Received 18/04/14

ON THE PROPERTIES OF THE s_2-GODUNOVA-LEVIN TYPE FUNCTION WITH ITS INEQUALITIES

M. EMIN OZDEMIR

Abstract. In this study, Firstly , it was defined a new class of function by modifying the definition of Sever S. Dragomir [1,def. 5] . Secondly , some properties of this class were obtained and finally, several integral inequalities for same class were established.

1. INTRODUCTION

Definition 1. *([1]) We say that the function $f : C \subset X \to [0,\infty)$ is of $s-$Godunova-Levin type, with $s \in [0,1]$, if*

$$f(tx + (1-t)y) \leq \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y)$$

for all $t \in (0,1)$ and $x, y \in C$.

We denote by $Q_s(C)$ the class of $s-$Godunova-Levin functions defined on C.

We observe that for $s = 0$, we obtain the class of p-functions while for $s = 1$ we obtain the class of Godunova-Levin.

To proof our main results, we need the follwing inequalities

Theorem 1. *(see [2]) $f, g : [a,b] \to \mathbb{R}$ be integrable functions, both increasing or both decreasing. Furthermore, let $p : [a,b] \to \mathbb{R}_+$ be an integrable function. Then*

$$\int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx \leq \int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx. \tag{1.1}$$

If one of the functions f or g is nonincreasing and the other nondecreasing then the inequality in (1.1) is reversed. Inequality (1.1) is known in the literature as Čebyšev's inequality and so are the following special cases of

$$\frac{1}{b-a}\int_a^b f(x)dx \int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx$$

and

$$\int_0^1 f(x)dx \int_0^1 g(x)dx \leq \int_0^1 f(x)g(x)dx.$$

Theorem 2. *(see [3])Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on $[a,b]$ and if $|f|^p$ and $|g|^q$ are integrable functions on $[a,b]$, then*

$$\int_a^b |f(x)g(x)|\, dx \leq \left(\int_a^b |f(x)|^p\, dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q\, dx\right)^{\frac{1}{q}} \tag{1.2}$$

1991 *Mathematics Subject Classification.* 26D15, 25D10.

Key words and phrases. $s-$Godunova- Levin, properties, inequality, jensen inequality.

with equality holding if and only if $A\left|f(x)\right|^{p}=B\left|g(x)\right|^{q}$ *almost everywhere, where* A *and* B *are constants.*

In [5] we established several new inequalities which euryhythmic by means of literature for $s-$Godunova-Levin type function and we also gave some error estimates for Trapezoidal formula. For some properties of this class of functions see [6]-[11].

The aim of this paper is to establish the several properties of $Q_{s_2}(I)$ and to write the new inequalities about $Q_{s_2}(I)$.

2. MAIN RESULTS

In this section, we discuss our main results.

The Properties of $\mathbf{Q}_{s_2}$

Definition 2. *The function* $f:C\subset X\rightarrow\mathbb{R}$ *is said to be Levin type function in the second sense, where* $s_2\in[0,1], r_1+r_2=1$ *if for every* $x,y\in I$ *we have*

$$f\left(r_1x+r_2y\right)\leq\frac{1}{r_1^{s_2}}f\left(x\right)+\frac{1}{r_2^{s_2}}f\left(y\right)\tag{2.1}$$

Denote by $Q_{s_2}(I)$ *the set of the* s_2-*Godunova- Levin type function on* I *for which* $f(0)\leq 0$.

The above definition has been modified from definition 1, otherwise , some results won't be valid.

Lemma 1. *If* $f\in Q_{s_2}(I)$, *then it is* s_2- *starshaped*

$$f\left(tx\right)\leq\frac{1}{r_1^{s_2}}f\left(x\right)$$

Proof. For any $x\in I$ and $t\in(0,1)$, $r_2+r_2=1, s_2\in[0,1]$, with $f(0)\leq 0$. we have

$$f\left(r_1x\right)=f\left(r_1x+r_20\right)\leq\frac{1}{r_1^{s_2}}f\left(x\right)+\frac{1}{r_2^{s_2}}f\left(0\right)\leq\frac{1}{r_1^{s_2}}f\left(x\right).$$

□

Proposition 1. *Let be* $g:I\subset X\rightarrow\mathbb{R}$, *if* $A^{-1}+B^{-1}=1, A,B>0$, *then* $g(x)=x^{s_2}$ *is* s_2- *Godunova-Levin type function in the second sense on I, for all* $s_2\in[0,1]$.

Proof. Since $(u+v)^{s_2}\leq u^{s_2}+v^{s_2}$, for all $u,v\geq 0$, $0\leq s_2\leq 1$ and if $A^{-1}+B^{-1}=1, A,B>0$, we have for the function $\mathrm{g}(x)=x^{s_2}$ and that

$$\begin{aligned}
g\left(A^{-1}x+B^{-1}y\right) &= \left(A^{-1}x+B^{-1}y\right)^{s_2} \\
&\leq \left(A^{-s_2}x^{s_2}+B^{-s_2}y^{s_2}\right) \\
&= \frac{1}{A^{s_2}}g\left(x\right)+\frac{1}{B^{s_2}}g\left(y\right) \\
&\leq \frac{1}{A^{-s_2}}g\left(x\right)+\frac{1}{B^{-s_2}}g\left(y\right)
\end{aligned}$$

□

which completes the proof.

Theorem 3. *If The function $f : C \subset X \to [0,\infty)$ is of a nondecresing s_2-Godunova-Levin type and g be a nonnegative convex function on $[0,\infty)$, for all $x, y \in [0,\infty)$. The composition fog of f with g is also s_2-Godunova-Levin type function.*

Proof. Let $h = fog$ Then we have , for all $t \in (0,1)$ and $s_2 \in [0,1]$

$$h\left(tx + (1-t)\,y\right) = f\left(g\left(tx + (1-t)\,y\right)\right)$$

Since f nondecreasing and s_2-Godunova-Levin type function, we obtain

$$f\left(g\left(tx + (1-t)\,y\right)\right) \leq f\left(tg\left(x\right) + (1-t)\,g\left(y\right)\right)$$

By using $f \in Q_{s_2}(I)$ and note that g is nonnegative, we get

$$\begin{aligned} f\left(tg\left(x\right) + (1-t)\,g\left(y\right)\right) &\leq \frac{1}{t^{s_2}} f\left(g\left(x\right)\right) + \frac{1}{(1-t)^{s_2}} f\left(g\left(y\right)\right) \\ &= \frac{1}{t^{s_2}} h\left(x\right) + \frac{1}{(1-t)^{s_2}} h\left(y\right) \end{aligned}$$

which completes the proof. □

Theorem 4. *Let $f, g \in Q_s\left(C\right)$, then the functions $f + g$ and λf are also s_2-Godunova-levin type function for all $\lambda > 0$, $s_2 \in [0,1]$, and $t \in (0,1)$.*

Proof. Since $f, g \in Q_{s_2}\left(C\right)$, then we have

$$\begin{aligned} f\left(tx + (1-t)\,y\right) &\leq \frac{1}{t^{s_2}} f(x) + \frac{1}{(1-t)^{s_2}} f(y) \\ g\left(tx + (1-t)\,y\right) &\leq \frac{1}{t^{s_2}} g(x) + \frac{1}{(1-t)^{s_2}} g(y) \end{aligned}$$

Adding the above inequalities we get

$$\begin{aligned} (f+g)\left(tx + (1-t)\,y\right) &= f\left(tx + (1-t)\,y\right) + g\left(tx + (1-t)\,y\right) \\ &\leq \frac{1}{t^{s_2}} f(x) + \frac{1}{(1-t)^{s_2}} f(y) + \frac{1}{t^{s_2}} g(x) + \frac{1}{(1-t)^{s_2}} g(y) \\ &= \frac{1}{t^{s_2}} \left[f(x) + g(x)\right] + \frac{1}{(1-t)^{s_2}} \left[f(y) + g(y)\right] \\ &= \frac{1}{t^{s_2}} \left[f + g\right](x) + \frac{1}{(1-t)^{s_2}} \left[f + g\right](y) \end{aligned}$$

and

$$\begin{aligned} \lambda f\left(ta + (1-t)\,b\right) &= f\left(t\left(\lambda a\right) + (1-t)\left(\lambda b\right)\right) \\ &\leq \frac{1}{t^{s_2}} f\left(\lambda a\right) + \frac{1}{(1-t)^{s_2}} f\left(\lambda b\right) \\ &= \frac{1}{t^{s_2}} \lambda f\left(a\right) + \frac{1}{(1-t)^{s_2}} \lambda f\left(b\right) \end{aligned}$$

which completes the required proofs. □

The following theorem is the Jensen inequality for an s_2-Godunova-levin type functions

Theorem 5. *6 Let $w_1, ...w_n$ be a positive numbers $(n \geq 2)$ and $x_i \in C($ $i = 1, 2, 3, ..., n)$. If $f \in Q_{s_2}$, Then*

$$f\left(\sum_{i=1}^{n} w_i x_i\right) \leq \sum_{i=1}^{n} \frac{1}{w_i^{s_2}} f(x_i) \tag{2.2}$$

where $W_n = \sum_{i=1}^{n} w_i,\ s \in [0,1]$.

If $n = 2$, then inequality (2.2) is equivalent to (2.1) with $r_1 = \frac{w_1}{W_2}$, $r_1 = \frac{w_2}{W_2}$, $(r_1 + r_2 = 1)$. let us suppose that inequality (2.2) holds for $n - 1$. Then, for $n-$tubles $(x_1, ..., x_n)$ and $(w_1, ..., w_n)$ we have

$$\begin{aligned} f\left(\sum_{i=1}^{n} w_i x_i\right) &= f\left(w_n x_n + \sum_{i=1}^{n-1} w_i x_i\right) \\ &\leq \frac{1}{w_n^{s_2}} f(x_n) + \sum_{i=1}^{n-1} \frac{1}{w_i^{s_2}} f(x_i) \\ &\leq \sum_{i=1}^{n} \frac{1}{w_i^{s_2}} f(x_i) \end{aligned}$$

The following theorem is also an Jensen type inequality for the class $Qs_2(C)$.

Theorem 6. *Let $w_1, ...w_n$ be a positive numbers $(n \geq 2)$ and $f \in Qs_2(C)$, $x_i \in C($ $i = 1, 2, 3, ..., n)$. Then*

$$f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \leq W_n^{s_2} \sum_{i=1}^{n} \frac{1}{w_i^{s_2}} f(x_i), \qquad s_2 \in [0,1] \tag{2.3}$$

where $W_n = \sum_{i=1}^{n} w_i$.

Proof. Let us suppose that $f \in Q_{s_2}(C)$, with $\frac{w_1}{W_n} + ... + \frac{w_n}{W_n} = 1$, then $n-$tubles $(x_1, ..., x_n)$ and $(w_1, ..., w_n)$ with theorem 6 we have

$$\begin{aligned} f\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) &= f\left(\frac{w_n}{W_n} x_n + \sum_{i=1}^{n-1} \frac{w_i}{W_n} x_i\right) \\ &\leq \frac{W_n^{s_2}}{w_n^{s_2}} f(x_n) + \left(\sum_{i=1}^{n-1} \frac{W_n^{s_2}}{w_i^{s_2}} f(x_i)\right) \\ &= \frac{W_n^{s_2}}{w_n^{s_2}} f(x_n) + W_n^{s_2} \sum_{i=1}^{n-1} \frac{1}{w_i^{s_2}} f(x_i) \\ &\leq W_n^{s_2}\left[\frac{1}{w_n^{s_2}} f(x_n) + \sum_{i=1}^{n-1} \frac{1}{w_i^{s_2}} f(x_i)\right] \\ &= W_n^{s_2} \sum_{i=1}^{n} \frac{1}{w_i^{s_2}} f(x_i) \end{aligned}$$

□

The New Inequalities About $\mathbf{Q}_{s_2}$

In [4], Özdemir et.al. used the following Lemma in order to establish several integral inequalities via some kinds of convexity.

Lemma 2. *Let $f : [a,b] \subset [0,\infty) \to \mathbb{R}$ be a continuous on $[a,b]$ such that $f \in L[a,b]$, $a < b$, Then the equality*

$$\int_a^b (x-a)^p (b-x)^q f(x)\,dx = (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q f(ta+(1-t)b)\,dt$$

holds for some fixed $p, q > 0$.

Theorem 7. *Let $f : [a,b] \subset [0,\infty) \to [0,\infty)$ be a continuous on $C = [a,b]$ such that $f \in L[a,b]$, $a < b$, If $f \in Q_{s_2}(C)$, $s_2 \in [0,1]$, $p, q > 0$. Then*

$$\begin{aligned} & \int_a^b (x-a)^p (b-x)^q f(x)\,dx \\ \leq \;& (b-a)^{p+q+1} \{\beta(q-s_2+1, p+1) f(a) + \beta(q+1, p-s_2+1) f(b)\}. \end{aligned}$$

Proof. from lemma 2 and $f \in Q_{s_2}(C)$, we have

$$\begin{aligned} & \int_a^b (x-a)^p (b-x)^q f(x)\,dx \\ \leq \;& (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q \left[\frac{1}{t^{s_2}} f(a) + \frac{1}{(1-t)^{s_2}} f(b)\right] dt \\ = \;& (b-a)^{p+q+1} \left\{ f(a) \int_0^1 (1-t)^p t^{q-s_2} dt + f(b) \int_0^1 (1-t)^{p-s_2} t^q dt \right\} \\ = \;& (b-a)^{p+q+1} \{\beta(q-s_2+1, p+1) f(a) + \beta(q+1, p-s_2+1) f(b)\} \end{aligned}$$

which completes the required proof. □

Theorem 8. *Let f and g be two functions s_2-Godunova-Levin type on $C = [a,b]$, such that f, g and $fg \in L[a,b]$, for some fixed $s_2 \in (0,1) \setminus \left\{\frac{1}{2}\right\}$, Then the following inequality holds:*

(2.4)

$$\int_a^b f(x)\,g(x)\,dx \leq \frac{b-a}{2} \left\{ K(,) \left(\frac{1}{1-2s_2}\right) + N(,) \left(\frac{2^{-1+2s_2}\sqrt{\pi}\,\Gamma(1-s_2)}{\Gamma\left(\frac{3}{2}-s_2\right)}\right) \right\}$$

where

$$\begin{aligned} K(,) &= \left(f^2+g^2\right)(a) + \left(f^2+g^2\right)(b), \\ N(,) &= 2\left(f(a)\,f(b) + g(a)\,g(b)\right), \\ & \quad \Gamma(.) \text{ is Gamma function.} \end{aligned}$$

Proof. Since f, g are two functions of Godunova-Levin type on $[a,b]$, we have

$$\begin{aligned} f(ta+(1-t)b) &\leq \frac{1}{t^{s_2}} f(a) + \frac{1}{(1-t)^{s_2}} f(b) \\ g(ta+(1-t)b) &\leq \frac{1}{t^{s_2}} g(a) + \frac{1}{(1-t)^{s_2}} g(b) \end{aligned}$$

for all $t \in (0,1)$.

Using the change of the variable $x = ta + (1-t)b$, we get

$$\int_a^b f(x)\,g(x)\,dx = (b-a)\int_0^1 f(ta+(1-t)b)\,g(ta+(1-t)b)\,dt$$

Now, Using the elementary inequality $cd \leq \frac{1}{2}\left(c^2+d^2\right)$ $(c,d \in \mathbb{R}^+)$ in the right hand side of above inequality with convexity , we get

$$\begin{aligned}
\int_a^b f(x)\,g(x)\,dx &\leq \frac{b-a}{2}\left[\int_0^1 \{f(ta+(1-t)b)\}^2\,dt + \int_0^1 \{g(ta+(1-t)b)\}^2\,dt\right] \\
&\leq \frac{b-a}{2}\left[\int_0^1 \left[\frac{1}{t^{s_2}}f(a) + \frac{1}{(1-t)^{s_2}}f(b)\right]^2 dt + \int_0^1 \left[\frac{1}{t^{s_2}}g(a) + \frac{1}{(1-t)^{s_2}}g(b)\right]^2 dt\right] \\
&= \frac{b-a}{2}\left[\int_0^1 t^{-2s_2}f^2(a)\,dt + 2\int_0^1 t^{-s_2}(1-t)^{-s_2} f(a)\,f(b)\,dt\right. \\
&\quad + \int_0^1 (1-t)^{-2s_2} f^2(b)\,dt + \int_0^1 t^{-2s_2} g^2(a)\,dt \\
&\quad \left. +2\int_0^1 t^{-s_2}(1-t)^{-s_2} g(a)\,g(b)\,dt + \int_0^1 (1-t)^{-2s_2} g^2(b)\,dt\right] \\
&= \frac{b-a}{2}\left\{\left[\left(f^2+g^2\right)(a) + \left(f^2+g^2\right)(b)\right]\frac{1}{1-s_2}\right. \\
&\quad \left. + \left[2\left(f(a)\,f(b) + g(a)\,g(b)\right)\right]\left(\frac{2^{-1+2s}\sqrt{\pi}\,\Gamma(1-s_2)}{\Gamma\left(\frac{3}{2}-s_2\right)}\right)\right\}
\end{aligned}$$

which completes the required proof. Here, we used the fact that

$$\begin{aligned}
\int_0^1 [t(1-t)]^{-s_2}\,dt &= \int_0^1 \left(t-t^2\right)^{-s_2} dt \\
&= \frac{2^{-1+2s_2}\sqrt{\pi}\,\Gamma(1-s_2)}{\Gamma\left(\frac{3}{2}-s_2\right)}
\end{aligned}$$

and

$$\int_0^1 t^{-2s_2}\,dt = \int_0^1 (1-t)^{-2s_2}\,dt = \frac{1}{1-2s_2}.$$

□

Corollary 1. *Under the conditions of Theorem 8 we also have the following inequality*

$$\int_a^b f(x)\,g(x)\,dx \leq \frac{(b-a)}{1-2s_2}\left\{\left(f^2(a)+f^2(b)\right) + \left(g^2(a)+g^2(b)\right)\right\}$$

Proof. As in the proof of Theorem 8, Firstly if we use the elementary $cd \leq \frac{1}{2}\left(c^2+d^2\right)$ for all $c,d \in \mathbb{R}^+$ we get

$$\begin{aligned}
\int_a^b f(x)\,g(x)\,dx &\leq \frac{b-a}{2}\left[\int_0^1 \{f(ta+(1-t)b)\}^2\,dt + \int_0^1 \{g(ta+(1-t)b)\}^2\,dt\right] \\
&\leq \frac{b-a}{2}\left[\int_0^1 \left[\frac{1}{t^{s_2}}f(a) + \frac{1}{(1-t)^{s_2}}f(b)\right]^2 dt + \int_0^1 \left[\frac{1}{t^{s_2}}g(a) + \frac{1}{(1-t)^{s_2}}g(b)\right]^2 dt\right]
\end{aligned}$$

secondly, we shall use the elementary inequality $(A+B)^2 \leq 2\left(A^2+B^2\right)$ in the right hand side of final inequality for all $A,B\in\mathbb{R}^+$, then we have

$$\begin{aligned}\int_a^b f(x)\,g(x)\,dx &\leq \frac{b-a}{2}\left[\int_0^1\left\{2\left(t^{-s_2}f^2(a)+(1-t)^{-2s_2}f^2(b)+2\left(t^{-2s_2}g^2(a)+(1-t)^{-2s_2}g^2(b)+\right)\right)\right\}\right]\\ &\leq \frac{(b-a)}{1-2s_2}\left\{f^2(a)+f^2(b)+g^2(a)+g^2(b)\right\}\end{aligned}$$

□

Lemma 3. *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}^+$ be a differentiable on $[a,b]$ such that $f'\in L[a,b]$, $a<b$, the inequality*

$$\frac{1}{b-a}\int_a^b\frac{(x-a)^q(b-x)^p}{(b-a)^{p+q}}f'(x)\,dx=\int_0^1(1-t)^p\,t^q f'(tb+(1-t)\,a)\,dt.$$

Proof. Using the change of variable $x=tb+(1-t)\,a$ on the right hand side of equalitiy, we obtain the required result. □

Theorem 9. *Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}^+$ be a differentiable on $C=[a,b]$ such that $f'\in L[a,b]$, $p,q>0$, $0\leq a<b<\infty$. If $f'\in Q_{s_2}(C)$, $s_2\in[0,1]$, then*

$$\frac{1}{b-a}\int_a^b\frac{(x-a)^q(b-x)^p}{(b-a)^{p+q}}f'(x)\,dx\leq f'(a)\,\beta(q+1,p-s_2+1)+f'(b)\,\beta(q-s_2+1,p+1)$$

where $\beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt,\ x,y>0$.

Proof. From Lemma 3 and since $f'\in Q_{s_2}(C)$, we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b\frac{(x-a)^q(b-x)^p}{(b-a)^{p+q}}f'(x)\,dx &\leq \int_0^1(1-t)^p\,t^q\left(\frac{1}{t^{s_2}}f'(a)+\frac{1}{(1-t)^{s_2}}f'(b)\right)dt\\ &= f'(b)\left[\int_0^1(1-t)^p\,t^{q-s_2}dt\right]+f'(a)\left[\int_0^1(1-t)^{p-s_2}\,t^q dt\right]\\ &= f'(b)\,\beta(q-s_2+1,p+1)+f'(a)\,\beta(q+1,p-s_2+1)\end{aligned}$$

which completes the required proof. □

Theorem 10. *Let $f:[a,b]\subseteq[0,\infty)\to[0,\infty)$ be a differentiable on $[a,b]$ such that $f'\in Q_{s_2}$, $f,g\in L[a,b]$, $0\leq a<b<\infty$. $[a,b]$, If one of the functions is increasing and the other decreasing then*

(2.5)

$$\int_0^1(1-t)^p\,t^q f'(tb+(1-t)\,a)\,dt\leq\frac{1}{b-a}\Gamma(1+p)\,\Gamma(1+q)\,\frac{1}{1-s_2}\left(f'(a)+f'(b)\right)$$

where $p,q>0, s_2\in[0,1)$ and $\Gamma$is Gamma function.

Proof. From Lemma 3 and 1.1 with $f' \in Q_{s_2}$ we obtain the proof,

$$\begin{aligned}\int_0^1 (1-t)^p t^q f'(tb+(1-t)a)\,dt &= \frac{1}{b-a}\int_a^b \frac{(x-a)^q (b-x)^p}{(b-a)^{p+q}} f'(x)\,dx \\ &\le \frac{1}{(b-a)^2}\int_a^b \frac{(x-a)^q (b-x)^p}{(b-a)^{p+q}}dx \int_a^b f'(x)\,dx \\ &= \frac{1}{b-a}\Gamma(1+p)\Gamma(1+q)\int_a^b f'(ta+(1-t)b)\,dx \\ &\le \frac{1}{b-a}\Gamma(1+p)\Gamma(1+q)\frac{1}{1-s_2}\left(f'(a)+f'(b)\right)\end{aligned}$$

□

Corollary 2. *let $f'$and $g(x) = \frac{(x-a)^q(b-x)^p}{(b-a)^{p+q}}$ be real and integrable functions on $[a,b]$ and let them both be either increasing and decreasing ,the inverse inequality holds.*

Theorem 11. *Let $f : [a,b] \subseteq [0,\infty) \to [0,\infty)$ be a differentiable on $[a,b]$ such that $f' \in L[a,b]$, $g \in L[a,b]$, $0 \le a < b < \infty$. $[a,b]$, $g(x) = \frac{(x-a)^q(b-x)^p}{(b-a)^{p+q}}$. If one function is increasing and the other decreasing, then one has the following inequality*

$$\int_0^1 (1-t)^p t^q f'(tb+(1-t)a)\,dt \le \frac{1}{(b-a)(p^2+1)^{\frac{1}{p}}(q^2+1)^{\frac{1}{q}}}\int_a^b \|f'\|\,dx$$

where $p^{-1}+q^{-1}=1$, and $|f'| = \int_a^b \|f'\|\,dx$.

Proof. from Lemma 3 and Čebyšev's inequalit (1.1) we have

$$\begin{aligned}\int_0^1 (1-t)^p t^q f'(tb+(1-t)a)\,dt &= \frac{1}{b-a}\int_a^b \frac{(x-a)^q (b-x)^p}{(b-a)^{p+q}} f'(x)\,dx \\ &\le \frac{1}{(b-a)^2}\int_a^b \frac{(x-a)^q (b-x)^p}{(b-a)^{p+q}}dx \int_a^b f'(x)\,dx\end{aligned}$$

Now , firstly, If we take modulus of final inequality and then use the Hölder inequality (1.2) in the right side of inequality, we get

$$\begin{aligned}\left|\int_0^1 (1-t)^p t^q f'(tb+(1-t)a)\,dt\right| &\le \frac{1}{(b-a)^{2+p+q}}\left(\int_a^b (x-a)^{q^2}dx\right)^{\frac{1}{q}} \\ &\quad\times\left(\int_a^b (b-x)^{p^2}dx\right)^{\frac{1}{p}}\int_a^b |f'(x)|\,dx \\ &= \frac{1}{(b-a)(p^2+1)^{\frac{1}{p}}(q^2+1)^{\frac{1}{q}}}\int_a^b \|f'\|\,dx\end{aligned}$$

which completes the required □

Theorem 12. *Under the conditions of Theorem 11 if $|f'| \in Q_{s_2}$ we get*

$$\int_0^1 (1-t)^p t^q f'(tb+(1-t)a)\,dt \le \frac{1}{(b-a)(p^2+1)^{\frac{1}{p}}(q^2+1)^{\frac{1}{q}}}\frac{1}{1-s_2}\left(|f'(a)|+|f'(b)|\right)$$

where $s \in [0,1)$.

Proof. Since

$$\int_a^b \|f'\| \, dx = \int_a^b |f'(x)| \, dx \leq \frac{1}{1-s_2} \left(|f'(a)| + |f'(b)| \right)$$

the proof is clear.

The final theorem is as the following. □

Theorem 13. *Let f be a s_2-Godunova-Levin type function on $\bar{I}_1$ and Let function g be s_2-Godunova- Levin function on $\bar{I}_2$and $\bar{I}_3 = \bar{I}_1 \cap \bar{I}_2$ under the condition that $\bar{I}_3$ has at least two points. If them both be either nondecreasing or nonincreasing, for all $x, y \in \bar{I}_3$,some fixeds $a, b \in \bar{I}_3$, $a < b$, $t \in (0,1)$, $s_2 \in [0,1]$, $fg \in L\left[\bar{I}_3\right]$, then the following inequality is true.*

$$f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \leq \frac{2^{2s_2+1}}{(b-a)} \left[\int_a^b f(x) g(x) \, dx + \int_a^b f(y) g(y) \, dy \right]$$

Proof. Since $f, g \in Q_s\left(\bar{I}_3\right)$, we have for all $x, y \in \bar{I}_3$ with $t = \frac{1}{2}$,

$$\begin{aligned} f\left(\frac{x+y}{2}\right) &\leq 2^{s_2}\left[f(x) + f(y)\right] \\ g\left(\frac{x+y}{2}\right) &\leq 2^{s_2}\left[g(x) + g(y)\right] \end{aligned}$$

by multiplying the above inequalities with $\left[f(x) - f(y)\right]\left[g(x) - g(y)\right] \geq 0$

$$\begin{aligned} f\left(\frac{x+y}{2}\right) g\left(\frac{x+y}{2}\right) &\leq 2^{2s_2}\left[f(x) + f(y)\right]\left[g(x) - g(y)\right] \\ &= 2^{2s_2+1}\left[f(x) g(x) + f(y) g(y)\right] \end{aligned}$$

Now, choosing $x = ta + (1-t) b$, $y = (1-t) + tb$,and afterwards, if we integrate over $[0,1]$ for t. We get the required inequality □

References

[1] S.S. Dragomir, Integral inequalities of Jensen type for $\lambda-$convex functions. RGMIA, Res. Rep. Coll. 17 (2014).

[2] D.S. Mitrinović, J.E. Pečarić and A.M. Fink, *Classical and new inequalities in analysis,* Math. and Its Applications, Kluwer Academic Publishers, Dordrecht, 1993

[3] D.S. Mitrinović, *Analytic inequalities*, Springer- Verlag, New-York, 1970.

[4] M.E. Özdemir, E. Set and M. Alomari, Integral inequalities via several kinds of convexity, Creat. Math. Inform., 20 (2011), 1, 62-73.

[5] M.E. Özdemir, Some inequalities on the s-Godunova Levin type functions, Submitted.

[6] S.S. Dragomir, Integral inequalities of Jensen type for $\lambda-$convex functions. RGMIA, Res. Rep. Coll. 17 (2014).

[7] S. S. Dragomir and B. Mond, On Hadamard's inequality for a class of functions of Godunova and Levin. Indian J. Math. 39 (1997), no. 1, 1–9.

[8] S. S. Dragomir and C. E. M. Pearce, On Jensen's inequality for a class of functions of Godunova and Levin. Period. Math. Hungar. 33 (1996), no. 2, 93–100.

[9] D. S. Mitrinovi ́c and J.E. Pečarić, Note on a class of functions of Godunova and Levin. C. R. Math. Rep. Acad. Sci. Canada 12 (1990), no. 1, 33–36.

[10] J.E. Pečarić, S. S. Dragomir, On an Inequality of Godunova and Levin and some refinements of Jensen integral inequality. Itinerant seminar of functional equations, Approximation and Convexity(cluj-Napoca, 1989), 263-268, preprint, 86-6, Univ. " Babeş-Bolyai", Cluj-Napoca,1989.

[11] M. Radulescu, S. Radulescu and P. Alexandrescu, On the Godunova-Levin-Schur class of functions. Math. Inequal. Appl. 12 (2009), no. 4, 853–862

Atatürk Üniversity, K.K. Education Faculty, Department of Mathematics, 25240 , Erzurum, Turkey
E-mail address: emos@atauni.edu.tr
URL: http://www.eminozdemir.com