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SOLUTIONS TO TWO OPEN PROBLEMS CONCERNING THE CONSTANTS OF LANDAU AND LEBESGUE

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ABSTRACT. The constants of Landau and Lebesgue are defined, for all integers $n \geq 0$, in order, by

$$G_n = \sum_{k=0}^{n} \frac{1}{16^k} \binom{2k}{k}^2 \quad \text{and} \quad L_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin\left((n+\frac{1}{2})t\right)}{\sin(\frac{1}{2}t)} \right| \mathrm{d}t,$$

which play important roles in the theories of complex analysis and Fourier series, respectively. Here we aim at giving solutions to two open problems concerning the constants of Landau and Lebesgue proposed by Chen and Choi, and Chen and Cheng.

1. Introduction

The constants of Landau and Lebesgue are defined, for all integers $n \geq 0$, in order, by

$$G_n = \sum_{k=0}^{n} \frac{1}{16^k} \binom{2k}{k}^2 \quad \text{and} \quad L_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin\left((n+\frac{1}{2})t\right)}{\sin(\frac{1}{2}t)} \right| \mathrm{d}t,$$

which play important roles in the theories of complex analysis and Fourier series, respectively.

Certain inequalities and asymptotic expansions for the constants G_n and L_n have been investigated by many authors (see, *e.g.*, [3, 4, 5, 6, 7, 9, 14, 17, 19, 20, 21, 22, 23, 27]). In this paper, we give solutions to two open problems concerning the constants of Landau and Lebesgue proposed by Chen and Cheng [5] and Chen and Choi [6]. See sections 2 and 3.

The following lemma is required in our present investigation.

Lemma 1.1 (see [8])**.** *Let $q_0 \neq 0$ and*

$$Q(x) \sim \sum_{j=0}^{\infty} q_j x^{-j}, \qquad x \to \infty \tag{1.1}$$

be a given asymptotical expansion. Then the following results hold:

(i) *The composition $A(x) = \ln(Q(x))$ has asymptotic expansion of the following form*

$$A(x) \sim \sum_{j=1}^{\infty} a_j x^{-j}, \qquad x \to \infty,$$

where

$$a_j = \frac{q_j}{q_0} - \frac{1}{jq_0} \sum_{k=1}^{j-1} k a_k q_{j-k}, \qquad j \in \mathbb{N}. \tag{1.2}$$

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(ii) *For all real r it holds*

$$[Q(x)]^r \sim \sum_{j=0}^{\infty} P_j(r) x^{-j}, \qquad x \to \infty,$$

where

$$P_0(r) = q_0^r, \quad P_j(r) = \frac{1}{jq_0} \sum_{k=1}^{j} [k(1+r) - j] q_k P_{j-k}(r), \qquad j \in \mathbb{N}. \tag{1.3}$$

2. The Landau constants

Landau himself studied the asymptotic behavior of G_n and showed that $G_n \sim \frac{1}{\pi} \ln n$, as $n \to \infty$. From then on, various approximation results are obtained for the Landau constants. The approximation of G_n goes to two related directions. One is to find sharper bounds of G_n for all positive integers n, and the other is to obtain large-n asymptotic approximations for the constants G_n. Watson [25] proved the asymptotic formula:

$$G_n = \frac{1}{\pi} \ln(n+1) + c_0 - \frac{1}{4\pi(n+1)} + O\left(\frac{1}{n^2}\right) \quad \text{as} \quad n \to \infty, \tag{2.1}$$

with

$$c_0 = \frac{1}{\pi}(\gamma + 4 \ln 2) = 1.0662758532089143543451101966157\ldots, \tag{2.2}$$

where γ denotes the Euler-Mascheroni constant. In what follows, c_0 is given in (2.2). Inspired by formula (2.1), Brutman [2] discovered upper and lower bounds for G_n:

$$1 + \frac{1}{\pi} \ln(n+1) \le G_n < c_0 + \frac{1}{\pi} \ln(n+1), \quad n \in \mathbb{N}_0. \tag{2.3}$$

Falaleev [11] presented new bounds for G_n:

$$c_0 + \frac{1}{\pi} \ln\left(n + \frac{3}{4}\right) < G_n \le 1.0976 + \frac{1}{\pi} \ln\left(n + \frac{3}{4}\right), \quad n \in \mathbb{N}_0. \tag{2.4}$$

Recently, some sharp inequalities and asymptotic expansions for G_n were established [3, 4, 5, 6, 7, 9, 14, 17, 19, 20, 21, 22, 23, 27]. For example, Mortici [19] improved the upper bound and proved the following inequality:

$$c_0 + \frac{1}{\pi} \ln\left(n + \frac{3}{4}\right) < G_n < c_0 + \frac{1}{\pi} \ln\left(n + \frac{3}{4} + \frac{11}{192n}\right), \quad n \in \mathbb{N}. \tag{2.5}$$

Moreover, the author pointed out that the constant 3/4 is the best possible. In [4], Chen improved the upper bound in (2.5) and presented the following inequality:

$$c_0 + \frac{1}{\pi} \ln\left(n + \frac{3}{4}\right) < G_n < c_0 + \frac{1}{\pi} \ln\left(n + \frac{3}{4} + \frac{11}{192(n+\frac{3}{4})}\right), \qquad n \in \mathbb{N}_0. \tag{2.6}$$

In fact, the following approximation formulas hold (see, *e.g.*, [3, 4]):

$$G_n = c_0 + \frac{1}{\pi} \ln\left(n + \frac{3}{4}\right) + O(n^{-2}), \tag{2.7}$$

$$G_n = c_0 + \frac{1}{\pi} \ln\left(n + \frac{3}{4} + \frac{11}{192n}\right) + O(n^{-3}), \tag{2.8}$$

$$G_n = c_0 + \frac{1}{\pi} \ln\left(n + \frac{3}{4} + \frac{11}{192(n+\frac{3}{4})}\right) + O(n^{-4}). \tag{2.9}$$

Also in [4], Chen presented the following better approximations than those in (2.7) to (2.9):

$$G_n = c_0 + \frac{1}{\pi} \ln\left(n + \frac{3}{4} + \frac{11}{192(n+\frac{3}{4})} - \frac{2009}{184320(n+\frac{3}{4})^3}\right) + O\big((n+\tfrac{3}{4})^{-6}\big) \tag{2.10}$$

and

$$G_n = c_0 + \frac{1}{\pi} \ln\left(n + \frac{3}{4} + \frac{11}{192(n+\frac{3}{4})} - \frac{2009}{184320(n+\frac{3}{4})^3} + \frac{2599153}{371589120(n+\frac{3}{4})^5}\right) + O\big((n+\tfrac{3}{4})^{-8}\big). \tag{2.11}$$

Nemes [21, Theorem 1.1] proved that for $0 < h < 3/2$, the Landau constants G_n have the following asymptotic expansions

$$G_n \sim \frac{1}{\pi}\ln(n+h) + c_0 - \sum_{k=1}^{\infty} \frac{g_k(h)}{(n+h)^k} \quad \text{as} \quad n \to \infty, \tag{2.12}$$

where the coefficients $g_k(h)$ are given by

$$g_k(h) = \frac{1}{\pi k} \sum_{j=0}^{k} \left(\binom{k}{j} B_{k-j}\left(h - \tfrac{1}{2}\right) \sum_{m=0}^{j} (-1)^{j+m} \binom{2m}{m}^2 \frac{m! S(j,m)}{16^m} \right). \tag{2.13}$$

Here, $S(j, m)$ denotes Stirling Numbers of the Second Kind defined by the generating functions:

$$\frac{(e^x - 1)^m}{m!} = \sum_{k=m}^{\infty} S(k, m) \frac{x^k}{k!},$$

and the $B_k(t)$ denotes Bernoulli Polynomials defined by the generating function:

$$\frac{z e^{tz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(t) \frac{z^n}{n!}, \qquad |z| < 2\pi. \tag{2.14}$$

Note that the Bernoulli numbers B_n $(n \in \mathbb{N}_0)$ are defined by $B_n := B_n(0)$ in (2.14).

In [21], the first few values of $g_k(h)$ are given by

$$\begin{aligned}
g_1(h) &= \frac{4h-3}{4\pi}, \\
g_2(h) &= \frac{96h^2 - 144h + 43}{192\pi}, \\
g_3(h) &= \frac{128h^3 - 288h^2 + 172h - 21}{384\pi}, \\
g_4(h) &= \frac{30720h^4 - 92160h^3 + 82560h^2 - 20160h - 619}{122880\pi}, \\
g_5(h) &= \frac{24576h^5 - 92160h^4 + 110080h^3 - 40320h^2 - 2476h + 1425}{122880\pi}.
\end{aligned}$$

In fact, (2.12) also holds for $h = 0$. Nemes [21] showed that for $0 < h < 3/2$,

$$g_k(h) = (-1)^k g_k(\tfrac{3}{2} - h) \quad \text{for} \quad k \in \mathbb{N},$$

which implies

$$g_{2k-1}(\tfrac{3}{4}) = 0 \quad \text{for} \quad k \in \mathbb{N}.$$

Moreover, the case $h = \frac{3}{4}$ is investigated in detail in Nemes' paper [21].

Let $0 \le h < \frac{3}{2}$. Based on the above result of Nemes, Chen and Cheng [5] extended the formulas (2.7) to (2.11) to the following full asymptotical expansion of the form:

$$G_n = c_0 + \frac{1}{\pi}\ln(n+h) + \frac{1}{\pi}\ln\left(1 + \sum_{j=1}^{m}\frac{a_j(h)}{(n+h)^j}\right) + O\left(\frac{1}{(n+h)^{m+1}}\right) \tag{2.15}$$

for $n \to \infty$ and $m \in \mathbb{N}$. Moreover, the authors gave a formula for successively determining the coefficients $a_j(h)$ in (2.15).

The gamma function may be defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t, \qquad x > 0.$$

The logarithmic derivative of the gamma function:

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \quad \text{or} \quad \ln\Gamma(x) = \int_1^x \psi(t)\mathrm{d}t$$

is known as the psi (or digamma) function.

There are bounds of other types involved. For example, Cvijović and Klinowski [10, Theorem 1] gave some estimates in terms of the psi function ψ, namely

$$c_0 + \frac{1}{\pi}\psi\left(n + \frac{5}{4}\right) < G_n < 1.0725 + \frac{1}{\pi}\psi\left(n + \frac{5}{4}\right), \quad n \in \mathbb{N}_0, \tag{2.16}$$

$$0.9883 + \frac{1}{\pi}\psi\left(n + \frac{3}{2}\right) < G_n < c_0 + \frac{1}{\pi}\psi\left(n + \frac{3}{2}\right), \quad n \in \mathbb{N}_0. \tag{2.17}$$

Alzer [1, Theorem 1] established sharp inequalities for G_n in terms of the psi function:

$$c_0 + \frac{1}{\pi}\psi\left(n + \alpha\right) < G_n \le c_0 + \frac{1}{\pi}\psi\left(n + \beta\right), \quad n \in \mathbb{N}_0 \tag{2.18}$$

with the best possible constants

$$\alpha = \frac{5}{4} \quad \text{and} \quad \beta = \psi^{-1}\big(\pi(1 - c_0)\big) = 1.26621\ldots.$$

In fact, we have

$$G_n = c_0 + \frac{1}{\pi}\psi\left(n + \frac{5}{4}\right) + O(n^{-2}) \tag{2.19}$$

(see, for example, [3, Theorem 4]). Recently, Chen [4] proved that for all integers $n \in \mathbb{N}_0$,

$$c_0 + \frac{1}{\pi}\psi\left(n + \frac{5}{4}\right) < G_n < c_0 + \frac{1}{\pi}\psi\left(n + \frac{5}{4} + \frac{1}{64(n+\frac{3}{4})}\right). \tag{2.20}$$

Moreover, the author pointed out that

$$G_n = c_0 + \frac{1}{\pi}\psi\left(n + \frac{5}{4} + \frac{1}{64(n+\frac{3}{4})}\right) + O(n^{-4}). \tag{2.21}$$

Alzer [1, Remark 1] pointed out that the lower bound in (2.20) is better than one in (2.6). Chen [4] showed that the upper bound in (2.20) is sharper than one in (2.6).

Very recently. Chen and Cheng [5] presented the following better approximations than that in (2.21):

$$G_n = c_0 + \frac{1}{\pi}\psi\left(n + \frac{5}{4} + \frac{1}{64n} - \frac{3}{256n^2} + \frac{61}{12288n^3}\right) + O\left(\frac{1}{n^5}\right) \tag{2.22}$$

and

$$G_n = c_0 + \frac{1}{\pi}\psi\left(n + \frac{5}{4} + \frac{\frac{1}{64}}{n+\frac{3}{4}} - \frac{\frac{47}{12288}}{(n+\frac{3}{4})^3} + \frac{\frac{17527}{5898240}}{(n+\frac{3}{4})^5}\right) + O\left(\frac{1}{(n+\frac{3}{4})^8}\right) \tag{2.23}$$

and posed the following natural question:

Open problem 2.1. *Find the constants $\beta_j(h)$ $(j \in \mathbb{N})$ such that*

$$G_n \sim c_0 + \frac{1}{\pi}\psi\left(n + \frac{5}{4} + \sum_{j=1}^{\infty}\frac{\beta_j(h)}{(n+h)^j}\right) \quad as \quad n \to \infty. \tag{2.24}$$

In particular, please consider two special cases $h = 0$, and $\frac{3}{4}$.

Theorem 2.1 gives a solution to the open problem 2.1. The proof of Theorem 2.1 makes use of the technology in [8].

Theorem 2.1. *The Landau constants G_n have the following asymptotic expansion:*

$$G_n \sim c_0 + \frac{1}{\pi}\psi\left(\sum_{j=0}^{\infty} q_j(h)(n+h)^{-j+1}\right) \quad as \quad n \to \infty, \tag{2.25}$$

where

$$q_0(h) = 1, \quad q_j(h) = \frac{1}{j}\sum_{k=1}^{j-1} k a_k q_{j-k}(h) + \sum_{k=1}^{j}\frac{(-1)^k B_k}{k}P_{j-k}(-k) - \pi g_j(h). \tag{2.26}$$

Here B_n are Bernoulli numbers, $g_k(h)$ are given in (2.13), $P_j(r)$ can be calculated by (1.3), and a_j can be calculated by (1.2).

Proof. Let us denote q_j instead of $q_j(h)$. We can let

$$\pi(G_{n-h} - c_0) \sim \psi\left(n\sum_{j=0}^{\infty} q_j n^{-j}\right), \qquad n \to \infty,$$

where q_j $(j \in \mathbb{N}_0)$ are real numbers to be determined. By using the following expansion (see [18, p. 33])

$$\psi(x) \sim \ln x - \sum_{k=1}^{\infty}\frac{(-1)^k B_k}{k}x^{-k}, \qquad x \to \infty, \tag{2.27}$$

we obtain

$$\psi\left(n\sum_{j=0}^{\infty} q_j n^{-j}\right) \sim \ln n + \ln\left(\sum_{j=0}^{\infty} q_j n^{-j}\right) - \sum_{k=1}^{\infty}\frac{(-1)^k B_k}{k}\left(n\sum_{j=0}^{\infty} q_j n^{-j}\right)^{-k}.$$

On the other hand, it follows from formula (2.12) that, as $n \to \infty$,

$$\pi(G_{n-h} - c_0) \sim \ln n - \sum_{j=1}^{\infty}\pi g_j(h) n^{-j}.$$

Hence, we have

$$\ln\left(\sum_{j=0}^{\infty} q_j n^{-j}\right) - \sum_{k=1}^{\infty}\frac{(-1)^k B_k}{k}\left(n\sum_{j=0}^{\infty} q_j n^{-j}\right)^{-k} \sim -\sum_{j=1}^{\infty}\pi g_j(h) n^{-j}. \tag{2.28}$$

Extracting the coefficients of the power n^0, it follows from here that $\ln q_0 = 0$, and hence $q_0 = 1$. Using (1.2) and (1.3), the left side of (2.28) can be written as

$$\sum_{k=1}^{\infty} a_k n^{-k} - \sum_{k=1}^{\infty} \frac{(-1)^k B_k}{k} n^{-k} \sum_{j=0}^{\infty} P_j(-k) n^{-j} = \sum_{j=1}^{\infty} \left(a_j - \sum_{k=1}^{j} \frac{B_k(1)}{k} P_{j-k}(-k) \right) n^{-j}. \tag{2.29}$$

The coefficient q_j which should be determined from here is hidden in the calculation of a_j. Using (1.2) we can write

$$a_j = q_j - \frac{1}{j} \sum_{k=1}^{j-1} k a_k q_{j-k}. \tag{2.30}$$

Linking together (2.28), (2.29) and (2.30) immediately follows (2.26), which proves Theorem 2.1. □

Remark 2.1. *Here we give explicit numerical values of some first terms of $q_j(h)$ by using the formula* (2.26). *This shows how easily we can determine the coefficients $q_j(h)$ in* (2.25). *Let $q_0 = 1$ in* (1.1). *By using* (1.2) *and* (1.3), *we obtain that*

$$a_1 = q_1, \quad a_2 = q_2 - \tfrac{1}{2} q_1^2$$

and

$$P_0(-1) = 1, \quad P_1(-1) = -q_1, \quad P_2(-1) = -q_2 + q_1^2,$$
$$P_0(-2) = 1, \quad P_1(-2) = -2q_1,$$

respectively. From (2.26), *we find that*

$$q_1(h) = -B_1 P_0(-1) - \pi g_1(h) = \tfrac{1}{2} - \tfrac{4h-3}{4} = \tfrac{5}{4} - h,$$
$$q_2(h) = \tfrac{1}{2} a_1 q_1(h) - B_1 P_1(-1) + \tfrac{1}{2} B_2 P_0(-2) - \pi g_2(h) = \tfrac{1}{64},$$
$$q_3(h) = \tfrac{1}{3}\big(a_1 q_2(h) + 2a_2 q_1(h)\big) - B_1 P_2(-1) + \tfrac{1}{2} B_2 P_1(-2) - \pi g_3(h) = -\tfrac{3}{256} + \tfrac{1}{64} h.$$

Likewise, we find from (2.26) *that*

$$q_4(h) = \tfrac{61}{12288} - \tfrac{3}{128} h + \tfrac{1}{64} h^2 \quad \textit{and} \quad q_5(h) = \tfrac{33}{16384} + \tfrac{61}{4096} h + \tfrac{1}{64} h^3 - \tfrac{9}{256} h^2.$$

Remark 2.2. *Observing* (2.24) *and* (2.25), *we let*

$$\beta_j(h) = q_{j+1}(h), \quad j \in \mathbb{N},$$

$q_j(h)$ can be calculated by (2.26). *This solves the open problem 2.1. In particular, we have*

$$\beta_1(0) = \frac{1}{64}, \quad \beta_2(0) = -\frac{3}{256}, \quad \beta_3(0) = \frac{61}{12288}, \quad \beta_4(0) = \frac{33}{16384}$$

and

$$\beta_1(\tfrac{3}{4}) = \frac{1}{64}, \quad \beta_2(\tfrac{3}{4}) = 0, \quad \beta_3(\tfrac{3}{4}) = -\frac{47}{12288}, \quad \beta_4(\tfrac{3}{4}) = 0.$$

We then obtain the following explicit asymptotical expansions:

$$G_n \sim c_0 + \frac{1}{\pi} \psi \left(n + \frac{5}{4} + \frac{1}{64n} - \frac{3}{256n^2} + \frac{61}{12288n^3} + \frac{33}{16384n^4} - \cdots \right) \tag{2.31}$$

and

$$G_n \sim c_0 + \frac{1}{\pi} \psi \left(n + \frac{5}{4} + \frac{1}{64(n + \frac{3}{4})} - \frac{47}{12288(n + \frac{3}{4})^3} + \cdots \right). \tag{2.32}$$

3. The Lebesgue Constants

The Lebesgue constants L_n have attracted the attention of several well-known mathematicians, for example, Fejér [12], Gronwall [15], Hardy [16], Szegö [24], and Watson [25], who established remarkable properties of these numbers. For instance, they presented monotonicity theorems as well as various series and integral representations for L_n. The following asymptotic formula is due to Watson [25]:

$$L_{n/2} = \frac{4}{\pi^2}\ln(n+1) + c_1 + O\left(\frac{1}{n^2}\right), \qquad n \to \infty, \tag{3.1}$$

with

$$c_1 := \frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{\ln k}{4k^2-1} + \frac{4}{\pi^2}(\gamma + 2\ln 2) = 0.989431273831146951741648 8\ldots, \tag{3.2}$$

where γ denotes the Euler-Mascheroni constant. Throughout this section, c_1 is referred to the constant in (3.2). Using (3.1) and (3.2), Galkin [13] obtained the following inequalities for $L_{n/2}$:

$$c_1 + \frac{4}{\pi^2}\ln(n+1) < L_{n/2} \le 1 + \frac{4}{\pi^2}\ln(n+1), \qquad n \in \mathbb{N}_0; \tag{3.3}$$

$$0.7190 + \frac{4}{\pi^2}\ln(n+2) < L_{n/2} \le c_1 + \frac{4}{\pi^2}\ln(n+2), \qquad n \in \mathbb{N}_0. \tag{3.4}$$

Zhao [27, Theorem 2] established the following two-sided inequalities:

$$\begin{aligned}&\frac{4}{\pi^2}\ln(n+1) + c_1 + \frac{d_0}{(n+1)^2} - \frac{d_1}{(n+1)^4} < L_{n/2} \\ &\quad < \frac{4}{\pi^2}\ln(n+1) + c_1 + \frac{d_0}{(n+1)^2} - \frac{d_1}{(n+1)^4} + \frac{d_2}{(n+1)^6}, \qquad n \in \mathbb{N}_0,\end{aligned} \tag{3.5}$$

where

$$\begin{aligned} d_0 &= \frac{12-\pi^2}{18\pi^2}, \\ d_1 &= \frac{7}{120\pi^2}\left(8 - \frac{2\pi^2}{3} - \frac{\pi^4}{90}\right) = -\frac{5040 - 420\pi^2 - 7\pi^4}{10800\pi^2}, \\ d_2 &= \frac{1}{16\pi^2}\left(32 - \frac{8\pi^2}{3} - \frac{2\pi^4}{45} - \frac{\pi^6}{945}\right) = \frac{30240 - 2520\pi^2 - 42\pi^4 - \pi^6}{15120\pi^2}. \end{aligned}$$

Recently, Chen and Choi [7] proved that for $n \in \mathbb{N}_0$ and $N \in \mathbb{N}_0$,

$$\frac{4}{\pi^2}\ln(n+1) + c_1 + \sum_{j=1}^{2N}\frac{\lambda_j}{(n+1)^{2j}} < L_{n/2} < \frac{4}{\pi^2}\ln(n+1) + c_1 + \sum_{j=1}^{2N+1}\frac{\lambda_j}{(n+1)^{2j}}, \tag{3.6}$$

with

$$\lambda_j = \frac{8}{\pi^2}\frac{B_{2j}}{2j}(2^{2j-1}-1)\left(1 + \sum_{k=1}^{j}\frac{(-1)^k}{(2k)!}B_{2k}\pi^{2k}\right), \tag{3.7}$$

where B_n ($n \in \mathbb{N}_0$) are the Bernoulli numbers. The inequality (3.6) implies the following asymptotic expansion:

$$L_{n/2} \sim \frac{4}{\pi^2}\ln(n+1) + c_1 + \sum_{j=1}^{\infty}\frac{\lambda_j}{(n+1)^{2j}}, \qquad n \to \infty. \tag{3.8}$$

The first few values of λ_j are given by

$$\begin{aligned}
\lambda_1 &= \frac{12-\pi^2}{18\pi^2},\\
\lambda_2 &= -\frac{5040-420\pi^2-7\pi^4}{10800\pi^2},\\
\lambda_3 &= \frac{937440-78120\pi^2-1302\pi^4-31\pi^6}{952560\pi^2},\\
\lambda_4 &= -\frac{153619200-12801600\pi^2-213360\pi^4-5080\pi^6-127\pi^8}{36288000\pi^2},\\
\lambda_5 &= \frac{17483558400-1456963200\pi^2-24282720\pi^4-578160\pi^6-14454\pi^8-365\pi^{10}}{564537600\pi^2},\\
\lambda_6 &= -\frac{\begin{array}{l}1849675317025536000-154139609752128000\pi^2-2568993495868800\pi^4\\ -61166511806400\pi^6-1529162795160\pi^8-38615222100\pi^{10}-977403607\pi^{12}\end{array}}{5354926536960000\pi^2},\\
\lambda_7 &= \frac{\begin{array}{l}21422321496576000-1785193458048000\pi^2-29753224300800\pi^4\\ -708410102400\pi^6-17710252560\pi^8-447228600\pi^{10}\\ -11319962\pi^{12}-286685\pi^{14}\end{array}}{3923023104000\pi^2}.
\end{aligned}$$

Remark 3.1. *Taking $N=1$ in* (3.6)*, we obtain the following inequalities:*

$$\begin{aligned}
&\frac{4}{\pi^2}\ln(n+1)+c_1+\frac{12-\pi^2}{18\pi^2(n+1)^2}-\frac{5040-420\pi^2-7\pi^4}{10800\pi^2(n+1)^4}<L_{n/2}\\
&\quad<\frac{4}{\pi^2}\ln(n+1)+c_1+\frac{12-\pi^2}{18\pi^2(n+1)^2}-\frac{5040-420\pi^2-7\pi^4}{10800\pi^2(n+1)^4}\\
&\quad+\frac{937440-78120\pi^2-1302\pi^4-31\pi^6}{952560\pi^2(n+1)^6},\qquad n\in\mathbb{N}_0,
\end{aligned}\tag{3.9}$$

improving the upper bound and confirming the lower in (3.5)*. The asymptotic formula* (3.8) *can be found in Wong* [26, pp. 40-42]*. In* [7]*, the authors gave a different proof of* (3.8) *from that in* [26, pp. 40-42].

Let the sequence $(b_j)_{j\in\mathbb{N}}$ be defined by

$$b_{2j}=\lambda_j \quad\text{and}\quad b_{2j-1}=0,\qquad j\in\mathbb{N}.\tag{3.10}$$

Then, the formula (3.8) can be rewritten as

$$L_{n/2}\sim\frac{4}{\pi^2}\ln(n+1)+c_1+\sum_{j=1}^{\infty}\frac{b_j}{(n+1)^j}+\cdots,\qquad n\to\infty.\tag{3.11}$$

By using the Bell polynomials, Chen and Choi [6, Theorem 3.2] extended Watson's formula (3.1) and obtained full asymptotical expansion:

$$L_{n/2}-\frac{4}{\pi^2}\ln(n+1)-c_1\sim\frac{4}{\pi^2}\ln\left(1+\sum_{j=1}^{\infty}\frac{\alpha_j}{(n+1)^j}\right),\qquad n\to\infty,\tag{3.12}$$

where the coefficients α_j $(j \in \mathbb{N})$ are given by using the following recursive formula:

$$\alpha_0 = 1, \quad \alpha_j = \sum_{\ell=0}^{j-1} \left(\frac{j-\ell}{j} \frac{\pi^2}{4} b_{j-\ell} \right) \alpha_\ell, \qquad n \in \mathbb{N},$$

and b_j $(j \in \mathbb{N})$ are given in (3.10).

For our later uses, we introduce the following set of partitions of an integer $n \in \mathbb{N}$:

$$\mathcal{A}_n := \{(k_1, k_2, \ldots, k_n) \in \mathbb{N}_0^n \, : \, k_1 + 2k_2 + \cdots + nk_n = n\}. \tag{3.13}$$

In number theory, the partition function $p(n)$ represents the number of possible partitions of $n \in \mathbb{N}$ (e.g., the number of distinct ways of representing n as a sum of natural numbers irregardless of order). By convention, $p(0) = 1$ and $p(n) = 0$ for n a negative integer. The first several values of the partition function $p(n)$ are (starting with $p(0) = 1$):

$$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, \ldots.$$

It is easy to see that the cardinality of the set $\mathcal{A}_n$ is equal to the partition function $p(n)$.

By mainly using the partition function, Chen and Choi [6, Theorem 3.3] provided an alternative representation formula to calculate the coefficients α_j $(j \in \mathbb{N})$ in (3.12) as follows:

$$\alpha_j = \sum_{(k_1, k_2, \ldots, k_j) \in \mathcal{A}_j} \frac{(\frac{\pi^2}{4})^{k_1+k_2+\cdots+k_j}}{k_1! k_2! \cdots k_j!} b_1^{k_1} b_2^{k_2} \cdots b_j^{k_j}, \tag{3.14}$$

where $\mathcal{A}_j$ $(j \in \mathbb{N})$ are given in (3.13) and b_j $(j \in \mathbb{N})$ are given in (3.10).

Here, from (3.12), we obtain the explicit asymptotic formula

$$\begin{aligned} L_{n/2} \sim c_1 + \frac{4}{\pi^2} \ln \Bigg(n + 1 + \frac{\frac{1}{6} - \frac{\pi^2}{72}}{n+1} - \frac{\frac{37}{360} - \frac{\pi^2}{135} - \frac{67\pi^4}{259200}}{(n+1)^3} \\ + \frac{\frac{10313}{45360} - \frac{3167\pi^2}{181440} - \frac{4721\pi^4}{10886400} - \frac{29719\pi^6}{2743372800}}{(n+1)^5} - \cdots \Bigg). \end{aligned} \tag{3.15}$$

There are bounds of other types involved. For example, Alzer [1, Theorem 4] gave an estimate in terms of the psi function ψ, namely,

$$c_1 + \frac{4}{\pi^2} \psi(n + \theta_1) \le L_{n/2} < c_1 + \frac{4}{\pi^2} \psi(n + \theta_2), \qquad n \in \mathbb{N}_0 \tag{3.16}$$

with the best possible constants

$$\theta_1 = \psi^{-1}\big(\pi^2(1 - c_1)/4\big) = 1.48891\ldots \quad \text{and} \quad \theta_2 = \frac{3}{2}.$$

Zhao [27] pointed out that

$$L_{n/2} = c_1 + \frac{4}{\pi^2} \psi\left(n + \frac{3}{2}\right) + O\left(\frac{1}{n^2}\right), \qquad n \to \infty. \tag{3.17}$$

Recently, Chen and Choi [6] presented a faster approximation formula than that in (3.17):

$$L_{n/2} = c_1 + \frac{4}{\pi^2} \psi\left(n + \frac{3}{2} + \frac{\frac{1}{8} - \frac{\pi^2}{72}}{n+1}\right) + O\left(\frac{1}{(n+1)^4}\right) \quad \text{as} \quad n \to \infty \tag{3.18}$$

and posed the following natural question:

Open problem 3.1. *Find the constants* r_j $(j \in \mathbb{N})$ *such that*

$$L_{n/2} \sim c_1 + \frac{4}{\pi^2}\psi\left(n + \frac{3}{2} + \sum_{j=1}^{\infty} \frac{r_j}{(n+1)^j}\right), \qquad n \to \infty. \tag{3.19}$$

Theorem 3.1 gives a solution to the open problem 3.1.

Theorem 3.1. *The Lebesgue constants* $L_{n/2}$ *have the following asymptotic expansion:*

$$L_{n/2} \sim c_0 + \frac{4}{\pi^2}\psi\left(\sum_{j=0}^{\infty} w_j (n+1)^{-j+1}\right) \quad as \quad n \to \infty, \tag{3.20}$$

where $w_0 = 1$, *and for* $j \in \mathbb{N}$,

$$\begin{aligned} w_{2j-1} &= \frac{1}{2j-1}\sum_{k=1}^{2j-2} k a_k w_{2j-k-1} + \sum_{k=1}^{2j-1} \frac{(-1)^k B_k}{k} P_{2j-k-1}(-k), \\ w_{2j} &= \frac{1}{2j}\sum_{k=1}^{2j-1} k a_k w_{2j-k} + \sum_{k=1}^{2j} \frac{(-1)^k B_k}{k} P_{2j-k}(-k) + \frac{\pi^2}{4}\lambda_j. \end{aligned} \tag{3.21}$$

Here B_n *are Bernoulli numbers,* λ_j *are given in* (3.7), a_j *can be calculated by*

$$a_j = w_j - \frac{1}{j}\sum_{k=1}^{j-1} k a_k w_{j-k}, \qquad j \in \mathbb{N},$$

$P_j(r)$ *can be calculated by*

$$P_0(r) = 1, \quad P_j(r) = \frac{1}{j}\sum_{k=1}^{j} [k(1+r) - j] w_k P_{j-k}(r), \qquad j \in \mathbb{N}.$$

Proof. A similar argument as in the proof of Theorem 2.1 will establish the result in Theorem 3.1. For completeness, we repeat to prove Theorem 3.1. We can let

$$\frac{\pi^2}{4}(L_{(n-1)/2} - c_1) \sim \psi\left(n\sum_{j=0}^{\infty} w_j n^{-j}\right), \qquad n \to \infty,$$

where w_j $(j \in \mathbb{N}_0)$ are real numbers to be determined. By using the expansion (2.27), we obtain

$$\psi\left(n\sum_{j=0}^{\infty} w_j n^{-j}\right) \sim \ln n + \ln\left(\sum_{j=0}^{\infty} w_j n^{-j}\right) - \sum_{k=1}^{\infty} \frac{(-1)^k B_k}{k}\left(n\sum_{j=0}^{\infty} w_j n^{-j}\right)^{-k}.$$

On the other hand, it follows from formula (3.8) that

$$\frac{\pi^2}{4}(L_{(n-1)/2} - c_1) \sim \ln n + \sum_{j=1}^{\infty} \frac{\frac{\pi^2}{4}\lambda_j}{n^{2j}}, \qquad n \to \infty,$$

with the coefficients λ_j $(j \in \mathbb{N})$ given in (3.7). Hence, we have

$$\ln\left(\sum_{j=0}^{\infty} w_j n^{-j}\right) - \sum_{k=1}^{\infty} \frac{(-1)^k B_k}{k}\left(n\sum_{j=0}^{\infty} w_j n^{-j}\right)^{-k} \sim \sum_{j=1}^{\infty} \frac{\frac{\pi^2}{4}\lambda_j}{n^{2j}}. \tag{3.22}$$

Extracting the coefficients of the power n^0, it follows from here that $\ln w_0 = 0$, and hence $w_0 = 1$. Using (1.2) and (1.3), the left side of (3.22) can be written as

$$\sum_{k=1}^{\infty} a_k n^{-k} - \sum_{k=1}^{\infty} \frac{(-1)^k B_k}{k} n^{-k} \sum_{j=0}^{\infty} P_j(-k) n^{-j} = \sum_{j=1}^{\infty} \left(a_j - \sum_{k=1}^{j} \frac{B_k(1)}{k} P_{j-k}(-k) \right) n^{-j}. \tag{3.23}$$

The coefficient w_j which should be determined from here is hidden in the calculation of a_j. By (1.2), here we can write

$$a_j = w_j - \frac{1}{j} \sum_{k=1}^{j-1} k a_k w_{j-k}. \tag{3.24}$$

Linking together (3.22), (3.23) and (3.24) immediately follows (3.21), which proves Theorem 3.1. □

Remark 3.2. *Observing* (3.19) *and* (3.20), *we let*

$$r_j = w_{j+1}, \quad j \in \mathbb{N},$$

w_j *can be calculated by* (3.21). *This solves the open problem 3.1. Here, we give the following explicit asymptotic expansion:*

$$L_{n/2} \sim c_1 + \frac{4}{\pi^2} \psi \left(n + \frac{3}{2} + \frac{\frac{1}{8} - \frac{\pi^2}{72}}{n+1} - \frac{\frac{35}{384} - \frac{59\pi^2}{8640} - \frac{67\pi^4}{259200}}{(n+1)^3} + \frac{\frac{9997}{46080} - \frac{48649\pi^2}{2903040} - \frac{139\pi^4}{322560} - \frac{29719\pi^6}{2743372800}}{(n+1)^5} - \cdots \right), \quad n \to \infty. \tag{3.25}$$

Remark 3.3. *From* (3.15) *and* (3.25), *we obtain the following approximation formulas*

$$L_{n/2} \approx c_1 + \frac{4}{\pi^2} \ln \left(n + 1 + \frac{\frac{1}{6} - \frac{\pi^2}{72}}{n+1} - \frac{\frac{37}{360} - \frac{\pi^2}{135} - \frac{67\pi^4}{259200}}{(n+1)^3} + \frac{\frac{10313}{45360} - \frac{3167\pi^2}{181440} - \frac{4721\pi^4}{10886400} - \frac{29719\pi^6}{2743372800}}{(n+1)^5} \right) = \nu_n \tag{3.26}$$

and

$$L_{n/2} \approx c_1 + \frac{4}{\pi^2} \psi \left(n + \frac{3}{2} + \frac{\frac{1}{8} - \frac{\pi^2}{72}}{n+1} - \frac{\frac{35}{384} - \frac{59\pi^2}{8640} - \frac{67\pi^4}{259200}}{(n+1)^3} + \frac{\frac{9997}{46080} - \frac{48649\pi^2}{2903040} - \frac{139\pi^4}{322560} - \frac{29719\pi^6}{2743372800}}{(n+1)^5} \right) = \mu_n, \tag{3.27}$$

respectively. Moreover, we find that

$$L_{n/2} = \nu_n + O(n^{-8}) \quad \text{and} \quad L_{n/2} = \mu_n + O(n^{-8}). \tag{3.28}$$

The following numerical computations (see Table 1) would show that, for $n \in \mathbb{N}$, *the formula* (3.27) *is a little sharper than the formula* (3.26).

Table 1. *Comparison between approximation formulas* (3.26) *and* (3.27).

n	$\lvert\nu_n - L_{n/2}\rvert$	$\lvert\mu_n - L_{n/2}\rvert$
1	2.99×10^{-6}	1.09×10^{-6}
10	4.97×10^{-12}	1.88×10^{-12}
100	9.99×10^{-20}	3.78×10^{-20}
200	4.07×10^{-22}	1.53×10^{-22}

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