{x}\right)\right|^2 f_A\left(\left|x^2\right|\right)\right],
\end{aligned}$$

which is equivalent to (2.22). □

We have the following result:

Theorem 4. *Assume that $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$ ($a_0 \neq 0$) is a function defined by power series with nonnegative coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$. If $x \in \mathbb{R}$ with $0 \leq x \leq 1$ and $0 \leq \alpha < R$, then*

$$0 \leq f(\alpha) f\left(\alpha x^2\right) - f^2(\alpha x) \leq \frac{1}{4} f^2(\alpha). \tag{2.23}$$

Proof. Let $n \geq 1$. Observe that

$$\begin{aligned}
&\sum_{j=0}^{n} a_j \alpha^j \left(x^j - \frac{f(\alpha x)}{f(\alpha)}\right)\left(x^j - \frac{1}{2}\right) \\
&= \sum_{j=0}^{n} a_j \alpha^j x^{2j} - \frac{f(\alpha x)}{f(\alpha)} \sum_{j=0}^{n} a_j \alpha^j x^j - \frac{1}{2} \sum_{j=0}^{n} a_j \alpha^j \left(x^j - \frac{f(\alpha x)}{f(\alpha)}\right)
\end{aligned} \tag{2.24}$$

for any $\alpha, x \in \mathbb{R}$.

Taking the modulus and utilizing the triangle inequality we have

$$\begin{aligned}
&\left|\sum_{j=0}^{n} a_j \alpha^j x^{2j} - \frac{f(\alpha x)}{f(\alpha)} \sum_{j=0}^{n} a_j \alpha^j x^j - \frac{1}{2} \sum_{j=0}^{n} a_j \alpha^j \left(x^j - \frac{f(\alpha x)}{f(\alpha)}\right)\right| \\
&\leq \sum_{j=0}^{n} a_j \alpha^j \left|x^j - \frac{f(\alpha x)}{f(\alpha)}\right| \left|x^j - \frac{1}{2}\right|
\end{aligned} \tag{2.25}$$

for any $\alpha, x \in \mathbb{R}$.

Since $0 \leq x \leq 1$, then $0 \leq x^j \leq 1$ for $j \in \{0, ..., n\}$, which implies that

$$\left|x^j - \frac{1}{2}\right| \leq \frac{1}{2} \text{ for } j \in \{0, ..., n\}.$$

Then by (2.25) we get

$$\begin{aligned}
&\left|\sum_{j=0}^{n} a_j \alpha^j x^{2j} - \frac{f(\alpha x)}{f(\alpha)} \sum_{j=0}^{n} a_j \alpha^j x^j - \frac{1}{2} \sum_{j=0}^{n} a_j \alpha^j \left(x^j - \frac{f(\alpha x)}{f(\alpha)}\right)\right| \\
&\leq \frac{1}{2} \sum_{j=0}^{n} a_j \alpha^j \left|x^j - \frac{f(\alpha x)}{f(\alpha)}\right|
\end{aligned} \tag{2.26}$$

for $0 \leq x \leq 1$ and $n \geq 1$.

Utilising the weighted discrete Cauchy-Bunyakovsky-Schwarz inequality, we have

$$\sum_{j=0}^{n} a_j \alpha^j \left| x^j - \frac{f(\alpha x)}{f(\alpha)} \right| \leq \left(\sum_{j=0}^{n} a_j \alpha^j \left(x^j - \frac{f(\alpha x)}{f(\alpha)} \right)^2 \right)^{1/2} \left(\sum_{j=0}^{n} a_j \alpha^j \right)^{1/2}. \tag{2.27}$$

Observe that

$$\begin{aligned} &\sum_{j=0}^{n} a_j \alpha^j \left(x^j - \frac{f(\alpha x)}{f(\alpha)} \right)^2 \\ &= \sum_{j=0}^{n} a_j \alpha^j \left[x^{2j} - 2 \left(x^j \frac{f(\alpha x)}{f(\alpha)} \right) + \frac{f^2(\alpha x)}{f^2(\alpha)} \right] \\ &= \sum_{j=0}^{n} a_j \alpha^j x^{2j} - 2 \left(\frac{f(\alpha x)}{f(\alpha)} \sum_{j=0}^{n} a_j \alpha^j x^j \right) + \frac{f^2(\alpha x)}{f^2(\alpha)} \sum_{j=0}^{n} a_j \alpha^j . \end{aligned} \tag{2.28}$$

From (2.25)-(2.28) we can state that

$$\begin{aligned} &\left| \sum_{j=0}^{n} a_j \alpha^j x^{2j} - \frac{f(\alpha x)}{f(\alpha)} \sum_{j=0}^{n} a_j \alpha^j x^j - \frac{1}{2} \sum_{j=0}^{n} a_j \alpha^j \left(x^j - \frac{f(\alpha x)}{f(\alpha)} \right) \right| \\ &\leq \frac{1}{2} \left(\sum_{j=0}^{n} a_j \alpha^j \right)^{1/2} \\ &\times \left[\sum_{j=0}^{n} a_j \alpha^j x^{2j} - 2 \left(\frac{f(\alpha x)}{f(\alpha)} \sum_{j=0}^{n} a_j \alpha^j x^j \right) + \frac{f^2(\alpha x)}{f^2(\alpha)} \sum_{j=0}^{n} a_j \alpha^j \right]^{1/2} \end{aligned} \tag{2.29}$$

for any $0 \leq x \leq 1$ and $n \geq 1$.

Since all series whose partial sums involved in the inequality (2.29) are convergent, then by letting $n \to \infty$ in (2.29) we deduce

$$\begin{aligned} &\left| f\left(\alpha x^2\right) - \frac{f(\alpha x)}{f(\alpha)} f(\alpha x) \right| \\ &\leq \frac{1}{2} [f(\alpha)]^{1/2} \left[f\left(\alpha x^2\right) - 2 \left(\frac{f(\alpha x)}{f(\alpha)} f(\alpha x) \right) + \frac{f^2(\alpha x)}{f^2(\alpha)} f(\alpha) \right]^{1/2}, \end{aligned}$$

namely

$$\left| f\left(\alpha x^2\right) - \frac{f^2(\alpha x)}{f(\alpha)} \right| \leq \frac{1}{2} [f(\alpha)]^{1/2} \left[f\left(\alpha x^2\right) - \frac{f^2(\alpha x)}{f(\alpha)} \right]^{1/2},$$

which is equivalent to the desired result (2.23). □

Corollary 3. *Let $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$ ($a_0 \neq 0$) be a function defined by power series with nonnegative coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. If $u, v \in \mathbb{R}$ with $0 \leq u \leq v$ and $0 < v^2 < R$, then*

$$0 \leq f\left(v^2\right) f\left(u^2\right) - f^2(uv) \leq \frac{1}{4} f^2\left(v^2\right). \tag{2.30}$$

Proof. If we take in (2.23) $\alpha = v^2$ and $x = \frac{u}{v}$, then we have the desired inequality (2.30). □

3. Applications

If we write the above inequalities for the exponential function $\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^n$, $\lambda \in \mathbb{C}$, then we have:

$$\begin{aligned} &\left|\exp\left[\alpha\left(1+z^2\right)\right] - \exp(2\alpha z)\right| \\ &\leq \exp\left[|\alpha|\left(1+|z|^2\right)\right] - |\exp(2|\alpha|z)|, \ \alpha, z \in \mathbb{C}, \end{aligned} \tag{3.1}$$

$$\left|\exp\left(x^2+y^2\right) - \exp(2xy)\right| \leq \exp\left(|x|^2+|y|^2\right) - |\exp(2\bar{x}y)|, \ x, y \in \mathbb{C}, \tag{3.2}$$

$$\begin{aligned} &\left|\exp\left(x^2+\sigma^2(x)y^2\right) - \exp(2\sigma(x)xy)\right| \\ &\leq \exp\left(|x|^2+|y|^2\right) - |\exp(2xy)|, \ x, y \in \mathbb{C} \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &\left|\exp\left[\alpha\left(1+|z|^2\right)\right] - \exp[2\alpha \operatorname{Re}(z)]\right|^2 \\ &\leq \exp\left(|\alpha||z|^2\right)\left[\exp\left(|\alpha||z|^2\right)|\exp(\alpha)|^2 + |\exp(\alpha z)|^2 \exp(|\alpha|)\right. \\ &\left. -2|\exp(\alpha)|^2 \operatorname{Re}\left(\exp\left(|\alpha| z + \overline{\alpha z} - \bar{\alpha}\right)\right)\right]. \end{aligned} \tag{3.4}$$

If we take $\alpha = 1$ in (3.1), then we get

$$\left|\exp\left(1+z^2\right) - \exp(2z)\right| \leq \exp\left(1+|z|^2\right) - |\exp(2z)|, \ z \in \mathbb{C}. \tag{3.5}$$

If we take $\alpha = 1$ in (3.4), then we get

$$\begin{aligned} &\left(\exp\left(1+|z|^2\right) - \exp[2\operatorname{Re}(z)]\right)^2 \\ &\leq \exp\left(|z|^2+1\right)\left[\exp\left(|z|^2+1\right) + |\exp(z)|^2 - 2\exp(2\operatorname{Re}(z))\right]. \end{aligned} \tag{3.6}$$

If $x \in \mathbb{R}$ with $0 \leq x \leq 1$ and $0 \leq \alpha$, then

$$0 \leq \exp\left(\alpha\left(1+x^2\right)\right) - \exp(2\alpha x) \leq \frac{1}{4}\exp(2\alpha). \tag{3.7}$$

If $0 \leq u \leq v$, then

$$0 \leq \exp\left(v^2+u^2\right) - \exp(2uv) \leq \frac{1}{4}\exp\left(2v^2\right). \tag{3.8}$$

If we write the above inequalities for the functions $\sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}$ and $\sum_{n=0}^{\infty} (-1)^n \lambda^n = \frac{1}{1+\lambda}$, $\lambda \in D(0,1)$, then we have

$$\begin{aligned} &\left|(1\pm\alpha)^{-1}\left(1\pm\alpha z^2\right)^{-1} - (1\pm\alpha z)^{-2}\right| \\ &\leq (1-|\alpha|)^{-1}\left(1-|\alpha||z|^2\right)^{-1} - |1-|\alpha| z|^{-2}, \ |\alpha|, |\alpha||z|^2 < 1, \end{aligned} \tag{3.9}$$

$$\begin{aligned} &\left|\left(1\pm x^2\right)^{-1}\left(1\pm y^2\right)^{-1} - (1\pm xy)^{-2}\right| \\ &\leq \left(1-|x|^2\right)^{-1}\left(1-|y|^2\right)^{-1} - |1-\bar{x}y|^{-2}, \ |x|, |y| < 1 \end{aligned} \tag{3.10}$$

and

$$\left|\left(1\pm x^{2}\right)^{-1}\left(1\pm\sigma^{2}(x)\,y^{2}\right)^{-1}-\left(1\pm\sigma(x)\,xy\right)^{-2}\right| \le\left(1-|x|^{2}\right)^{-1}\left(1-|y|^{2}\right)^{-1}-|1-xy|^{-2},\ \ |x|,|y|<1. \tag{3.11}$$

If $u,v\in\mathbb{R}$ with $0\le u\le v<1$, then

$$0\le\left(1-v^{2}\right)^{-1}\left(1-u^{2}\right)^{-1}-(1-uv)^{-2}\le\frac{1}{4}\left(1-v^{2}\right)^{-2}. \tag{3.12}$$

If we write the above inequalities for $\sum_{n=1}^{\infty}\frac{1}{n}\lambda^{n}=\ln\frac{1}{1-\lambda}$ and $\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n}\lambda^{n}=\ln\frac{1}{1+\lambda}$, $\lambda\in D(0,1)$, then we have

$$\left|\ln(1\pm\alpha)^{-1}\ln\left(1\pm\alpha z^{2}\right)^{-1}-\left[\ln(1\pm\alpha z)^{-1}\right]^{2}\right| \le\ln(1-|\alpha|)^{-1}\ln\left(1-|\alpha|\,|z|^{2}\right)^{-1}-\left|\ln(1-|\alpha|\,z)^{-1}\right|^{2},\ \ |\alpha|,|\alpha|\,|z|^{2}<1, \tag{3.13}$$

$$\left|\ln\left(1\pm x^{2}\right)^{-1}\ln\left(1\pm y^{2}\right)^{-1}-\left[\ln(1\pm xy)^{-1}\right]^{2}\right| \le\ln\left(1-|x|^{2}\right)^{-1}\ln\left(1-|y|^{2}\right)^{-1}-\left|\ln(1-\bar{x}y)^{-1}\right|^{2},\ \ |x|,|y|<1 \tag{3.14}$$

and

$$\left|\ln\left(1\pm x^{2}\right)^{-1}\ln\left(1\pm\sigma^{2}(x)\,y^{2}\right)^{-1}-\left[\ln(1\pm\sigma(x)\,xy)^{-1}\right]^{2}\right| \le\ln\left(1-|x|^{2}\right)^{-1}\ln\left(1-|y|^{2}\right)^{-1}-\left|\ln(1-xy)^{-1}\right|^{2},\ \ |x|,|y|<1. \tag{3.15}$$

If $u,v\in\mathbb{R}$ with $0\le u\le v<1$, then

$$0\le\ln\left(1-v^{2}\right)^{-1}\ln\left(1-u^{2}\right)^{-1}-\left[\ln(1-uv)^{-1}\right]^{2} \le\frac{1}{4}\left[\ln\left(1-v^{2}\right)^{-1}\right]^{2}. \tag{3.16}$$

The *polylogarithm* $Li_{n}(z)$, also known as the *de Jonquières function* is the function defined by

$$Li_{n}(z):=\sum_{k=1}^{\infty}\frac{z^{k}}{k^{n}} \tag{3.17}$$

defined in the complex plane over the unit disk $D(0,1)$ for all complex values of the order n.

The special case $z=1$ reduces to $Li_{s}(1)=\zeta(s)$, where ζ is the *Riemann zeta function.*

The polylogarithm of nonnegative integer order arises in the sums of the form

$$\sum_{k=1}^{\infty}k^{n}r^{k}=Li_{-n}(r)=\frac{1}{(1-r)^{n+1}}\sum_{i=0}^{n}\left\langle{n\atop i}\right\rangle k^{n-i}$$

where $\left\langle{n\atop i}\right\rangle$ is an *Eulerian number*, namely, we recall that

$$\left\langle{n\atop k}\right\rangle:=\sum_{j=0}^{k+1}(-1)^{j}\binom{n+1}{i}(k-j+1)^{n}.$$

Polylogarithms also arise in sums of generalized harmonic numbers $H_{n,r}$ as

$$\sum_{n=1}^{\infty} H_{n,r} z^n = \frac{Li_r(z)}{1-z} \quad \text{for} \quad z \in D(0,1),$$

where, we recall that

$$H_{n,r} := \sum_{k=1}^{n} \frac{1}{k^r} \quad \text{and} \quad H_{n,1} := H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Special forms of low-order polylogarithms include

$$Li_{-2}(z) = \frac{z(z+1)}{(1-z)^3}, \qquad Li_{-1}(z) = \frac{z}{(1-z)^2},$$
$$Li_0(z) = \frac{z}{1-z} \quad \text{and} \quad Li_1(z) = -\ln(1-z), \quad z \in D(0,1).$$

At argument $z = -1$, the general polylogarithms become $Li_x(-1) = -\eta(x)$, where $\eta(x)$ is the *Dirichlet eta function.*

If we use the inequality (2.10) for *polylogarithm* $Li_n(z)$ we can state that

$$\left|Li_n(\alpha) Li_n(\alpha z^2) - Li_n^2(\alpha z)\right| \le Li_n(|\alpha|) Li_n\left(|\alpha| |z|^2\right) - |Li_n(|\alpha| z)|^2 \tag{3.18}$$

for $\alpha, z \in \mathbb{C}$ with $|\alpha|, |z| < 1$ and n is a negative or a positive integer.

If $u, v \in \mathbb{R}$ with $0 \le u \le v < 1$, then

$$0 \le Li_n(v^2) Li_n(u^2) - [Li_n(uv)]^2 \le \frac{1}{4} Li_n(v^2), \tag{3.19}$$

where n is a negative or a positive integer.

Similar inequalities can be stated for *hypergeometric functions* or for *modified Bessel functions of the first kind*, see [4]-[6]. The details are omitted.

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