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FURTHER INEQUALITIES FOR POWER SERIES WITH NONNEGATIVE COEFFICIENTS VIA A REVERSE OF JENSEN INEQUALITY

S. S. DRAGOMIR[1,2]

Abstract. Some inequalities for power series with nonnegative coefficients via a new reverse of Jensen inequality are given. Applications for some fundamental functions defined by power series are also provided.

1. Introduction

In 1994, Dragomir & Ionescu obtained the following *reverse of Jensen's discrete inequality*:

Let $\Phi : I \to \mathbb{R}$ be a differentiable convex function on the interior $\mathring{I}$ of the interval I. If $x_i \in \mathring{I}$ and $w_i \geq 0$ $(i = 1, \dots, n)$ with $W_n := \sum_{i=1}^{n} w_i = 1$, then one has the inequality:

$$0 \leq \sum_{i=1}^{n} w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} w_i x_i\right) \leq \sum_{i=1}^{n} w_i \Phi'(x_i) x_i - \sum_{i=1}^{n} w_i \Phi'(x_i) \sum_{i=1}^{n} w_i x_i. \tag{1.1}$$

In order to improve Grüss' discrete inequality, Cerone & Dragomir established in 2002 the following result [1]:

$$\left|\sum_{i=1}^{n} w_i a_i b_i - \sum_{i=1}^{n} w_i a_i \sum_{i=1}^{n} w_i b_i\right| \leq \frac{1}{2}(A - a) \sum_{i=1}^{n} w_i \left|b_i - \sum_{j=1}^{n} w_j b_j\right| \leq \frac{1}{2}(A - a)\left[\sum_{i=1}^{n} w_i b_i^2 - \left(\sum_{i=1}^{n} w_i b_i\right)^2\right]^{1/2}, \tag{1.2}$$

provided $\infty < a \leq a_i \leq A < \infty$, and $w_i \geq 0$ $(i = 1, \dots, n)$ with $W_n := \sum_{i=1}^{n} w_i = 1$.

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In addition, if $\infty < b \leq b_i \leq B < \infty$, $(i = 1, \dots, n)$ then we have the string of inequalities

$$\begin{aligned}
(1.3)\qquad & \left| \sum_{i=1}^{n} w_i a_i b_i - \sum_{i=1}^{n} w_i a_i \sum_{i=1}^{n} w_i b_i \right| \\
& \leq \frac{1}{2} (A - a) \sum_{i=1}^{n} w_i \left| b_i - \sum_{j=1}^{n} w_j b_j \right| \\
& \leq \frac{1}{2} (A - a) \left[\sum_{i=1}^{n} w_i b_i^2 - \left(\sum_{i=1}^{n} w_i b_i \right)^2 \right]^{1/2} \\
& \leq \frac{1}{4} (A - a) (B - b) .
\end{aligned}$$

Utilising these results, we observe that if Φ is differentiable convex on a finite interval, say $[m, M]$, then we have the inequalities:

$$\begin{aligned}
(1.4)\qquad 0 & \leq \sum_{i=1}^{n} w_i \Phi (x_i) - \Phi \left(\sum_{i=1}^{n} w_i x_i \right) \\
& \leq \sum_{i=1}^{n} w_i \Phi' (x_i) x_i - \sum_{i=1}^{n} w_i \Phi' (x_i) \sum_{i=1}^{n} w_i x_i \\
& \leq \frac{1}{2} (M - m) \sum_{i=1}^{n} w_i \left| \Phi' (x_i) - \sum_{j=1}^{n} w_j \Phi' (x_j) \right| \\
& \leq \frac{1}{2} (M - m) \left[\sum_{i=1}^{n} w_i [\Phi' (x_i)]^2 - \left(\sum_{i=1}^{n} w_i \Phi' (x_i) \right)^2 \right]^{1/2}
\end{aligned}$$

for $x_i \in (m, M)$ $(i = 1, \dots, n)$.

If the lateral derivatives $\Phi'_+ (m)$ and $\Phi'_- (M)$ are finite, then we also have

$$\begin{aligned}
(1.5)\qquad 0 & \leq \sum_{i=1}^{n} w_i \Phi (x_i) - \Phi \left(\sum_{i=1}^{n} w_i x_i \right) \\
& \leq \sum_{i=1}^{n} w_i \Phi' (x_i) x_i - \sum_{i=1}^{n} w_i \Phi' (x_i) \sum_{i=1}^{n} w_i x_i \\
& \leq \frac{1}{2} \left[\Phi'_- (M) - \Phi'_+ (m) \right] \sum_{i=1}^{n} w_i \left| x_i - \sum_{j=1}^{n} w_j x_j \right| \\
& \leq \frac{1}{2} \left[\Phi'_- (M) - \Phi'_+ (m) \right] \left[\sum_{i=1}^{n} w_i x_i^2 - \left(\sum_{i=1}^{n} w_i x_i \right)^2 \right]^{1/2} \\
& \leq \frac{1}{4} (M - m) \left[\Phi'_- (M) - \Phi'_+ (m) \right]
\end{aligned}$$

for $x_i \in [m, M]$ $(i = 1, \dots, n)$.

The most important power series with nonnegative coefficients are:

$$\exp(z)=\sum_{n=0}^{\infty}\frac{1}{n!}z^n,\ z\in\mathbb{C},\ \frac{1}{1-z}=\sum_{n=0}^{\infty}z^n,\ z\in D(0,1),\tag{1.6}$$
$$\ln\frac{1}{1-z}=\sum_{n=1}^{\infty}\frac{1}{n}z^n,\ z\in D(0,1),\ \cosh z=\sum_{n=0}^{\infty}\frac{1}{(2n)!}z^{2n},\ z\in\mathbb{C},$$
$$\sinh z=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1},\ z\in\mathbb{C}.$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\ z\in D(0,1),\tag{1.7}$$
$$\sin^{-1}(z)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}z^{2n+1},\ z\in D(0,1),$$
$$\tanh^{-1}(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\ z\in D(0,1),$$
$${}_2F_1(\alpha,\beta,\gamma,z):=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}z^n,\alpha,\beta,\gamma>0$$
$$z\in D(0,1),$$

where Γ is *Gamma function.*

On utilizing the above reverses of Jensen inequality we obtained in [5]:

Theorem 1. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$. If $p\geq 1$, $0<\alpha<R$ and $x>0$ with $\alpha x^p,\alpha x^{p-1}<R$, then*

$$0\leq\frac{f(\alpha x^p)}{f(\alpha)}-\left[\frac{f(\alpha x)}{f(\alpha)}\right]^p\leq p\left[\frac{f(\alpha x^p)}{f(\alpha)}-\frac{f\left(\alpha x^{p-1}\right)}{f(\alpha)}\frac{f(\alpha x)}{f(\alpha)}\right].\tag{1.8}$$

Moreover, if $0<x\leq 1$, then

$$0\leq\frac{f(\alpha x^p)}{f(\alpha)}-\left[\frac{f(\alpha x)}{f(\alpha)}\right]^p\leq p\left[\frac{f(\alpha x^p)}{f(\alpha)}-\frac{f\left(\alpha x^{p-1}\right)}{f(\alpha)}\frac{f(\alpha x)}{f(\alpha)}\right]\tag{1.9}$$
$$\leq\frac{1}{2}p\left(\frac{f\left(\alpha x^{2(p-1)}\right)}{f(\alpha)}-\left[\frac{f\left(\alpha x^{p-1}\right)}{f(\alpha)}\right]^2\right)^{1/2}\leq\frac{1}{4}p$$

and

$$0\leq\frac{f(\alpha x^p)}{f(\alpha)}-\left[\frac{f(\alpha x)}{f(\alpha)}\right]^p\leq p\left[\frac{f(\alpha x^p)}{f(\alpha)}-\frac{f\left(\alpha x^{p-1}\right)}{f(\alpha)}\frac{f(\alpha x)}{f(\alpha)}\right]\tag{1.10}$$
$$\leq\frac{1}{2}p\left(\frac{f\left(\alpha x^2\right)}{f(\alpha)}-\left[\frac{f(\alpha x)}{f(\alpha)}\right]^2\right)^{1/2}\leq\frac{1}{4}p.$$

Corollary 1. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$. If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $u,v>0$ with $v^p\le u^q<R$, then*

$$\left[\frac{f(uv)}{f(u^q)}\right]^p\le\frac{f(v^p)}{f(u^q)}\le\frac{1}{4}p+\left[\frac{f(uv)}{f(u^q)}\right]^p \tag{1.11}$$

and

$$0\le[f(v^p)]^{1/p}[f(u^q)]^{1/q}-f(uv)\le\frac{1}{4^{1/p}}p^{1/p}f(u^q). \tag{1.12}$$

For some similar exponential and logarithmic inequalities see [5].

For other recent results for power series with nonnegative coefficients, see [2], [7], [11] and [12]. For more results on power series inequalities, see [2] and [7]-[10].

Motivated by the above results and utilizing a new reverse of Jensen inequality we provide in this paper other inequalities for power series with nonnegative coefficients. Applications for some fundamental functions are given as well.

2. Reverses of Jensen's Inequality

The following reverse of the Jensen's inequality holds:

Theorem 2. *Let $f:I\to\mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m,M\in\mathbb{R}$, $m<M$ with $[m,M]\subset\mathring{I}$, $\mathring{I}$ is the interior of I. If $x_i\in[m,M]$ and $w_i\ge0$ $(i=1,\dots,n)$ with $W_n:=\sum_{i=1}^n w_i=1$ and $\sum_{i=1}^n w_ix_i\in(m,M)$, then*

$$\begin{aligned}0&\le\sum_{i=1}^n w_if(x_i)-f\left(\sum_{i=1}^n w_ix_i\right)\\&\le\frac{\left(M-\sum_{i=1}^n w_ix_i\right)\left(\sum_{i=1}^n w_ix_i-m\right)}{M-m}\Psi_f\left(\sum_{i=1}^n w_ix_i;m,M\right)\\&\le\frac{\left(M-\sum_{i=1}^n w_ix_i\right)\left(\sum_{i=1}^n w_ix_i-m\right)}{M-m}\sup_{t\in(m,M)}\Psi_f(t;m,M)\\&\le\left(M-\sum_{i=1}^n w_ix_i\right)\left(\sum_{i=1}^n w_ix_i-m\right)\frac{f'_-(M)-f'_+(m)}{M-m}\\&\le\frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right],\end{aligned} \tag{2.1}$$

where $\Psi_f(\cdot;m,M):(m,M)\to\mathbb{R}$ is defined by

$$\Psi_f(t;m,M)=\frac{f(M)-f(t)}{M-t}-\frac{f(t)-f(m)}{t-m}.$$

We also have the inequality

$$
\begin{aligned}
(2.2)\qquad 0 &\leq \sum_{i=1}^{n} w_i f(x_i) - f\left(\sum_{i=1}^{n} w_i x_i\right) \\
&\leq \frac{\left(M - \sum_{i=1}^{n} w_i x_i\right)\left(\sum_{i=1}^{n} w_i x_i - m\right)}{M-m} \Psi_f\left(\sum_{i=1}^{n} w_i x_i; m, M\right) \\
&\leq \frac{1}{4}(M-m)\Psi_f\left(\sum_{i=1}^{n} w_i x_i; m, M\right) \\
&\leq \frac{1}{4}(M-m) \sup_{t\in(m,M)} \Psi_f(t;m,M) \leq \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right],
\end{aligned}
$$

provided that $\sum_{i=1}^{n} w_i x_i \in (m, M)$.

Proof. By the convexity of f we have that

$$
\begin{aligned}
(2.3)\qquad & \sum_{i=1}^{n} w_i f(x_i) - f\left(\sum_{i=1}^{n} w_i x_i\right) \\
&= \sum_{i=1}^{n} w_i f\left[\frac{m(M-x_i) + M(x_i - m)}{M-m}\right] \\
&\quad - f\left(\sum_{i=1}^{n} w_i \left[\frac{m(M-x_i) + M(x_i-m)}{M-m}\right]\right) \\
&\leq \sum_{i=1}^{n} w_i \frac{(M-x_i) f(m) + (x_i - m) f(M)}{M-m} \\
&\quad - f\left(\frac{m\left(M - \sum_{i=1}^{n} w_i x_i\right) + M\left(\sum_{i=1}^{n} w_i x_i - m\right)}{M-m}\right) \\
&= \frac{\left(M - \sum_{i=1}^{n} w_i x_i\right) f(m) + \left(\sum_{i=1}^{n} w_i x_i - m\right) f(M)}{M-m} \\
&\quad - f\left(\frac{m\left(M - \sum_{i=1}^{n} w_i x_i\right) + M\left(\sum_{i=1}^{n} w_i x_i - m\right)}{M-m}\right) := B.
\end{aligned}
$$

By denoting

$$\Delta_f(t;m,M) := \frac{(t-m) f(M) + (M-t) f(m)}{M-m} - f(t), \quad t \in [m, M]$$

we have

$$
\begin{aligned}
(2.4)\qquad \Delta_f(t;m,M) &= \frac{(t-m) f(M) + (M-t) f(m) - (M-m) f(t)}{M-m} \\
&= \frac{(t-m) f(M) + (M-t) f(m) - (M-t+t-m) f(t)}{M-m} \\
&= \frac{(t-m)\left[f(M) - f(t)\right] - (M-t)\left[f(t) - f(m)\right]}{M-m} \\
&= \frac{(M-t)(t-m)}{M-m}\Psi_f(t;m,M)
\end{aligned}
$$

for any $t \in (m, M)$.

Therefore we have the equality

$$B=\frac{\left(M-\sum_{i=1}^{n}w_ix_i\right)\left(\sum_{i=1}^{n}w_ix_i-m\right)}{M-m}\Psi_f\left(\sum_{i=1}^{n}w_ix_i;m,M\right) \tag{2.5}$$

provided that $\sum_{i=1}^{n}w_ix_i\in(m,M)$.

If $\sum_{i=1}^{n}w_ix_i\in(m,M)$, then

$$\begin{aligned}\Psi_f\left(\sum_{i=1}^{n}w_ix_i;m,M\right) &\le \sup_{t\in(m,M)}\Psi_f(t;m,M)\\ &=\sup_{t\in(m,M)}\left[\frac{f(M)-f(t)}{M-t}-\frac{f(t)-f(m)}{t-m}\right]\\ &\le \sup_{t\in(m,M)}\left[\frac{f(M)-f(t)}{M-t}\right]+\sup_{t\in(m,M)}\left[-\frac{f(t)-f(m)}{t-m}\right]\\ &=\sup_{t\in(m,M)}\left[\frac{f(M)-f(t)}{M-t}\right]-\inf_{t\in(m,M)}\left[\frac{f(t)-f(m)}{t-m}\right]\\ &=f'_-(M)-f'_+(m),\end{aligned}$$

which by (2.3) and (2.5) produces the desired result (2.1).

Since, obviously

$$\frac{\left(M-\sum_{i=1}^{n}w_ix_i\right)\left(\sum_{i=1}^{n}w_ix_i-m\right)}{M-m}\le\frac{1}{4}(M-m),$$

then by (2.3) and (2.5) we deduce the second inequality (2.2).

The last part is clear. □

For similar integral versions see [4].

Remark 1. *a) For $p>1$ and $0<m<M<\infty$ consider the function $\Psi_p(\cdot;m,M):(m,M)\to\mathbb{R}$ defined by*

$$\begin{aligned}\Psi_p(t;m,M) &= \frac{M^p-t^p}{M-t}-\frac{t^p-m^p}{t-m}\\ &=\frac{t(M^p-m^p)-t^p(M-m)-mM\left(M^{p-1}-m^{p-1}\right)}{(M-t)(t-m)}.\end{aligned}$$

If $x_i \in [m, M]$ *and* $w_i \geq 0 \quad (i = 1, \ldots, n)$ *with* $W_n := \sum_{i=1}^n w_i = 1$ *and* $\sum_{i=1}^n w_i x_i \in (m, M)$, *then*

$$
\begin{aligned}
(2.6) \qquad 0 &\leq \sum_{i=1}^n w_i x_i^p - \left(\sum_{i=1}^n w_i x_i\right)^p \\
&\leq \frac{\left(M - \sum_{i=1}^n w_i x_i\right)\left(\sum_{i=1}^n w_i x_i - m\right)}{M - m} \Psi_p\left(\sum_{i=1}^n w_i x_i; m, M\right) \\
&\leq \frac{\left(M - \sum_{i=1}^n w_i x_i\right)\left(\sum_{i=1}^n w_i x_i - m\right)}{M - m} \sup_{t \in (m,M)} \Psi_p(t; m, M) \\
&\leq p\left(M - \sum_{i=1}^n w_i x_i\right)\left(\sum_{i=1}^n w_i x_i - m\right) \frac{M^{p-1} - m^{p-1}}{M - m} \\
&\leq \frac{1}{4} p (M - m)\left(M^{p-1} - m^{p-1}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
(2.7) \qquad 0 &\leq \sum_{i=1}^n w_i x_i^p - \left(\sum_{i=1}^n w_i x_i\right)^p \\
&\leq \frac{\left(M - \sum_{i=1}^n w_i x_i\right)\left(\sum_{i=1}^n w_i x_i - m\right)}{M - m} \Psi_p\left(\sum_{i=1}^n w_i x_i; m, M\right) \\
&\leq \frac{1}{4}(M - m) \Psi_p\left(\sum_{i=1}^n w_i x_i; m, M\right) \\
&\leq \frac{1}{4}(M - m) \sup_{t \in (m,M)} \Psi_p(t; m, M) \leq \frac{1}{4} p (M - m)\left(M^{p-1} - m^{p-1}\right).
\end{aligned}
$$

For $0 < m < M < \infty$ *consider the function* $\Psi_{-\ln}(\cdot; m, M) : (m, M) \to \mathbb{R}$ *defined by*

$$
\begin{aligned}
\Psi_{-\ln}(t; m, M) &= \frac{-\ln M + \ln t}{M - t} - \frac{-\ln t + \ln m}{t - m} \\
&= \frac{(M - m)\ln t - (M - t)\ln m - (t - m)\ln M}{(M - t)(t - m)} \\
&= \ln\left(\frac{t^{M-m}}{m^{M-t} M^{t-m}}\right)^{\frac{1}{(M-t)(t-m)}}
\end{aligned}
$$

b) If $x_i \in [m, M]$ *and* $w_i \geq 0 \quad (i = 1, \dots, n)$ *with* $W_n := \sum_{i=1}^n w_i = 1$ *and* $\sum_{i=1}^n w_i x_i \in (m, M)$, *then*

$$0 \leq \ln\left(\sum_{i=1}^n w_i x_i\right) - \sum_{i=1}^n w_i \ln x_i \tag{2.8}$$
$$\leq \ln\left(\frac{\sum_{i=1}^n w_i x_i}{m^{\frac{M-\sum_{i=1}^n w_i x_i}{M-m}} M^{\frac{\sum_{i=1}^n w_i x_i - m}{M-m}}}\right)$$
$$\leq \frac{\left(M - \sum_{i=1}^n w_i x_i\right)\left(\sum_{i=1}^n w_i x_i - m\right)}{M-m} \sup_{t\in(m,M)} \Psi_{-\ln}(t; m, M)$$
$$\leq \frac{1}{Mm}\left(M - \sum_{i=1}^n w_i x_i\right)\left(\sum_{i=1}^n w_i x_i - m\right) \leq \frac{1}{4}\frac{(M-m)^2}{Mm},$$

and

$$0 \leq \ln\left(\sum_{i=1}^n w_i x_i\right) - \sum_{i=1}^n w_i \ln x_i \tag{2.9}$$
$$\leq \ln\left(\frac{\sum_{i=1}^n w_i x_i}{m^{\frac{M-\sum_{i=1}^n w_i x_i}{M-m}} M^{\frac{\sum_{i=1}^n w_i x_i - m}{M-m}}}\right)$$
$$\leq \frac{1}{4}(M-m)$$
$$\times \ln\left(\frac{\left(\sum_{i=1}^n w_i x_i\right)^{M-m}}{m^{M-\sum_{i=1}^n w_i x_i} M^{\sum_{i=1}^n w_i x_i - m}}\right)^{\frac{1}{\left(M-\sum_{i=1}^n w_i x_i\right)\left(\sum_{i=1}^n w_i x_i - m\right)}}$$
$$\leq \frac{1}{4}(M-m) \sup_{t\in(m,M)} \Psi_{-\ln}(t; m, M) \leq \frac{1}{4}\frac{(M-m)^2}{Mm}.$$

3. Power Inequalities

For $p > 1$, $f(t) := t^p$, $m = 0$ and $M = 1$ we have

$$\Psi_f(t; m, M) = \frac{t^p - 1}{t - 1} - t^{p-1} = \frac{1 - t^{p-1}}{1 - t} =: B_p(t).$$

If $p \in (1, 2)$, the function $\Gamma(t) = t^{p-1}$ is concave on $(0, 1)$ and then $B_p(\cdot)$ is decreasing on $(0, 1)$. Therefore

$$\sup_{t\in(0,1)} B_p(t) = \lim_{t\to 0+} B_p(t) = 1.$$

If $p = 2$, then $B_p(t) = 1$ for $t \in (0, 1)$. If $p \in (2, \infty)$, the function $\Gamma(t) = t^{p-1}$ is convex on $(0, 1)$ and then $B_p(\cdot)$ is increasing on $(0, 1)$. Therefore

$$\sup_{t\in(0,1)} B_p(t) = \lim_{t\to 1-} B_p(t) = p - 1.$$

In conclusion

$$M_p := \sup_{t\in(0,1)} B_p(t) = \begin{cases} 1 \text{ if } p \in (1, 2], \\ \\ p - 1 \text{ if } p \in (2, \infty). \end{cases}$$

If $z_i \in [0,1]$ and $w_i \geq 0$ $(i = 1, \dots, n)$ with $W_n := \sum_{i=1}^n w_i = 1$ and $\sum_{i=1}^n w_i z_i \in (0,1)$, then from (2.1) and (2.2) we have the inequalities:

$$0 \leq \sum_{i=1}^n w_i z_i^p - \left(\sum_{i=1}^n w_i z_i\right)^p \leq M_p \left(1 - \sum_{i=1}^n w_i z_i\right) \sum_{i=1}^n w_i z_i \leq \frac{1}{4} M_p \tag{3.1}$$

and

$$0 \leq \sum_{i=1}^n w_i z_i^p - \left(\sum_{i=1}^n w_i z_i\right)^p \leq \frac{1}{4} \cdot \frac{1 - \left(\sum_{i=1}^n w_i z_i\right)^{p-1}}{1 - \sum_{i=1}^n w_i z_i} \leq \frac{1}{4} M_p. \tag{3.2}$$

Proposition 1. *If $x_i \geq 0$, $y_i > 0$ for $i \in \{1, \dots, n\}$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and such that*

$$0 \leq \frac{x_i}{y_i^{q-1}} \leq 1 \text{ for } i \in \{1, \dots, n\}, \tag{3.3}$$

then we have

$$\begin{aligned} 0 &\leq \frac{\sum_{i=1}^n x_i^p}{\sum_{i=1}^n y_i^q} - \left(\frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n y_i^q}\right)^p \\ &\leq M_p \left(1 - \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n y_i^q}\right) \left(\frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n y_i^q}\right) \leq \frac{1}{4} M_p \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} 0 &\leq \frac{\sum_{i=1}^n x_i^p}{\sum_{i=1}^n y_i^q} - \left(\frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n y_i^q}\right)^p \\ &\leq \frac{1}{4} \cdot \frac{1 - \left(\frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n y_i^q}\right)^{p-1}}{1 - \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n y_i^q}} \leq \frac{1}{4} M_p, \end{aligned} \tag{3.5}$$

where M_p is defined above.

Proof. The inequalities (3.4) and (3.5) follow from (3.1) and (3.2) by choosing

$$z_i = \frac{x_i}{y_i^{q-1}} \text{ and } w_i = \frac{y_i^q}{\sum_{j=1}^n y_j^q}.$$

The details are omitted. □

Remark 2. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Assume that*

$$0 \leq \frac{a_i}{b_i^{q-1}} \leq 1, \text{ for } i \in \{1, \dots, n\}. \tag{3.6}$$

If $p_i > 0$ for $i \in \{1, \dots, n\}$, then for $x_i := p_i^{1/p} a_i$ and $y_i := p_i^{1/q} b_i$ we have

$$\frac{x_i}{y_i^{q-1}} = \frac{p_i^{1/p} a_i}{\left(p_i^{1/q} b_i\right)^{q-1}} = \frac{p_i^{1/p} a_i}{p_i^{(q-1)/q} b_i^{q-1}} = \frac{p_i^{1/p} a_i}{p_i^{1/p} b_i^{q-1}} = \frac{a_i}{b_i^{q-1}} \in [0,1]$$

for $i \in \{1, \dots, n\}$.

If we write the inequalities (3.4) and (3.5) for these choices, we get the weighted inequalities

$$0 \leq \frac{\sum_{i=1}^{n} p_i a_i^p}{\sum_{i=1}^{n} p_i b_i^q} - \left(\frac{\sum_{i=1}^{n} p_i a_i b_i}{\sum_{i=1}^{n} p_i b_i^q}\right)^p \leq M_p \left(1 - \frac{\sum_{i=1}^{n} p_i a_i b_i}{\sum_{i=1}^{n} p_i b_i^q}\right)\left(\frac{\sum_{i=1}^{n} p_i a_i b_i}{\sum_{i=1}^{n} p_i b_i^q}\right) \leq \frac{1}{4} M_p \tag{3.7}$$

and

$$0 \leq \frac{\sum_{i=1}^{n} p_i a_i^p}{\sum_{i=1}^{n} p_i b_i^q} - \left(\frac{\sum_{i=1}^{n} p_i a_i b_i}{\sum_{i=1}^{n} p_i b_i^q}\right)^p \leq \frac{1}{4} \cdot \frac{1 - \left(\frac{\sum_{i=1}^{n} p_i a_i b_i}{\sum_{i=1}^{n} p_i b_i^q}\right)^{p-1}}{1 - \frac{\sum_{i=1}^{n} p_i a_i b_i}{\sum_{i=1}^{n} p_i b_i^q}} \leq \frac{1}{4} M_p. \tag{3.8}$$

Theorem 3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0, R)$ with $R > 0$ or $R = \infty$. If $p > 1$, $0 < \alpha < R$ and $0 < x \leq 1$, then*

$$0 \leq \frac{f(\alpha x^p)}{f(\alpha)} - \left(\frac{f(\alpha x)}{f(\alpha)}\right)^p \leq M_p \left(1 - \frac{f(\alpha x)}{f(\alpha)}\right) \frac{f(\alpha x)}{f(\alpha)} \leq \frac{1}{4} M_p \tag{3.9}$$

and

$$0 \leq 0 \leq \frac{f(\alpha x^p)}{f(\alpha)} - \left(\frac{f(\alpha x)}{f(\alpha)}\right)^p \leq \frac{1}{4} \cdot \frac{1 - \left(\frac{f(\alpha x)}{f(\alpha)}\right)^{p-1}}{1 - \frac{f(\alpha x)}{f(\alpha)}} \leq \frac{1}{4} M_p. \tag{3.10}$$

Proof. Let $m \geq 1$ and $0 < \alpha < R$, $0 < x \leq 1$. If we write the inequality (3.1) for

$$w_j = \frac{a_j \alpha^j}{\sum_{k=0}^{m} a_k \alpha^k} \text{ and } z_j := x^j \in [0, 1], \; j \in \{0, ..., m\},$$

then we get

$$\begin{aligned} 0 &\leq \frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j x^{pj} - \left(\frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j x^j\right)^p \\ &\leq M_p \left(1 - \frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j x^j\right) \frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j x^j \\ &\leq \frac{1}{4} M_p. \end{aligned} \tag{3.11}$$

Since all series whose partial sums involved in the inequality (3.11) are convergent, then by letting $m \to \infty$ in (3.11) we deduce (3.9).

The inequality (3.10) follows from (3.2) in a similar way and the details are omitted. □

Remark 3. *We observe that from (1.9) we have for $p > 1$*

$$0 \leq \frac{f(\alpha x^p)}{f(\alpha)} - \left[\frac{f(\alpha x)}{f(\alpha)}\right]^p \leq \frac{1}{4} p, \tag{3.12}$$

which is not as good as the inequality

$$0 \leq \frac{f(\alpha x^p)}{f(\alpha)} - \left(\frac{f(\alpha x)}{f(\alpha)}\right)^p \leq \frac{1}{4} \times \begin{cases} 1 \text{ if } p \in (1,2], \\ \\ p-1 \text{ if } p \in (2,\infty). \end{cases} \tag{3.13}$$

that has been obtained in (3.9).

Corollary 2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R > 0$ or $R = \infty$. If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $u, v > 0$ with $v^p \leq u^q < R$, then*

$$0 \leq \frac{f(v^p)}{f(u^q)} - \left(\frac{f(uv)}{f(u^q)}\right)^p \leq M_p \left(1 - \frac{f(uv)}{f(u^q)}\right) \frac{f(uv)}{f(u^q)} \leq \frac{1}{4} M_p \tag{3.14}$$

and

$$0 \leq \frac{f(v^p)}{f(u^q)} - \left(\frac{f(uv)}{f(u^q)}\right)^p \leq \frac{1}{4} \cdot \frac{1 - \left(\frac{f(uv)}{f(u^q)}\right)^{p-1}}{1 - \frac{f(uv)}{f(u^q)}} \leq \frac{1}{4} M_p. \tag{3.15}$$

Proof. Follows by taking into (3.9) and (3.10) $\alpha = u^q$ and $x = \frac{v}{u^{q/p}}$. The details are omitted. □

Remark 4. *From (3.14) we have*

$$\left(\frac{f(uv)}{f(u^q)}\right)^p \leq \frac{f(v^p)}{f(u^q)} \leq \left(\frac{f(uv)}{f(u^q)}\right)^p + \frac{1}{4} M_p \tag{3.16}$$

and

$$0 \leq [f(v^p)]^{1/p} [f(u^q)]^{1/q} - f(uv) \leq \frac{1}{4^{1/p}} M_p^{1/p} f(u^q) \tag{3.17}$$

provided that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $u, v > 0$ with $v^p \leq u^q < R$.

These inequalities are better that the corresponding ones from Corollary 1.

If we take $p = q = 2$ in (3.16) and (3.17, then we get

$$\left(\frac{f(uv)}{f(u^2)}\right)^2 \leq \frac{f(v^2)}{f(u^2)} \leq \left(\frac{f(uv)}{f(u^2)}\right)^2 + \frac{1}{4} \tag{3.18}$$

and

$$0 \leq \left[f\left(v^2\right)\right]^{1/2} \left[f\left(u^2\right)\right]^{1/2} - f(uv) \leq \frac{1}{2} f\left(u^2\right), \tag{3.19}$$

provided that $u, v > 0$ with $v^2 \leq u^2 < R$.

Example 1. *a) If we write the inequalities (3.9) and (3.10) for the function $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $z \in D(0,1)$, then we have*

$$0 \leq \frac{1-\alpha}{1-\alpha x^p} - \left(\frac{1-\alpha}{1-\alpha x}\right)^p \leq M_p \frac{\alpha(1-\alpha)(1-x)}{(1-\alpha x)^2} \leq \frac{1}{4} M_p \tag{3.20}$$

and

$$\begin{aligned} 0 &\leq \frac{1-\alpha}{1-\alpha x^p} - \left(\frac{1-\alpha}{1-\alpha x}\right)^p \\ &\leq \frac{1}{4} \cdot \frac{1-\alpha x}{\alpha(1-x)} \left[1 - \left(\frac{1-\alpha}{1-\alpha x}\right)^{p-1}\right] \leq \frac{1}{4} M_p \end{aligned} \tag{3.21}$$

for any $\alpha, x \in (0,1)$ *and* $p > 1$.

b) If we write the inequalities (3.9) and (3.10) for the function $\exp z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$, $z \in \mathbb{C}$, *then we have*

$$0 \le \exp[\alpha(x^p - 1)] - \exp[p\alpha(x-1)] \le M_p(1 - \exp[\alpha(x-1)])\exp[\alpha(x-1)] \le \frac{1}{4}M_p \tag{3.22}$$

and

$$0 \le \exp[\alpha(x^p - 1)] - \exp[p\alpha(x-1)] \le \frac{1}{4}\cdot\frac{1 - \exp[\alpha(p-1)(x-1)]}{1 - \exp[\alpha(x-1)]} \le \frac{1}{4}M_p \tag{3.23}$$

for any $\alpha > 0$, $p > 1$ *and* $x \in (0,1)$.

4. Logarithmic Inequalities

If we consider the convex function $f(t) = t\ln t$, $t > 0$, then

$$\Psi_{\cdot\ln(\cdot)}(t; m, M) = \frac{M\ln M - t\ln t}{M - t} - \frac{t\ln t - m\ln m}{t - m} \tag{4.1}$$

for $0 < m < M < \infty$.

If we take $M = 1$ and $m \to 0+$ in (4.1) then we have

$$\begin{aligned}\lim_{m\to 0+}\Psi_{\cdot\ln(\cdot)}(t; m, 1) &= \lim_{m\to 0+}\left[\frac{-t\ln t}{1-t} - \frac{t\ln t - m\ln m}{t - m}\right] \\ &= \frac{-t\ln t}{1-t} - \frac{t\ln t}{t} = \frac{\ln t}{t-1}\end{aligned}$$

for $t \in (0,1)$.

From (2.2) we have

$$0 \le \sum_{i=1}^{n} w_i x_i \ln x_i - \sum_{i=1}^{n} w_i x_i \ln\left(\sum_{i=1}^{n} w_i x_i\right) \le \frac{1}{4}\frac{\ln\left(\sum_{i=1}^{n} w_i x_i\right)^{-1}}{1 - \sum_{i=1}^{n} w_i x_i} \tag{4.2}$$

for any $x_i \in (0,1)$, $w_i \ge 0$ $(i = 1, \ldots, n)$ with $W_n := \sum_{i=1}^{n} w_i = 1$.

Theorem 4. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a power series with nonnegative coefficients and convergent on the open disk* $D(0,R)$ *with* $R > 0$ *or* $R = \infty$. *If* $0 < \alpha < R$, $p > 0$ *and* $x \in (0,1)$, *then*

$$0 \le \frac{p\alpha x^p f'(\alpha x^p)}{f(\alpha)}\ln x - \frac{f(\alpha x^p)}{f(\alpha)}\ln\left(\frac{f(\alpha x^p)}{f(\alpha)}\right) \le \frac{1}{4}\frac{\ln\left(\frac{f(\alpha)}{f(\alpha x^p)}\right)}{1 - \frac{f(\alpha x^p)}{f(\alpha)}}. \tag{4.3}$$

Proof. If $0<\alpha<R$ and $m\geq 1$, then by (4.2) for $x_j=(x^p)^j$, we have

$$0\leq \frac{1}{\sum_{k=0}^m a_k\alpha^k}\sum_{j=0}^m a_j\alpha^j x^{pj}\ln x^{pj} - \frac{1}{\sum_{k=0}^m a_k\alpha^k}\sum_{j=0}^m a_j\alpha^j x^{pj}\ln\left(\frac{1}{\sum_{k=0}^m a_k\alpha^k}\sum_{j=0}^m a_j\alpha^j x^{pj}\right) \leq \frac{1}{4}\frac{\ln\left(\frac{1}{\sum_{k=0}^m a_k\alpha^k}\sum_{j=0}^m a_j\alpha^j x^{pj}\right)^{-1}}{1-\frac{1}{\sum_{k=0}^m a_k\alpha^k}\sum_{j=0}^m a_j\alpha^j x^{pj}},$$

where $p>0$ and $x\in(0,1)$.

This is equivalent to

$$0\leq \frac{\ln x^p}{\sum_{k=0}^m a_k\alpha^k}\sum_{j=0}^m ja_j\alpha^j (x^p)^j - \frac{1}{\sum_{k=0}^m a_k\alpha^k}\sum_{j=0}^m a_j\alpha^j (x^p)^j\ln\left(\frac{1}{\sum_{k=0}^m a_k\alpha^k}\sum_{j=0}^m a_j\alpha^j (x^p)^j\right) \leq \frac{1}{4}\frac{\ln\left(\frac{1}{\sum_{k=0}^m a_k\alpha^k}\sum_{j=0}^m a_j\alpha^j (x^p)^j\right)^{-1}}{1-\frac{1}{\sum_{k=0}^m a_k\alpha^k}\sum_{j=0}^m a_j\alpha^j (x^p)^j}. \tag{4.4}$$

Since all series whose partial sums involved in the inequality (4.4) are convergent, then by letting $m\to\infty$ in (4.4) we deduce (4.3). □

Example 2. *a) If we write the inequality (4.3) for the function* $\frac{1}{1-z}=\sum_{n=0}^\infty z^n$, $z\in D(0,1)$, *then we have for* $\alpha,x\in(0,1)$ *and* $p>0$ *that*

$$0\leq \frac{p\alpha x^p(1-\alpha)}{(1-\alpha x^p)^2}\ln x-\frac{1-\alpha}{(1-\alpha x^p)}\ln\left(\frac{1-\alpha}{1-\alpha x^p}\right) \leq \frac{1}{4}\frac{(1-\alpha x^p)\ln\left(\frac{1-\alpha x^p}{1-\alpha}\right)}{\alpha(1-x^p)}. \tag{4.5}$$

b) If we write the inequality (4.3) for the function $\exp z=\sum_{n=0}^\infty\frac{1}{n!}z^n$, $z\in\mathbb{C}$, *then we have*

$$0\leq[p\alpha x^p\ln x-\alpha(x^p-1)]\exp[\alpha(x^p-1)]\leq\frac{1}{4}\frac{\alpha(1-x^p)}{1-\exp[\alpha(x^p-1)]} \tag{4.6}$$

for $x\in(0,1)$ *and* $\alpha,p>0$.

5. Exponential Inequalities

If we consider the exponential function $f:\mathbb{R}\to(0,\infty)$, $f(t)=\exp(\beta t)$ with $\beta>0$ then

$$\Psi_{\exp(\beta\cdot)}(t;m,M)=\frac{\exp(\beta M)-\exp(\beta t)}{M-t}-\frac{\exp(\beta t)-\exp(m)}{t-m}.$$

If we take $M=0$ we have

$$\Psi_{\exp(\beta\cdot)}(t;m,0)=\frac{1-\exp(\beta t)}{-t}-\frac{\exp(\beta t)-\exp(m)}{t-m}$$

and letting $m \to -\infty$, then we get

$$\lim_{m\to-\infty} \Psi_{\exp(\beta\cdot)}(t;m,0) = \frac{\exp(\beta t)-1}{t} =: \Psi_{\exp(\beta\cdot)}(t)$$

with $t \in (-\infty, 0)$.

Since $\exp(\beta\cdot)$ is convex on $(-\infty,0)$, then $\Psi_{\exp(\beta\cdot)}(\cdot)$ is monotonic nondecreasing on $(-\infty, 0)$ and then

$$\sup_{t\in(-\infty,0)} \Psi_{\exp(\beta\cdot)}(t) = \lim_{t\to 0-} \frac{\exp(\beta t)-1}{t} = \beta.$$

From (2.1) we have

$$0 \le \sum_{i=1}^{n} w_i \exp(\beta x_i) - \exp\left(\beta\sum_{i=1}^{n} w_i x_i\right) \le -\beta \sum_{i=1}^{n} w_i x_i \tag{5.1}$$

for any $x_i \le 0$, $w_i \ge 0$ $(i = 1,\ldots,n)$ with $W_n := \sum_{i=1}^{n} w_i = 1$.

Theorem 5. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R > 0$ or $R = \infty$. If $x \le 0$, $\beta > 0$ with $\exp(\beta x) < R$ and $0 < \alpha < R$, then*

$$0 \le \frac{f(\alpha\exp(\beta x))}{f(\alpha)} - \exp\left[\frac{\alpha\beta x f'(\alpha)}{f(\alpha)}\right] \le -\frac{\alpha\beta x f'(\alpha)}{f(\alpha)}. \tag{5.2}$$

Proof. If $0 < \alpha < R$ and $m \ge 1$, then by (5.1) for $x_j = jx$, we have

$$\begin{aligned} 0 &\le \frac{1}{\sum_{j=0}^{m} a_j\alpha^j}\sum_{j=0}^{m} a_j\alpha^j[\exp(\beta x)]^j - \exp\left(\frac{\beta x}{\sum_{j=0}^{m} a_j\alpha^j}\sum_{j=0}^{m} j a_j\alpha^j\right) \\ &\le \frac{-\beta x}{\sum_{j=0}^{m} a_j\alpha^j}\sum_{j=0}^{m} j a_j\alpha^j \end{aligned} \tag{5.3}$$

for $x \in (-\infty, 0)$.

Since all series whose partial sums involved in the inequality (5.3) are convergent, then by letting $m \to \infty$ in (5.3) we deduce (5.2). □

Example 3. *a) If we write the inequality (5.2) for the function $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $z \in D(0,1)$, then we have for $x \le 0$, $\beta > 0$ and $0 < \alpha < 1$, that*

$$0 \le \frac{1-\alpha}{1-\alpha\exp(\beta x)} - \exp\left(\frac{\alpha\beta x}{1-\alpha}\right) \le -\frac{\alpha\beta x}{1-\alpha}. \tag{5.4}$$

b) If we write the inequality (5.2) for the function $\exp z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$, $z \in \mathbb{C}$, then we have

$$0 \le \exp(\alpha[\exp(\beta x) - 1]) - \exp(\alpha\beta x) \le -\alpha\beta x \tag{5.5}$$

for any $\alpha > 0$ and $x \le 0$, $\beta > 0$.

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[1]Mathematics, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa