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A NEW GENERALIZATION OF THE MIDPOINT FORMULA FOR $n-$TIME DIFFERENTIABLE MAPPINGS WHICH ARE CONVEX

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Abstract. In this paper, we establish several new inequalities for $n-$ time differentiable mappings that are connected with the celebrated Hermite-Hadamard integral inequality.

1. INRODUCTION

On November 22, 1881, Hermite (1822-1901) sent a letter to the Journal Mathesis. This letter was published in Mathesis 3 (1883, p: 82) and in this letter an inequality presented which is well-known in the literature as Hermite-Hadamard integral inequality:

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2} \tag{1.1}$$

where $f:I\subseteq\mathbb{R}\rightarrow\mathbb{R}$ is a convex function on the interval I of a real numbers and $a,b\in I$ with $a<b$. If the function f is concave, the inequality in (1.1) is reversed.

The inequalities (1.1) have become an important cornerstone in mathematical anlysis and optimization. Many uses of these inequalities have been discovered in a variety of settings. Moreover , many inequalities of special means can be obtained for a particular choice of the function f. Due to the rich geometrical significance of Hermite-Hadamard's inequlity, there is growing literature providing its new proofs, extensions, refinements and generalizations, see for example ([1], [5], [8]-[11], [15]-[18]) and the references therein.

Definition 1. *A function $f:[a,b]\subset\mathbb{R}\rightarrow\mathbb{R}$ is said to be convex if whenever $x,y\in[a,b]$ and $t\in[0,1]$, the following inequality holds:*

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y).$$

We say that f is concave if $(-f)$ is convex. This definition has its origins in Jensen's results from [7] and has opened up the most extended, useful and multidisciplinary domain of mathematics, namely, convex analysis. Convex curves and convex bodies have appeared in mathematical literature since antiquity and there are many important results related to them.

For other recent results concerning the $n-$time differentiable functions see [2]-[4], [6], [8], [12], [17] where further references are given.

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The main purpose of the present paper is to establish several new inequalities for $n-$ time differantiable mappings that are connected with the celebrated Hermite-Hadamard integral inequality.

2. MAIN RESULTS

Lemma 1. *For $n \in \mathbb{N}$; let $f : I \subset \mathbb{R} \to \mathbb{R}$ be n-time differentiable. If $a, b \in I$ with $a < b$ and $f^{(n)} \in L[a,b]$, then*

$$\begin{aligned} \int_a^b f(t)dt &= \sum_{k=0}^{n-1} \left(\frac{1+(-1)^k}{2^{k+1}(k+1)!} \right) (b-a)^{k+1} f^{(k)} \left(\frac{a+b}{2} \right) \\ &+ (b-a)^{n+1} \int_0^1 M_n(t) f^{(n)}(ta+(1-t)b)dt \end{aligned} \tag{2.1}$$

where

$$M_n(t) = \begin{cases} \frac{t^n}{n!}, & t \in \left[0, \frac{1}{2}\right] \\ \frac{(t-1)^n}{n!}, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

and n natural number, $n \geq 1$.

Proof. The proof is by mathematical induction.

The case $n = 1$ is [[9], Lemma 2.1].

Assume that (2.1) holds for "n" and let us prove it for "$n+1$". That is, we have to prove the equality

$$\begin{aligned} \int_a^b f(t)dt &= \sum_{k=0}^{n} \left(\frac{1+(-1)^k}{2^{k+1}(k+1)!} \right) (b-a)^{k+1} f^{(k)} \left(\frac{a+b}{2} \right) \\ &+ (b-a)^{n+2} \int_0^1 M_{n+1}(t) f^{(n+1)}(ta+(1-t)b)dt \end{aligned} \tag{2.2}$$

where, obviously,

$$M_{n+1}(t) = \begin{cases} \frac{t^{n+1}}{(n+1)!}, & t \in \left[0, \frac{1}{2}\right] \\ \frac{(t-1)^{n+1}}{(n+1)!}, & t \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

We have

$$\begin{aligned} & (b-a)^{n+2} \int_0^1 M_{n+1}(t) f^{(n+1)}(ta+(1-t)b)dt \\ = \ & (b-a)^{n+2} \left\{ \int_0^{\frac{1}{2}} \frac{t^{n+1}}{(n+1)!} f^{(n+1)}(ta+(1-t)b)dt \right. \\ & \left. + \int_{\frac{1}{2}}^1 \frac{(t-1)^{n+1}}{(n+1)!} f^{(n+1)}(ta+(1-t)b)dt \right\} \end{aligned}$$

and integrating by parts gives

$$
\begin{aligned}
& (b-a)^{n+2}\int_0^1 M_{n+1}(t)f^{(n+1)}(ta+(1-t)b)dt \\
= \;& (b-a)^{n+2}\left\{\frac{t^{n+1}}{(n+1)!}\,\frac{f^{(n)}(ta+(1-t)b)}{a-b}\bigg|_0^{\frac{1}{2}}-\frac{1}{a-b}\int_0^{\frac{1}{2}}\frac{t^n}{n!}f^{(n)}(ta+(1-t)b)dt\right. \\
& \left.+\frac{(t-1)^{n+1}}{(n+1)!}\,\frac{f^{(n)}(ta+(1-t)b)}{a-b}\bigg|_{\frac{1}{2}}^1-\frac{1}{a-b}\int_{\frac{1}{2}}^1\frac{(t-1)^n}{n!}f^{(n)}(ta+(1-t)b)dt\right\} \\
= \;& -\frac{1+(-1)^n}{2^{n+1}(n+1)!}f^{(n)}\left(\frac{a+b}{2}\right)(b-a)^{n+1}+(b-a)^{n+1}\int_0^1 M_n(t)f^{(n)}(ta+(1-t)b)dt.
\end{aligned}
$$

That is

$$
\begin{aligned}
(b-a)^{n+1}\int_0^1 M_n(t)f^{(n)}(ta+(1-t)b)dt \;=\; & \frac{1+(-1)^n}{2^{n+1}(n+1)!}f^{(n)}\left(\frac{a+b}{2}\right)(b-a)^{n+1} \\
& +(b-a)^{n+2}\int_0^1 M_{n+1}(t)f^{(n+1)}(ta+(1-t)b)dt.
\end{aligned}
$$

Now, using the mathematical induction hypothesis, we get

$$
\begin{aligned}
\int_a^b f(t)dt \;=\; & \sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right) \\
& +\frac{1+(-1)^n}{2^{n+1}(n+1)!}(b-a)^{n+1}f^{(n)}\left(\frac{a+b}{2}\right)+(b-a)^{n+2}\int_0^1 M_{n+1}(t)f^{(n+1)}(ta+(1-t)b)dt \\
= \;& \sum_{k=0}^{n}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right) \\
& +(b-a)^{n+2}\int_0^1 M_{n+1}(t)f^{(n+1)}(ta+(1-t)b)dt.
\end{aligned}
$$

That is, the identity (2.2) and the theorem is thus proved. □

Theorem 1. *For $n \geq 1$, let $f : I \subset \mathbb{R} \to \mathbb{R}$ be $n-$time differentiable and $a < b$. If $f^{(n)} \in L[a,b]$ and $\left|f^{(n)}\right|$ is convex on $[a,b]$, then the following inequality holds:*

$$
\begin{aligned}
& \left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \\
\leq \;& \frac{(b-a)^{n+1}}{2^n(n+1)!}\left(\frac{\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|}{2}\right).
\end{aligned}
\tag{2.3}
$$

Proof. Since $\left|f^{(n)}\right|$ is convex on $[a,b]$, from Lemma 1 and Hölder integral inequality, it follows that

$$\begin{aligned}
&\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \\
\leq\ & (b-a)^{n+1}\int_0^1 |M_n(t)|\left|f^{(n)}(ta+(1-t)b)\right|dt \\
\leq\ & (b-a)^{n+1}\left\{\int_0^{\frac{1}{2}}\frac{t^n}{n!}\left[t\left|f^{(n)}(a)\right|+(1-t)\left|f^{(n)}(b)\right|\right]dt\right. \\
&\left.+\int_{\frac{1}{2}}^1\frac{(1-t)^n}{n!}\left[t\left|f^{(n)}(a)\right|+(1-t)\left|f^{(n)}(b)\right|\right]dt\right\} \\
=\ & \frac{(b-a)^{n+1}}{2^n(n+1)!}\left(\frac{\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|}{2}\right).
\end{aligned}$$

This completes the proof. □

Theorem 2. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be $n-$time differentiable and $a<b$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^q$ is convex on $[a,b]$, then we have*

$$\begin{aligned}
(2.4)\qquad &\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{(b-a)^{n+1}}{2^{n+1}n!}\left(\frac{1}{np+1}\right)^{\frac{1}{p}} \\
&\times\left\{\left(\frac{\left|f^{(n)}(a)\right|^q+3\left|f^{(n)}(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(b)\right|^q}{4}\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1 and Hölder integral inequality, we obtain

$$\begin{aligned}
&\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \\
\leq\ & (b-a)^{n+1}\int_0^1 |M_n(t)|\left|f^{(n)}(ta+(1-t)b)\right|dt \\
\leq\ & \frac{(b-a)^{n+1}}{n!}\left\{\left(\int_0^{\frac{1}{2}}t^{np}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1(1-t)^{np}dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Since $\left|f^{(n)}\right|^q$ is convex on $[a,b]$, then

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^{n+1}}{2^{n+1}n!}\left(\frac{1}{np+1}\right)^{\frac{1}{p}}$$
$$\times\left\{\left(\frac{\left|f^{(n)}(a)\right|^q+3\left|f^{(n)}(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(b)\right|^q}{4}\right)^{\frac{1}{q}}\right\}$$

which completes the proof. □

Theorem 3. *For $n \geq 1$, let $f : I \subset \mathbb{R} \to \mathbb{R}$ be $n-$time differentiable and $a < b$. If $f^{(n)} \in L[a,b]$ and $\left|f^{(n)}\right|^q$ is convex on $[a,b]$, for $q \geq 1$, then the following inequality holds:*

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \tag{2.5}$$
$$\leq \frac{(b-a)^{n+1}}{2^{n+1}(n+1)!}\left\{\left[\frac{n+1}{2n+4}\left|f^{(n)}(a)\right|^q+\frac{n+3}{2n+4}\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\left[\frac{n+3}{2n+4}\left|f^{(n)}(a)\right|^q+\frac{n+1}{2n+4}\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right]\right\}.$$

Proof. From Lemma 1 and using the well known power-mean integral inequality, we have

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq (b-a)^{n+1}\int_0^1 |M_n(t)|\left|f^{(n)}(ta+(1-t)b)\right|dt$$
$$\leq \frac{(b-a)^{n+1}}{n!}\left\{\left(\int_0^{\frac{1}{2}} t^n dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}} t^n\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1 (1-t)^n dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1 (1-t)^n\left|f^{(n)}(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right\}.$$

Since $\left|f^{(n)}\right|^q$ is convex on $[a,b]$, for $q \geq 1$, then we obtain

$$\left|\int_a^b f(t)dt - \sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^{n+1}}{2^{n+1}(n+1)!}\left\{\left[\frac{n+1}{2n+4}\left|f^{(n)}(a)\right|^q+\frac{n+3}{2n+4}\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\left[\frac{n+3}{2n+4}\left|f^{(n)}(a)\right|^q+\frac{n+1}{2n+4}\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right]\right\}$$

whisch completes the proof. □

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