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Multivariate error function based neural network approximations

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Abstract

Here we present multivariate quantitative approximations of real and complex valued continuous multivariate functions on a box or $\mathbb{R}^N$, $N \in \mathbb{N}$, by the multivariate quasi-interpolation, Baskakov type and quadrature type neural network operators. We treat also the case of approximation by iterated operators of the last three types. These approximations are derived by establishing multidimensional Jackson type inequalities involving the multivariate modulus of continuity of the engaged function or its high order partial derivatives. Our multivariate operators are defined by using a multidimensional density function induced by the Gaussian error special function. The approximations are pointwise and uniform. The related feed-forward neural network is with one hidden layer.

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1 Introduction

The author in [2] and [3], see chapters 2-5, was the first to establish neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliagnet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support. Also in [3] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see chapters 4-5 there.

For this article the author is motivated by the article [12] of Z. Chen and F. Cao, also by [4], [5], [6], [7], [8], [9], [10], [13], [14].

The author here performs multivariate error function based neural network approximations to continuous functions over boxes or over the whole $\mathbb{R}^N$, $N \in \mathbb{N}$, then he extends his results to complex valued multivariate functions. Also he does iterated approximation. All convergences here are with rates expressed via the multivariate modulus of continuity of the involved function or its high order partial derivative and given by very tight multidimensional Jackson type inequalities.

The author here comes up with the "right" precisely defined multivariate quasi-interpolation neural network operators related to boxes or $\mathbb{R}^N$, as well as Baskakov type and quadrature type related operators on $\mathbb{R}^N$. Our boxes are not necessarily symmetric to the origin. In preparation to prove our results we establish important properties of the basic multivariate density function induced by error function and defining our operators.

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle a_j \cdot x \rangle + b_j\right), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental network models, the activation function is the error function. About neural networks read [15], [16], [17].

2 Basics

We consider here the (Gauss) error special function ([1], [11])

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \quad x \in \mathbb{R}, \tag{1}$$

which is a sigmoidal type function and is a strictly increasing function.

It has the basic properties

$$\operatorname{erf}(0) = 0, \quad \operatorname{erf}(-x) = -\operatorname{erf}(x), \quad \operatorname{erf}(+\infty) = 1, \quad \operatorname{erf}(-\infty) = -1.$$

We consider the activation function ([10])

$$\chi(x) = \frac{1}{4}\left(\operatorname{erf}(x+1) - \operatorname{erf}(x-1)\right) > 0, \text{ any } x \in \mathbb{R}, \tag{2}$$

which is an even function.

Next we follow [10] on χ. We got there $\chi(0) \simeq 0.4215$, and that χ is strictly decreasing on $[0, \infty)$ and strictly increasing on $(-\infty, 0]$, and the x-axis is the horizontal asymptote on χ, i.e. χ is a bell symmetric function.

Theorem 1 *([10]) We have that*

$$\sum_{i=-\infty}^{\infty} \chi(x-i) = 1, \quad \textit{all } x \in \mathbb{R}, \tag{3}$$

$$\sum_{i=-\infty}^{\infty} \chi(nx-i) = 1, \quad \textit{all } x \in \mathbb{R}, \ n \in \mathbb{N}, \tag{4}$$

and

$$\int_{-\infty}^{\infty} \chi(x)\, dx = 1, \tag{5}$$

that is $\chi(x)$ is a density function on $\mathbb{R}$.

We need the important

Theorem 2 *([10]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} \geq 3$. It holds*

$$\sum_{\left\{\begin{array}{c} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{array}\right.}^{\infty} \chi(nx-k) < \frac{1}{2\sqrt{\pi}\left(n^{1-\alpha}-2\right) e^{\left(n^{1-\alpha}-2\right)^2}}. \tag{6}$$

Denote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.

Theorem 3 *([10]) Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k)} < \frac{1}{\chi(1)} \simeq 4.019, \quad \forall\, x \in [a,b]. \tag{7}$$

Also from [10] we get

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k) \neq 1, \tag{8}$$

at least for some $x \in [a,b]$.

For large enough n we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. In general it holds by (4) that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi\left(nx-k\right) \leq 1. \tag{9}$$

We introduce

$$Z\left(x_1,...,x_N\right) := Z\left(x\right) := \prod_{i=1}^{N} \chi\left(x_i\right), \quad x=\left(x_1,...,x_N\right) \in \mathbb{R}^N, \ N \in \mathbb{N}. \tag{10}$$

It has the properties:

(i) $Z\left(x\right) > 0, \ \forall \ x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} Z\left(x-k\right) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} Z\left(x_1-k_1,...,x_N-k_N\right) = 1, \tag{11}$$

where $k := \left(k_1,...,k_n\right) \in \mathbb{Z}^N$, $\forall \ x \in \mathbb{R}^N$,

hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z\left(nx-k\right) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} Z\left(nx_1-k_1,...,nx_N-k_N\right) = 1, \tag{12}$$

$\forall \ x \in \mathbb{R}^N$; $n \in \mathbb{N}$,

and

(iv)

$$\int_{\mathbb{R}^N} Z\left(x\right) dx = 1, \tag{13}$$

that is Z is a multivariate density function.

Here $\left\|x\right\|_{\infty} := \max\left\{\left|x_1\right|,...,\left|x_N\right|\right\}$, $x \in \mathbb{R}^N$, also set $\infty := \left(\infty,...,\infty\right)$, $-\infty := \left(-\infty,...,-\infty\right)$ upon the multivariate context, and

$$\begin{aligned} \lceil na \rceil &: = \left(\lceil na_1 \rceil,...,\lceil na_N \rceil\right), \\ \lfloor nb \rfloor &: = \left(\lfloor nb_1 \rfloor,...,\lfloor nb_N \rfloor\right), \end{aligned} \tag{14}$$

where $a := \left(a_1,...,a_N\right)$, $b := \left(b_1,...,b_N\right)$.

We obviously see that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\prod_{i=1}^{N} \chi(nx_i - k_i) \right) =$$

$$\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \dots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} \left(\prod_{i=1}^{N} \chi(nx_i - k_i) \right) = \prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \chi(nx_i - k_i) \right). \quad (15)$$

For $0 < \beta < 1$ and $n \in \mathbb{N}$, a fixed $x \in \mathbb{R}^N$, we have that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k) =$$

$$\sum_{\substack{k=\lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} \chi(nx-k) + \sum_{\substack{k=\lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} \chi(nx-k). \quad (16)$$

In the last two sums the counting is over disjoint vector sets of k's, because the condition $\left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}$ implies that there exists at least one $\left| \frac{k_r}{n} - x_r \right| > \frac{1}{n^\beta}$, where $r \in \{1, ..., N\}$.

We treat

$$\sum_{\substack{k=\lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} Z(nx-k) = \prod_{i=1}^{N} \left(\sum_{\substack{k_i=\lceil na_i \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb_i \rfloor} \chi(nx_i - k_i) \right)$$

$$\leq \left(\prod_{\substack{i=1 \\ i \neq r}}^{N} \left(\sum_{k_i=-\infty}^{\infty} \chi(nx_i - k_i) \right) \right) \cdot \left(\sum_{\substack{k_r=\lceil na_r \rceil \\ \left| \frac{k_r}{n} - x_r \right| > \frac{1}{n^\beta}}}^{\lfloor nb_r \rfloor} \chi(nx_r - k_r) \right) = \quad (17)$$

$$\left(\sum_{\substack{k_r=\lceil na_r \rceil \\ \left| \frac{k_r}{n} - x_r \right| > \frac{1}{n^\beta}}}^{\lfloor nb_r \rfloor} \chi(nx_r - k_r) \right) \leq \sum_{\substack{k_r=-\infty \\ \left| \frac{k_r}{n} - x_r \right| > \frac{1}{n^\beta}}}^{\infty} \chi(nx_r - k_r)$$

$$= \sum_{\substack{k_r = -\infty \\ : |nx_r - k_r| > n^{1-\beta}}}^{\infty} \chi\left(nx_r - k_r\right) \overset{(6)}{\leq} \frac{1}{2\sqrt{\pi}\left(n^{1-\beta} - 2\right) e^{\left(n^{1-\beta}-2\right)^2}}, \tag{18}$$

when $n^{1-\beta} \geq 3$.

We have proved that

(v)

$$\sum_{\substack{k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\lfloor nb \rfloor} Z\left(nx - k\right) \leq \frac{1}{2\sqrt{\pi}\left(n^{1-\beta} - 2\right) e^{\left(n^{1-\beta}-2\right)^2}}, \tag{19}$$

$0 < \beta < 1$, $n \in \mathbb{N}; n^{1-\beta} \geq 3$, $x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$.

By Theorem 3 clearly we obtain

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z\left(nx - k\right)} = \frac{1}{\prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \chi\left(nx_i - k_i\right)\right)} \tag{20}$$

$$< \frac{1}{\left(\chi\left(1\right)\right)^N} \simeq \left(4.019\right)^N.$$

That is,

(vi) it holds

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z\left(nx - k\right)} < \frac{1}{\left(\chi\left(1\right)\right)^N} \simeq \left(4.019\right)^N, \tag{21}$$

$\forall\ x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$, $n \in \mathbb{N}$.

It is also clear that

(vii)

$$\sum_{\substack{k = -\infty \\ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\infty} Z\left(nx - k\right) \leq \frac{1}{2\sqrt{\pi}\left(n^{1-\beta} - 2\right) e^{\left(n^{1-\beta}-2\right)^2}}, \tag{22}$$

$0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} \geq 3$, $x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$.

Also we get that

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z\left(nx - k\right) \neq 1, \tag{23}$$

for at least some $x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

Let $f \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$ and $n \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

We introduce and define the multivariate positive linear neural network operator ($x := (x_1, ..., x_N) \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$)

$$A_n(f, x_1, ..., x_N) := A_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)} \tag{24}$$

$$:= \frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} ... \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, ..., \frac{k_N}{n}\right)\left(\prod_{i=1}^{N} \chi(nx_i - k_i)\right)}{\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \chi(nx_i - k_i)\right)}.$$

For large enough n we always obtain $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$. Also $a_i \leq \frac{k_i}{n} \leq b_i$, iff $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

For convinience we call

$$A_n^*(f, x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx - k) \tag{25}$$

$$:= \sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} ... \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, ..., \frac{k_N}{n}\right)\left(\prod_{i=1}^{N} \chi(nx_i - k_i)\right),$$

$\forall\ x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

That is

$$A_n(f, x) := \frac{A_n^*(f, x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)}, \tag{26}$$

$\forall\ x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $n \in \mathbb{N}$.

Hence

$$A_n(f, x) - f(x) = \frac{A_n^*(f, x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)}. \tag{27}$$

Consequently we derive

$$|A_n(f, x) - f(x)| \leq (4.019)^N \left| A_n^*(f, x) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) \right|, \tag{28}$$

$\forall\ x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

We will estimate the right hand side of (28).

For the last we need, for $f \in C\left(\prod_{i=1}^{N}\left[a_i, b_i\right]\right)$ the first multivariate modulus of continuity

$$\omega_1(f, h) := \sup_{\substack{x, y \in \prod_{i=1}^{N}[a_i, b_i] \\ \|x-y\|_\infty \le h}} |f(x) - f(y)|, \quad h > 0. \tag{29}$$

It holds that

$$\lim_{h \to 0} \omega_1(f, h) = 0. \tag{30}$$

Similarly it is defined for $f \in C_B\left(\mathbb{R}^N\right)$ (continuous and bounded functions on $\mathbb{R}^N$) the $\omega_1(f, h)$, and it has the property (30), given that $f \in C_U\left(\mathbb{R}^N\right)$ (uniformly continuous functions on $\mathbb{R}^N$).

When $f \in C_B\left(\mathbb{R}^N\right)$ we define,

$$B_n(f, x) := B_n(f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx - k) := \tag{31}$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n}, \frac{k_2}{n}, ..., \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} \chi(nx_i - k_i)\right),$$

$n \in \mathbb{N}$, $\forall\ x \in \mathbb{R}^N$, $N \in \mathbb{N}$, the multivariate quasi-interpolation neural network operator.

Also for $f \in C_B\left(\mathbb{R}^N\right)$ we define the multivariate Kantorovich type neural network operator

$$C_n(f, x) := C_n(f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty} \left(n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\, dt\right) Z(nx - k) := \tag{32}$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \left(n^N \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} ... \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1, ..., t_N)\, dt_1 ... dt_N\right)$$

$$\cdot \left(\prod_{i=1}^{N} \chi(nx_i - k_i)\right),$$

$n \in \mathbb{N}$, $\forall\ x \in \mathbb{R}^N$.

Again for $f \in C_B\left(\mathbb{R}^N\right)$, $N \in \mathbb{N}$, we define the multivariate neural network operator of quadrature type $D_n(f, x)$, $n \in \mathbb{N}$, as follows. Let $\theta = (\theta_1, ..., \theta_N) \in \mathbb{N}^N$, $r = (r_1, ..., r_N) \in \mathbb{Z}_+^N$, $w_r = w_{r_1, r_2, ... r_N} \ge 0$, such that $\sum_{r=0}^{\theta} w_r = \sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} ... \sum_{r_N=0}^{\theta_N} w_{r_1, r_2, ... r_N} = 1$; $k \in \mathbb{Z}^N$ and

$$\delta_{nk}(f) := \delta_{n, k_1, k_2, ..., k_N}(f) := \sum_{r=0}^{\theta} w_r f\left(\frac{k}{n} + \frac{r}{n\theta}\right) :=$$

$$\sum_{r_1=0}^{\theta_1}\sum_{r_2=0}^{\theta_2}...\sum_{r_N=0}^{\theta_N} w_{r_1,r_2,...r_N} f\left(\frac{k_1}{n}+\frac{r_1}{n\theta_1},\frac{k_2}{n}+\frac{r_2}{n\theta_2},...,\frac{k_N}{n}+\frac{r_N}{n\theta_N}\right), \quad (33)$$

where $\frac{r}{\theta} := \left(\frac{r_1}{\theta_1},\frac{r_2}{\theta_2},...,\frac{r_N}{\theta_N}\right)$.

We put

$$D_n(f,x) := D_n(f,x_1,...,x_N) := \sum_{k=-\infty}^{\infty}\delta_{nk}(f)\, Z(nx-k) := \quad (34)$$

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty}\delta_{n,k_1,k_2,...,k_N}(f)\left(\prod_{i=1}^{N}\chi(nx_i-k_i)\right),$$

$\forall\ x\in\mathbb{R}^N$.

Let fixed $j\in\mathbb{N}$, $0<\beta<1$, and $A,B>0$. For large enough $n\in\mathbb{N}: n^{1-\beta}\geq 3$, in the linear combination $\left(\frac{A}{n^{\beta j}}+\frac{B}{(n^{1-\beta}-2)e^{(n^{1-\beta}-2)^2}}\right)$, the dominant rate of convergence, as $n\to\infty$, is $n^{-\beta j}$. The closer β is to 1 we get faster and better rate of convergence to zero.

Let $f\in C^m\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $m,N\in\mathbb{N}$. Here f_α denotes a partial derivative of f, $\alpha:=(\alpha_1,...,\alpha_N)$, $\alpha_i\in\mathbb{Z}_+$, $i=1,...,N$, and $|\alpha|:=\sum_{i=1}^{N}\alpha_i=l$, where $l=0,1,...,m$. We write also $f_\alpha:=\frac{\partial^\alpha f}{\partial x^\alpha}$ and we say it is of order l.

We denote

$$\omega_{1,m}^{\max}(f_\alpha,h) := \max_{\alpha:|\alpha|=m}\omega_1(f_\alpha,h). \quad (35)$$

Call also

$$\|f_\alpha\|_{\infty,m}^{\max} := \max_{|\alpha|=m}\{\|f_\alpha\|_\infty\}, \quad (36)$$

$\|\cdot\|_\infty$ is the supremum norm.

In this article we study the basic approximation properties of A_n, B_n, C_n, D_n neural network operators and as well of their iterates. That is, the quantitative pointwise and uniform convergence of these operators to the unit operator I. We study also the complex functions related approximation.

3 Multidimensional Real Neural Network Approximations

Here we present a series of neural network approximations to a function given with rates.

We give

Theorem 4 *Let* $f \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $0<\beta<1$, $x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $N, n \in \mathbb{N}$ *with* $n^{1-\beta} \geq 3$. *Then*

1)

$$|A_n(f,x) - f(x)| \leq (4.019)^N \left[\omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{\|f\|_\infty}{\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}\right] =: \lambda_1, \tag{37}$$

and

2)

$$\|A_n(f) - f\|_\infty \leq \lambda_1. \tag{38}$$

We notice that $\lim_{n\to\infty} A_n(f) = f$, *pointwise and uniformly.*

Proof. We observe that

$$\Delta(x) := A_n^*(f,x) - f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k) =$$

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right) Z(nx-k) - \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f(x) Z(nx-k) =$$

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \left(f\left(\frac{k}{n}\right) - f(x)\right) Z(nx-k). \tag{39}$$

Thus

$$|\Delta(x)| \leq \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \left|f\left(\frac{k}{n}\right) - f(x)\right| Z(nx-k) =$$

$$\sum_{\left\{\begin{array}{c} k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty \leq \frac{1}{n^\beta}\end{array}\right.}^{\lfloor nb\rfloor} \left|f\left(\frac{k}{n}\right) - f(x)\right| Z(nx-k) +$$

$$\sum_{\left\{\begin{array}{c} k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}\end{array}\right.}^{\lfloor nb\rfloor} \left|f\left(\frac{k}{n}\right) - f(x)\right| Z(nx-k) \overset{\text{(by (12))}}{\leq}$$

$$\omega_1\left(f, \frac{1}{n^\beta}\right) + 2\|f\|_\infty \sum_{\left\{\begin{array}{c} k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}\end{array}\right.}^{\lfloor nb\rfloor} Z(nx-k) \overset{\text{(by (19))}}{\leq}$$

$$\omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{\|f\|_\infty}{\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}. \tag{40}$$

So that

$$|\Delta| \leq \omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{\|f\|_\infty}{\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}.$$

Now using (28) we finish proof. ∎

We continue with

Theorem 5 *Let $f \in C_B\left(\mathbb{R}^N\right)$, $0<\beta<1$, $x \in \mathbb{R}^N$, $N, n \in \mathbb{N}$ with $n^{1-\beta} \geq 3$. Then*

1)

$$\left|B_n(f,x) - f(x)\right| \leq \omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{\|f\|_\infty}{\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}} =: \lambda_2, \tag{41}$$

2)

$$\left\|B_n(f) - f\right\|_\infty \leq \lambda_2. \tag{42}$$

Given that $f \in \left(C_U\left(\mathbb{R}^N\right) \cap C_B\left(\mathbb{R}^N\right)\right)$, we obtain $\lim_{n\to\infty} B_n(f) = f$, uniformly.

Proof. We have that

$$B_n(f,x) - f(x) \overset{(12)}{=} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx-k) - f(x)\sum_{k=-\infty}^{\infty} Z(nx-k) = \tag{43}$$

$$\sum_{k=-\infty}^{\infty}\left(f\left(\frac{k}{n}\right) - f(x)\right) Z(nx-k).$$

Hence

$$\left|B_n(f,x) - f(x)\right| \leq \sum_{k=-\infty}^{\infty}\left|f\left(\frac{k}{n}\right) - f(x)\right| Z(nx-k) =$$

$$\sum_{\substack{k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty \leq \frac{1}{n^\beta}}}^{\infty}\left|f\left(\frac{k}{n}\right) - f(x)\right| Z(nx-k) +$$

$$\sum_{\substack{k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}}}^{\infty}\left|f\left(\frac{k}{n}\right) - f(x)\right| Z(nx-k) \overset{(12)}{\leq}$$

$$\omega_1\left(f, \frac{1}{n^\beta}\right) + 2\|f\|_\infty \sum_{\substack{k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}}}^{\infty} Z(nx-k) \overset{(19)}{\leq}$$

$$\omega_1\left(f,\frac{1}{n^\beta}\right)+\frac{\|f\|_\infty}{\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}, \tag{44}$$

proving the claim. ■

We give

Theorem 6 *Let $f\in C_B\left(\mathbb{R}^N\right)$, $0<\beta<1$, $x\in\mathbb{R}^N$, $N,n\in\mathbb{N}$ with $n^{1-\beta}\geq 3$. Then*

1)

$$\left|C_n(f,x)-f(x)\right|\leq\omega_1\left(f,\frac{1}{n}+\frac{1}{n^\beta}\right)+\frac{\|f\|_\infty}{\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}=:\lambda_3, \tag{45}$$

2)

$$\left\|C_n(f)-f\right\|_\infty\leq\lambda_3. \tag{46}$$

Given that $f\in\left(C_U\left(\mathbb{R}^N\right)\cap C_B\left(\mathbb{R}^N\right)\right)$, we obtain $\lim_{n\to\infty}C_n(f)=f$, uniformly.

Proof. We notice that

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}}f(t)\,dt=\int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}}\int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}}\ldots\int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}}f(t_1,t_2,\ldots,t_N)\,dt_1dt_2\ldots dt_N=$$

$$\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}}\ldots\int_0^{\frac{1}{n}}f\left(t_1+\frac{k_1}{n},t_2+\frac{k_2}{n},\ldots,t_N+\frac{k_N}{n}\right)dt_1\ldots dt_N=\int_0^{\frac{1}{n}}f\left(t+\frac{k}{n}\right)dt. \tag{47}$$

Thus it holds

$$C_n(f,x)=\sum_{k=-\infty}^{\infty}\left(n^N\int_0^{\frac{1}{n}}f\left(t+\frac{k}{n}\right)dt\right)Z(nx-k). \tag{48}$$

We observe that

$$\left|C_n(f,x)-f(x)\right|=$$

$$\left|\sum_{k=-\infty}^{\infty}\left(n^N\int_0^{\frac{1}{n}}f\left(t+\frac{k}{n}\right)dt\right)Z(nx-k)-\sum_{k=-\infty}^{\infty}f(x)\,Z(nx-k)\right|=$$

$$\left|\sum_{k=-\infty}^{\infty}\left(\left(n^N\int_0^{\frac{1}{n}}f\left(t+\frac{k}{n}\right)dt\right)-f(x)\right)Z(nx-k)\right|=$$

$$\left|\sum_{k=-\infty}^{\infty}\left(n^N\int_0^{\frac{1}{n}}\left(f\left(t+\frac{k}{n}\right)-f(x)\right)dt\right)Z(nx-k)\right|\leq \tag{49}$$

$$\sum_{k=-\infty}^{\infty}\left(n^N\int_0^{\frac{1}{n}}\left|f\left(t+\frac{k}{n}\right)-f(x)\right|dt\right)Z(nx-k)=$$

$$\sum_{\left\{\substack{k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty \le \frac{1}{n^\beta}}\right.}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left| f\left(t+\frac{k}{n}\right) - f(x) \right| dt\right) Z(nx-k) +$$

$$\sum_{\left\{\substack{k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}}\right.}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left| f\left(t+\frac{k}{n}\right) - f(x) \right| dt\right) Z(nx-k) \le$$

$$\sum_{\left\{\substack{k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty \le \frac{1}{n^\beta}}\right.}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \omega_1\left(f, \|t\|_\infty + \left\|\frac{k}{n}-x\right\|_\infty\right) dt\right) Z(nx-k) +$$

$$2\|f\|_\infty \left(\sum_{\left\{\substack{k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}}\right.}^{\infty} Z(|nx-k|) \right) \le$$

$$\omega_1\left(f, \frac{1}{n}+\frac{1}{n^\beta}\right) + \frac{\|f\|_\infty}{\sqrt{\pi}\left(n^{1-\beta}-2\right) e^{\left(n^{1-\beta}-2\right)^2}}, \tag{50}$$

proving the claim. ■

We also present

Theorem 7 *Let $f \in C_B\left(\mathbb{R}^N\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $N, n \in \mathbb{N}$ with $n^{1-\beta} \ge 3$. Then*

1)

$$|D_n(f,x) - f(x)| \le \omega_1\left(f, \frac{1}{n}+\frac{1}{n^\beta}\right) + \frac{\|f\|_\infty}{\sqrt{\pi}\left(n^{1-\beta}-2\right) e^{\left(n^{1-\beta}-2\right)^2}} = \lambda_3, \tag{51}$$

2)

$$\|D_n(f) - f\|_\infty \le \lambda_3. \tag{52}$$

Given that $f \in \left(C_U\left(\mathbb{R}^N\right) \cap C_B\left(\mathbb{R}^N\right)\right)$, we obtain $\lim_{n\to\infty} D_n(f) = f$, uniformly.

Proof. We have that

$$|D_n(f,x) - f(x)| = \left| \sum_{k=-\infty}^{\infty} \delta_{nk}(f) Z(nx-k) - \sum_{k=-\infty}^{\infty} f(x) Z(nx-k) \right| = \tag{53}$$

$$\left|\sum_{k=-\infty}^{\infty}\left(\delta_{nk}\left(f\right)-f\left(x\right)\right)Z\left(nx-k\right)\right|=$$

$$\left|\sum_{k=-\infty}^{\infty}\left(\sum_{r=0}^{\theta}w_r\left(f\left(\frac{k}{n}+\frac{r}{n\theta}\right)-f\left(x\right)\right)\right)Z\left(nx-k\right)\right|\leq$$

$$\sum_{k=-\infty}^{\infty}\left(\sum_{r=0}^{\theta}w_r\left|f\left(\frac{k}{n}+\frac{r}{n\theta}\right)-f\left(x\right)\right|\right)Z\left(nx-k\right)=$$

$$\sum_{\begin{cases}k=-\infty\\ \left\|\frac{k}{n}-x\right\|_{\infty}\leq\frac{1}{n^{\beta}}\end{cases}}^{\infty}\left(\sum_{r=0}^{\theta}w_r\left|f\left(\frac{k}{n}+\frac{r}{n\theta}\right)-f\left(x\right)\right|\right)Z\left(nx-k\right)+$$

$$\sum_{\begin{cases}k=-\infty\\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}\end{cases}}^{\infty}\left(\sum_{r=0}^{\theta}w_r\left|f\left(\frac{k}{n}+\frac{r}{n\theta}\right)-f\left(x\right)\right|\right)Z\left(nx-k\right)\leq$$

$$\sum_{\begin{cases}k=-\infty\\ \left\|\frac{k}{n}-x\right\|_{\infty}\leq\frac{1}{n^{\beta}}\end{cases}}^{\infty}\left(\sum_{r=0}^{\theta}w_r\omega_1\left(f,\left\|\frac{k}{n}-x\right\|_{\infty}+\left\|\frac{r}{n\theta}\right\|_{\infty}\right)\right)Z\left(nx-k\right)+$$

$$2\left\|f\right\|_{\infty}\left(\sum_{\begin{cases}k=-\infty\\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}\end{cases}}^{\infty}Z\left(nx-k\right)\right)\leq$$

$$\omega_1\left(f,\frac{1}{n}+\frac{1}{n^{\beta}}\right)+\frac{\left\|f\right\|_{\infty}}{\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}},\tag{54}$$

proving the claim. ■

In the next we discuss high order of approximation by using the smoothness of f.

We give

Theorem 8 *Let* $f\in C^m\left(\prod_{i=1}^{N}\left[a_i,b_i\right]\right)$, $0<\beta<1$, $n,m,N\in\mathbb{N}$, $n^{1-\beta}\geq 3$, $x\in\left(\prod_{i=1}^{N}\left[a_i,b_i\right]\right)$. *Then*

i)

$$\left|A_n\left(f,x\right)-f\left(x\right)-\sum_{j=1}^{m}\left(\sum_{|\alpha|=j}\left(\frac{f_{\alpha}\left(x\right)}{\prod_{i=1}^{N}\alpha_i!}\right)A_n\left(\prod_{i=1}^{N}\left(\cdot-x_i\right)^{\alpha_i},x\right)\right)\right|\leq\tag{55}$$

$$(4.019)^{N}\cdot\left\{\frac{N^{m}}{m!n^{m\beta}}\omega_{1,m}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)+\right.$$

$$\left.\left(\frac{\|b-a\|_{\infty}^{m}\|f_{\alpha}\|_{\infty,m}^{\max}N^{m}}{m!}\right)\frac{1}{\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^{2}}}\right\},$$

ii)

$$|A_{n}(f,x)-f(x)|\leq(4.019)^{N}\cdot \tag{56}$$

$$\left\{\sum_{j=1}^{m}\left(\sum_{|\alpha|=j}\left(\frac{|f_{\alpha}(x)|}{\prod_{i=1}^{N}\alpha_{i}!}\right)\left[\frac{1}{n^{\beta j}}+\left(\prod_{i=1}^{N}(b_{i}-a_{i})^{\alpha_{i}}\right)\cdot\right.\right.\right.$$

$$\left.\left.\frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^{2}}}\right]\right)+\frac{N^{m}}{m!n^{m\beta}}\omega_{1,m}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)$$

$$\left.+\left(\frac{\|b-a\|_{\infty}^{m}\|f_{\alpha}\|_{\infty,m}^{\max}N^{m}}{m!}\right)\frac{1}{\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^{2}}}\right\},$$

iii)

$$\|A_{n}(f)-f\|_{\infty}\leq(4.019)^{N}\cdot \tag{57}$$

$$\left\{\sum_{j=1}^{m}\left(\sum_{|\alpha|=j}\left(\frac{|f_{\alpha}(x)|}{\prod_{i=1}^{N}\alpha_{i}!}\right)\left[\frac{1}{n^{\beta j}}+\left(\prod_{i=1}^{N}(b_{i}-a_{i})^{\alpha_{i}}\right)\cdot\right.\right.\right.$$

$$\left.\left.\frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^{2}}}\right]\right)+\frac{N^{m}}{m!n^{m\beta}}\omega_{1,m}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)$$

$$\left.+\left(\frac{\|b-a\|_{\infty}^{m}\|f_{\alpha}\|_{\infty,m}^{\max}N^{m}}{m!}\right)\frac{1}{\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^{2}}}\right\}=:K_{n},$$

iv) Assume $f_{\alpha}(x_{0})=0$, for all $\alpha:|\alpha|=1,...,m$; $x_{0}\in\left(\prod_{i=1}^{N}[a_{i},b_{i}]\right)$. Then

$$|A_{n}(f,x_{0})-f(x_{0})|\leq(4.019)^{N}\left\{\frac{N^{m}}{m!n^{m\beta}}\omega_{1}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)+\right. \tag{58}$$

$$\left.\left(\frac{\|b-a\|_{\infty}^{m}\|f_{\alpha}\|_{\infty,m}^{\max}N^{m}}{m!}\right)\frac{1}{\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^{2}}}\right\},$$

notice in the last the extremely high rate of convergence at $n^{-\beta(m+1)}$.

Proof. Consider $g_{z}(t):=f(x_{0}+t(z-x_{0}))$, $t\geq 0$; $x_{0},z\in\prod_{i=1}^{N}[a_{i},b_{i}]$. Then

$$g_{z}^{(j)}(t)=\left[\left(\sum_{i=1}^{N}(z_{i}-x_{0i})\frac{\partial}{\partial x_{i}}\right)^{j}f\right](x_{01}+t(z_{1}-x_{01}),...,x_{0N}+t(z_{N}-x_{0N})), \tag{59}$$

for all $j = 0, 1, ..., m$.

We have the multivariate Taylor's formula

$$f(z_1, ..., z_N) = g_z(1) =$$

$$\sum_{j=0}^{m} \frac{g_z^{(j)}(0)}{j!} + \frac{1}{(m-1)!} \int_0^1 (1-\theta)^{m-1} \left(g_z^{(m)}(\theta) - g_z^{(m)}(0) \right) d\theta. \quad (60)$$

Notice $g_z(0) = f(x_0)$. Also for $j = 0, 1, ..., m$, we have

$$g_z^{(j)}(0) = \sum_{\substack{\alpha:=(\alpha_1,...,\alpha_N),\ \alpha_i \in \mathbb{Z}^+, \\ i=1,...,N,\ |\alpha|:=\sum_{i=1}^N \alpha_i = j}} \left(\frac{j!}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N (z_i - x_{0i})^{\alpha_i} \right) f_\alpha(x_0). \quad (61)$$

Furthermore

$$g_z^{(m)}(\theta) =$$

$$\sum_{\substack{\alpha:=(\alpha_1,...,\alpha_N),\ \alpha_i \in \mathbb{Z}^+, \\ i=1,...,N,\ |\alpha|:=\sum_{i=1}^N \alpha_i = m}} \left(\frac{m!}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N (z_i - x_{0i})^{\alpha_i} \right) f_\alpha(x_0 + \theta(z - x_0)), \quad (62)$$

$0 \leq \theta \leq 1$.

So we treat $f \in C^m\left(\prod_{i=1}^N [a_i, b_i]\right)$.

Thus, we have for $\frac{k}{n}, x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$ that

$$f\left(\frac{k_1}{n}, ..., \frac{k_N}{n}\right) - f(x) =$$

$$\sum_{j=1}^{m} \sum_{\substack{\alpha:=(\alpha_1,...,\alpha_N),\ \alpha_i \in \mathbb{Z}^+, \\ i=1,...,N,\ |\alpha|:=\sum_{i=1}^N \alpha_i = j}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left(\frac{k_i}{n} - x_i \right)^{\alpha_i} \right) f_\alpha(x) + R, \quad (63)$$

where

$$R := m \int_0^1 (1-\theta)^{m-1} \sum_{\substack{\alpha:=(\alpha_1,...,\alpha_N),\ \alpha_i \in \mathbb{Z}^+, \\ i=1,...,N,\ |\alpha|:=\sum_{i=1}^N \alpha_i = m}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left(\frac{k_i}{n} - x_i \right)^{\alpha_i} \right)$$

$$\cdot \left[f_\alpha\left(x + \theta\left(\frac{k}{n} - x \right) \right) - f_\alpha(x) \right] d\theta. \quad (64)$$

We see that

$$|R| \leq m \int_0^1 (1-\theta)^{m-1} \sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left| \frac{k_i}{n} - x_i \right|^{\alpha_i} \right) \cdot$$

$$\left|f_{\alpha}\left(x+\theta\left(\frac{k}{n}-x\right)\right)-f_{\alpha}(x)\right| d\theta \leq m\int_{0}^{1}(1-\theta)^{m-1}\cdot \tag{65}$$

$$\left(\sum_{|\alpha|=m}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}\left|\frac{k_i}{n}-x_i\right|^{\alpha_i}\right)\omega_1\left(f_{\alpha},\theta\left\|\frac{k}{n}-x\right\|_{\infty}\right)\right)d\theta \leq (*) .$$

Notice here that

$$\left\|\frac{k}{n}-x\right\|_{\infty} \leq \frac{1}{n^{\beta}} \Leftrightarrow \left|\frac{k_i}{n}-x_i\right| \leq \frac{1}{n^{\beta}},\ \ i=1,...,N. \tag{66}$$

We further see that

$$(*) \leq m\cdot\omega_{1,m}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)\int_{0}^{1}(1-\theta)^{m-1}\left(\sum_{|\alpha|=m}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}\left(\frac{1}{n^{\beta}}\right)^{\alpha_i}\right)\right)d\theta =$$

$$\left(\frac{\omega_{1,m}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)}{(m!)\,n^{m\beta}}\right)\left(\sum_{|\alpha|=m}\frac{m!}{\prod_{i=1}^{N}\alpha_i!}\right)=\left(\frac{\omega_{1,m}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right)}{(m!)\,n^{m\beta}}\right)N^{m}. \tag{67}$$

Conclusion: When $\left\|\frac{k}{n}-x\right\|_{\infty} \leq \frac{1}{n^{\beta}}$, we proved that

$$|R| \leq \left(\frac{N^m}{m!n^{m\beta}}\right)\omega_{1,m}^{\max}\left(f_{\alpha},\frac{1}{n^{\beta}}\right). \tag{68}$$

In general we notice that

$$|R| \leq m\int_{0}^{1}(1-\theta)^{m-1}\left(\sum_{|\alpha|=m}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)2\left\|f_{\alpha}\right\|_{\infty}\right)d\theta =$$

$$2\sum_{|\alpha|=m}\frac{1}{\prod_{i=1}^{N}\alpha_i!}\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)\left\|f_{\alpha}\right\|_{\infty} \leq$$

$$\left(\frac{2\left\|b-a\right\|_{\infty}^{m}\left\|f_{\alpha}\right\|_{\infty,m}^{\max}}{m!}\right)\left(\sum_{|\alpha|=m}\frac{m!}{\prod_{i=1}^{N}\alpha_i!}\right)=\frac{2\left\|b-a\right\|_{\infty}^{m}\left\|f_{\alpha}\right\|_{\infty,m}^{\max}N^m}{m!}. \tag{69}$$

We proved in general that

$$|R| \leq \frac{2\left\|b-a\right\|_{\infty}^{m}\left\|f_{\alpha}\right\|_{\infty,m}^{\max}N^m}{m!} := \rho. \tag{70}$$

Next we see that

$$U_n := \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)\,R =$$

$$\sum_{\substack{k=\lceil na\rceil \\ : \left\|\frac{k}{n}-x\right\|_\infty \leq \frac{1}{n^\beta}}}^{\lfloor nb\rfloor} Z(nx-k)\,R + \sum_{\substack{k=\lceil na\rceil \\ : \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb\rfloor} Z(nx-k)\,R.$$

Consequently

$$|U_n| \leq \left(\sum_{\substack{k=\lceil na\rceil \\ : \left\|\frac{k}{n}-x\right\|_\infty \leq \frac{1}{n^\beta}}}^{\lfloor nb\rfloor} Z(nx-k) \right) \frac{N^m}{m! n^{m\beta}} \omega_{1,m}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right)$$

$$+\rho \frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right) e^{\left(n^{1-\beta}-2\right)^2}}$$

$$\leq \frac{N^m}{m! n^{m\beta}} \omega_{1,m}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) + \rho \frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right) e^{\left(n^{1-\beta}-2\right)^2}}. \tag{71}$$

We have established that

$$|U_n| \leq \frac{N^m}{m! n^{m\beta}} \omega_{1,m}^{\max}\left(f_\alpha, \frac{1}{n^\beta}\right) +$$

$$\left(\frac{\|b-a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) \frac{1}{\sqrt{\pi}\left(n^{1-\beta}-2\right) e^{\left(n^{1-\beta}-2\right)^2}}. \tag{72}$$

We observe that

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right) Z(nx-k) - f(x) \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k) =$$

$$\sum_{j=1}^{m} \left(\sum_{|\alpha|=j} \left(\frac{f_\alpha(x)}{\prod_{i=1}^N \alpha_i!} \right) \left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k) \left(\prod_{i=1}^{N} \left(\frac{k_i}{n} - x_i \right)^{\alpha_i} \right) \right) \right)$$

$$+ \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)\,R. \tag{73}$$

The last says

$$A_n^*(f,x) - f(x) \left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k) \right) -$$

$$\sum_{j=1}^{m} \left(\sum_{|\alpha|=j} \left(\frac{f_\alpha(x)}{\prod_{i=1}^N \alpha_i!} \right) A_n^*\left(\prod_{i=1}^{N} (\cdot - x_i)^{\alpha_i}, x \right) \right) = U_n. \tag{74}$$

Clearly A_n^* is a positive linear operator.

Thus (here $\alpha_i \in \mathbb{Z}^+ : |\alpha| = \sum_{i=1}^N \alpha_i = j$)

$$\left| A_n^* \left(\prod_{i=1}^N (\cdot - x_i)^{\alpha_i}, x \right) \right| \leq A_n^* \left(\prod_{i=1}^N |\cdot - x_i|^{\alpha_i}, x \right) =$$

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\prod_{i=1}^N \left| \frac{k_i}{n} - x_i \right|^{\alpha_i} \right) Z(nx-k) =$$

$$\sum_{\substack{k=\lceil na \rceil \\ : \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} \left(\prod_{i=1}^N \left| \frac{k_i}{n} - x_i \right|^{\alpha_i} \right) Z(nx-k) +$$

$$\sum_{\substack{k=\lceil na \rceil \\ : \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} \left(\prod_{i=1}^N \left| \frac{k_i}{n} - x_i \right|^{\alpha_i} \right) Z(nx-k) \leq$$

$$\frac{1}{n^{\beta j}} + \prod_{i=1}^N (b_i - a_i)^{\alpha_i} \left(\sum_{\substack{k=\lceil na \rceil \\ : \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} Z(nx-k) \right) \leq$$

$$\frac{1}{n^{\beta j}} + \left(\prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) \frac{1}{2\sqrt{\pi} \left(n^{1-\beta} - 2 \right) e^{\left(n^{1-\beta} - 2 \right)^2}}. \tag{75}$$

So we have proved that

$$\left| A_n^* \left(\prod_{i=1}^N (\cdot - x_i)^{\alpha_i}, x \right) \right| \leq \frac{1}{n^{\beta j}} + \left(\prod_{i=1}^N (b_i - a_i)^{\alpha_i} \right) \frac{1}{2\sqrt{\pi} \left(n^{1-\beta} - 2 \right) e^{\left(n^{1-\beta} - 2 \right)^2}}, \tag{76}$$

for all $j = 1, ..., m$.

At last we observe

$$\left| A_n(f,x) - f(x) - \sum_{j=1}^m \left(\sum_{|\alpha|=j} \left(\frac{f_\alpha(x)}{\prod_{i=1}^N \alpha_i!} \right) A_n \left(\prod_{i=1}^N (\cdot - x_i)^{\alpha_i}, x \right) \right) \right| \leq$$

$$(4.019)^N \cdot \left| A_n^*(f,x) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) - \right.$$

$$\left. \sum_{j=1}^m \left(\sum_{|\alpha|=j} \left(\frac{f_\alpha(x)}{\prod_{i=1}^N \alpha_i!} \right) A_n^* \left(\prod_{i=1}^N (\cdot - x_i)^{\alpha_i}, x \right) \right) \right|. \tag{77}$$

Putting all of the above together we prove theorem. ■

We make

Definition 9 *Let* $f \in C_B(\mathbb{R}^N)$, $N \in \mathbb{N}$. *We define the general neural network operator*

$$F_n(f,x) := \sum_{k=-\infty}^{\infty} l_{nk}(f) Z(nx-k) =$$

$$\begin{cases} B_n(f,x), & \text{if } l_{nk}(f) = f\left(\frac{k}{n}\right), \\ C_n(f,x), & \text{if } l_{nk}(f) = n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt, \\ D_n(f,x), & \text{if } l_{nk}(f) = \delta_{nk}(f). \end{cases} \tag{78}$$

Clearly $l_{nk}(f)$ is a positive linear functional such that $|l_{nk}(f)| \leq \|f\|_\infty$.

Hence $F_n(f)$ is a positive linear operator with $\|F_n(f)\|_\infty \leq \|f\|_\infty$, a continuous bounded linear operator.

We need

Theorem 10 *Let* $f \in C_B(\mathbb{R}^N)$, $N \geq 1$. *Then* $F_n(f) \in C_B(\mathbb{R}^N)$.

Proof. Clearly $F_n(f)$ is a bounded function.

Next we prove the continuity of $F_n(f)$. Notice for $N = 1$, $Z = \chi$ by (10).

We will use the Weierstrass M test: If a sequence of positive constants $M_1, M_2, M_3, ...$, can be found such that in some interval

(a) $|u_n(x)| \leq M_n$, $n = 1, 2, 3, ...$

(b) $\sum M_n$ converges,

then $\sum u_n(x)$ is uniformly and absolutely convergent in the interval.

Also we will use:

If $\{u_n(x)\}$, $n = 1, 2, 3, ...$ are continuous in $[a,b]$ and if $\sum u_n(x)$ converges uniformly to the sum $S(x)$ in $[a,b]$, then $S(x)$ is continuous in $[a,b]$. I.e. a uniformly convergent series of continuous functions is a continuous function. First we prove claim for $N = 1$.

We will prove that $\sum_{k=-\infty}^{\infty} l_{nk}(f) \chi(nx-k)$ is continuous in $x \in \mathbb{R}$.

There always exists $\lambda \in \mathbb{N}$ such that $nx \in [-\lambda, \lambda]$.

Since $nx \leq \lambda$, then $-nx \geq -\lambda$ and $k - nx \geq k - \lambda \geq 0$, when $k \geq \lambda$. Therefore

$$\sum_{k=\lambda}^{\infty} \chi(nx-k) = \sum_{k=\lambda}^{\infty} \chi(k-nx) \leq \sum_{k=\lambda}^{\infty} \chi(k-\lambda) = \sum_{k'=0}^{\infty} \chi(k') \leq 1. \tag{79}$$

So for $k \geq \lambda$ we get

$$|l_{nk}(f)| \chi(nx-k) \leq \|f\|_\infty \chi(k-\lambda),$$

and

$$\|f\|_\infty \sum_{k=\lambda}^{\infty} \chi(k-\lambda) \leq \|f\|_\infty .$$

Hence by Weierstrass M test we obtain that $\sum_{k=\lambda}^{\infty} l_{nk}(f)\chi(nx-k)$ is uniformly and absolutely convergent on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

Since $l_{nk}(f)\chi(nx-k)$ is continuous in x, then $\sum_{k=\lambda}^{\infty} l_{nk}(f)\chi(nx-k)$ is continuous on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

Because $nx \geq -\lambda$, then $-nx \leq \lambda$, and $k-nx \leq k+\lambda \leq 0$, when $k \leq -\lambda$. Therefore

$$\sum_{k=-\infty}^{-\lambda} \chi(nx-k) = \sum_{k=-\infty}^{-\lambda} \chi(k-nx) \leq \sum_{k=-\infty}^{-\lambda} \chi(k+\lambda) = \sum_{k'=-\infty}^{0} \chi(k') \leq 1.$$

So for $k \leq -\lambda$ we get

$$|l_{nk}(f)|\chi(nx-k) \leq \|f\|_\infty \chi(k+\lambda), \tag{80}$$

and

$$\|f\|_\infty \sum_{k=-\infty}^{-\lambda} \chi(k+\lambda) \leq \|f\|_\infty .$$

Hence by Weierstrass M test we obtain that $\sum_{k=-\infty}^{-\lambda} l_{nk}(f)\chi(nx-k)$ is uniformly and absolutely convergent on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

Since $l_{nk}(f)\chi(nx-k)$ is continuous in x, then $\sum_{k=-\infty}^{-\lambda} l_{nk}(f)\chi(nx-k)$ is continuous on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

So we proved that $\sum_{k=\lambda}^{\infty} l_{nk}(f)\chi(nx-k)$ and $\sum_{k=-\infty}^{-\lambda} l_{nk}(f)\chi(nx-k)$ are continuous on $\mathbb{R}$. Since $\sum_{k=-\lambda+1}^{\lambda-1} l_{nk}(f)\chi(nx-k)$ is a finite sum of continuous functions on $\mathbb{R}$, it is also a continuous function on $\mathbb{R}$.

Writing

$$\sum_{k=-\infty}^{\infty} l_{nk}(f)\chi(nx-k) = \sum_{k=-\infty}^{-\lambda} l_{nk}(f)\chi(nx-k) +$$

$$\sum_{k=-\lambda+1}^{\lambda-1} l_{nk}(f)\chi(nx-k) + \sum_{k=\lambda}^{\infty} l_{nk}(f)\chi(nx-k) \tag{81}$$

we have it as a continuous function on $\mathbb{R}$. Therefore $F_n(f)$, when $N=1$, is a continuous function on $\mathbb{R}$.

When $N=2$ we have

$$F_n(f, x_1, x_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} l_{nk}(f)\chi(nx_1-k_1)\chi(nx_2-k_2) =$$

$$\sum_{k_1=-\infty}^{\infty} \chi(nx_1 - k_1) \left(\sum_{k_2=-\infty}^{\infty} l_{nk}(f) \chi(nx_2 - k_2) \right)$$

(there always exist $\lambda_1, \lambda_2 \in \mathbb{N}$ such that $nx_1 \in [-\lambda_1, \lambda_1]$ and $nx_2 \in [-\lambda_2, \lambda_2]$)

$$= \sum_{k_1=-\infty}^{\infty} \chi(nx_1 - k_1) \left[\sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f) \chi(nx_2 - k_2) + \right.$$

$$\left. \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} l_{nk}(f) \chi(nx_2 - k_2) + \sum_{k_2=\lambda_2}^{\infty} l_{nk}(f) \chi(nx_2 - k_2) \right] =$$

$$= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f) \chi(nx_1 - k_1) \chi(nx_2 - k_2) +$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} l_{nk}(f) \chi(nx_1 - k_1) \chi(nx_2 - k_2) +$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=\lambda_2}^{\infty} l_{nk}(f) \chi(nx_1 - k_1) \chi(nx_2 - k_2) =: (*) .$$

(For convenience call

$$F(k_1, k_2, x_1, x_2) := l_{nk}(f) \chi(nx_1 - k_1) \chi(nx_2 - k_2) .)$$

Thus

$$(*) = \sum_{k_1=-\infty}^{-\lambda_1} \sum_{k_2=-\infty}^{-\lambda_2} F(k_1, k_2, x_1, x_2) + \sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=-\infty}^{-\lambda_2} F(k_1, k_2, x_1, x_2) +$$

$$\sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} F(k_1, k_2, x_1, x_2) + \sum_{k_1=-\infty}^{-\lambda_1} \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} F(k_1, k_2, x_1, x_2) +$$

$$\sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} F(k_1, k_2, x_1, x_2) + \sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} F(k_1, k_2, x_1, x_2) +$$

$$\sum_{k_1=-\infty}^{-\lambda_1} \sum_{k_2=\lambda_2}^{\infty} F(k_1, k_2, x_1, x_2) + \sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=\lambda_2}^{\infty} F(k_1, k_2, x_1, x_2) + \tag{82}$$

$$\sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=\lambda_2}^{\infty} F(k_1, k_2, x_1, x_2) .$$

Notice that the finite sum of continuous functions $F(k_1, k_2, x_1, x_2)$, $\sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} F(k_1, k_2, x_1, x_2)$ is a continuous function.

The rest of the summands of $F_n(f, x_1, x_2)$ are treated all the same way and similarly to the case of $N = 1$. The method is demonstrated as follows.

We will prove that $\sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f)\, \chi(nx_1 - k_1)\, \chi(nx_2 - k_2)$ is continuous in $(x_1, x_2) \in \mathbb{R}^2$.

The continuous function

$$|l_{nk}(f)|\, \chi(nx_1 - k_1)\, \chi(nx_2 - k_2) \leq \|f\|_\infty\, \chi(k_1 - \lambda_1)\, \chi(k_2 + \lambda_2),$$

and

$$\|f\|_\infty \sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} \chi(k_1 - \lambda_1)\, \chi(k_2 + \lambda_2) =$$

$$\|f\|_\infty \left(\sum_{k_1=\lambda_1}^{\infty} \chi(k_1 - \lambda_1) \right) \left(\sum_{k_2=-\infty}^{-\lambda_2} \chi(k_2 + \lambda_2) \right) \leq$$

$$\|f\|_\infty \left(\sum_{k_1'=0}^{\infty} \chi(k_1') \right) \left(\sum_{k_2'=-\infty}^{0} \chi(k_2') \right) \leq \|f\|_\infty .$$

So by the Weierstrass M test we get that $\sum_{k_1=\lambda_1}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f)\, \chi(nx_1 - k_1)\, \chi(nx_2 - k_2)$ is uniformly and absolutely convergent. Therefore it is continuous on $\mathbb{R}^2$.

Next we prove continuity on $\mathbb{R}^2$ of $\sum_{k_1=-\lambda_1+1}^{\lambda_1-1} \sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f)\, \chi(nx_1 - k_1)\, \chi(nx_2 - k_2)$.

Notice here that

$$|l_{nk}(f)|\, \chi(nx_1 - k_1)\, \chi(nx_2 - k_2) \leq \|f\|_\infty\, \chi(nx_1 - k_1)\, \chi(k_2 + \lambda_2)$$

$$\leq \|f\|_\infty\, \chi(0)\, \chi(k_2 + \lambda_2) = 0.4215 \cdot \|f\|_\infty\, \chi(k_2 + \lambda_2),$$

and

$$0.4215 \cdot \|f\|_\infty \left(\sum_{k_1=-\lambda_1+1}^{\lambda_1-1} 1 \right) \left(\sum_{k_2=-\infty}^{-\lambda_2} \chi(k_2 + \lambda_2) \right) =$$

$$0.4215 \cdot \|f\|_\infty (2\lambda_1 - 1) \left(\sum_{k_2'=-\infty}^{0} \chi(k_2') \right) \leq 0.4215 \cdot (2\lambda_1 - 1)\, \|f\|_\infty . \quad (83)$$

So the double series under consideration is uniformly convergent and continuous. Clearly $F_n(f, x_1, x_2)$ is proved to be continuous on $\mathbb{R}^2$.

Similarly reasoning one can prove easily now, but with more tedious work, that $F_n(f, x_1, ..., x_N)$ is continuous on $\mathbb{R}^N$, for any $N \geq 1$. We choose to omit this similar extra work. ■

Remark 11 *By (24) it is obvious that* $\|A_n(f)\|_\infty \leq \|f\|_\infty < \infty$*, and* $A_n(f) \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$*, given that* $f \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$*.*

Call L_n *any of the operators* A_n, B_n, C_n, D_n*.*

Clearly then

$$\left\|L_n^2(f)\right\|_\infty = \|L_n(L_n(f))\|_\infty \leq \|L_n(f)\|_\infty \leq \|f\|_\infty, \tag{84}$$

etc.

Therefore we get

$$\left\|L_n^k(f)\right\|_\infty \leq \|f\|_\infty, \ \ \forall\ k \in \mathbb{N}, \tag{85}$$

the contraction property.

Also we see that

$$\left\|L_n^k(f)\right\|_\infty \leq \left\|L_n^{k-1}(f)\right\|_\infty \leq ... \leq \|L_n(f)\|_\infty \leq \|f\|_\infty. \tag{86}$$

Also $L_n(1) = 1$*,* $L_n^k(1) = 1$*,* $\forall\ k \in \mathbb{N}$*.*

Here L_n^k *are positive linear operators.*

Notation 12 *Here* $N \in \mathbb{N}$*,* $0 < \beta < 1$*. Denote by*

$$c_N := \begin{cases} (4.019)^N, & \text{if } L_n = A_n, \\ 1, & \text{if } L_n = B_n, C_n, D_n, \end{cases} \tag{87}$$

$$\varphi(n) := \begin{cases} \frac{1}{n^\beta}, & \text{if } L_n = A_n,\ B_n, \\ \frac{1}{n} + \frac{1}{n^\beta}, & \text{if } L_n = C_n, D_n, \end{cases} \tag{88}$$

$$\Omega := \begin{cases} C\left(\prod_{i=1}^N [a_i, b_i]\right), & \text{if } L_n = A_n, \\ C_B\left(\mathbb{R}^N\right), & \text{if } L_n = B_n, C_n, D_n, \end{cases} \tag{89}$$

and

$$Y := \begin{cases} \prod_{i=1}^N [a_i, b_i], & \text{if } L_n = A_n, \\ \mathbb{R}^N, & \text{if } L_n = B_n, C_n, D_n. \end{cases} \tag{90}$$

We give the condensed

Theorem 13 *Let* $f \in \Omega$*,* $0 < \beta < 1$*,* $x \in Y$*;* n*,* $N \in \mathbb{N}$ *with* $n^{1-\beta} \geq 3$*. Then*

(i)

$$|L_n(f,x) - f(x)| \leq c_N \left[\omega_1(f, \varphi(n)) + \frac{\|f\|_\infty}{\sqrt{\pi}\left(n^{1-\beta} - 2\right) e^{\left(n^{1-\beta}-2\right)^2}}\right] =: \tau, \tag{91}$$

(ii)

$$\left\| L_n(f) - f \right\|_\infty \leq \tau. \tag{92}$$

For f uniformly continuous and in Ω we obtain

$$\lim_{n\to\infty} L_n(f) = f,$$

pointwise and uniformly.

Proof. By Theorems 4-7. ■

Next we do iterated neural network approximation (see also [9]).

We make

Remark 14 *Let $r \in \mathbb{N}$ and L_n as above. We observe that*

$$L_n^r f - f = \left(L_n^r f - L_n^{r-1} f\right) + \left(L_n^{r-1} f - L_n^{r-2} f\right) +$$

$$\left(L_n^{r-2} f - L_n^{r-3} f\right) + \ldots + \left(L_n^2 f - L_n f\right) + \left(L_n f - f\right).$$

Then

$$\left\| L_n^r f - f \right\|_\infty \leq \left\| L_n^r f - L_n^{r-1} f \right\|_\infty + \left\| L_n^{r-1} f - L_n^{r-2} f \right\|_\infty +$$

$$\left\| L_n^{r-2} f - L_n^{r-3} f \right\|_\infty + \ldots + \left\| L_n^2 f - L_n f \right\|_\infty + \left\| L_n f - f \right\|_\infty =$$

$$\left\| L_n^{r-1}\left(L_n f - f\right) \right\|_\infty + \left\| L_n^{r-2}\left(L_n f - f\right) \right\|_\infty + \left\| L_n^{r-3}\left(L_n f - f\right) \right\|_\infty$$

$$+ \ldots + \left\| L_n\left(L_n f - f\right) \right\|_\infty + \left\| L_n f - f \right\|_\infty \leq r \left\| L_n f - f \right\|_\infty. \tag{93}$$

That is

$$\left\| L_n^r f - f \right\|_\infty \leq r \left\| L_n f - f \right\|_\infty. \tag{94}$$

We give

Theorem 15 *All here as in Theorem 13 and $r \in \mathbb{N}$, τ as in (91). Then*

$$\left\| L_n^r f - f \right\|_\infty \leq r\tau. \tag{95}$$

So that the speed of convergence to the unit operator of L_n^r is not worse than of L_n.

Proof. By (94) and (92). ■

We make

Remark 16 *Let $m_1, \ldots, m_r \in \mathbb{N} : m_1 \leq m_2 \leq \ldots \leq m_r$, $0 < \beta < 1$, $f \in \Omega$. Then $\varphi(m_1) \geq \varphi(m_2) \geq \ldots \geq \varphi(m_r)$, φ as in (88).*

Therefore

$$\omega_1\left(f, \varphi(m_1)\right) \geq \omega_1\left(f, \varphi(m_2)\right) \geq \ldots \geq \omega_1\left(f, \varphi(m_r)\right). \tag{96}$$

Assume further that $m_i^{1-\beta} \geq 3$, $i = 1, ..., r$. *Then*

$$\frac{1}{\left(m_1^{1-\beta} - 2\right) e^{\left(m_1^{1-\beta}-2\right)^2}} \geq \frac{1}{\left(m_2^{1-\beta} - 2\right) e^{\left(m_2^{1-\beta}-2\right)^2}}$$

$$\geq ... \geq \frac{1}{\left(m_r^{1-\beta} - 2\right) e^{\left(m_r^{1-\beta}-2\right)^2}}. \tag{97}$$

Let L_{m_i} *as above,* $i = 1, ..., r$, *all of the same kind.*

We write

$$L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right) - f =$$
$$L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right) - L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}f\right)\right) +$$
$$L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}f\right)\right) - L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_3}f\right)\right) +$$
$$L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_3}f\right)\right) - L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_4}f\right)\right) + ... + \tag{98}$$
$$L_{m_r}\left(L_{m_{r-1}}f\right) - L_{m_r}f + L_{m_r}f - f =$$
$$L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\right)\right)\left(L_{m_1}f - f\right) + L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_3}\right)\right)\left(L_{m_2}f - f\right) +$$
$$L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_4}\right)\right)\left(L_{m_3}f - f\right) + ... + L_{m_r}\left(L_{m_{r-1}}f - f\right) + L_{m_r}f - f.$$

Hence by the triangle inequality property of $\|\cdot\|_\infty$ *we get*

$$\left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right) - f\right\|_\infty \leq$$
$$\left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\right)\right)\left(L_{m_1}f - f\right)\right\|_\infty + \left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_3}\right)\right)\left(L_{m_2}f - f\right)\right\|_\infty +$$
$$\left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_4}\right)\right)\left(L_{m_3}f - f\right)\right\|_\infty + ... +$$
$$\left\|L_{m_r}\left(L_{m_{r-1}}f - f\right)\right\|_\infty + \left\|L_{m_r}f - f\right\|_\infty$$

(repeatedly applying (84))

$$\leq \left\|L_{m_1}f - f\right\|_\infty + \left\|L_{m_2}f - f\right\|_\infty + \left\|L_{m_3}f - f\right\|_\infty + ... +$$
$$\left\|L_{m_{r-1}}f - f\right\|_\infty + \left\|L_{m_r}f - f\right\|_\infty = \sum_{i=1}^{r} \left\|L_{m_i}f - f\right\|_\infty. \tag{99}$$

That is, we proved

$$\left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right) - f\right\|_\infty \leq \sum_{i=1}^{r} \left\|L_{m_i}f - f\right\|_\infty. \tag{100}$$

We give

Theorem 17 *Let $f \in \Omega$; N, $m_1, m_2, ..., m_r \in \mathbb{N} : m_1 \leq m_2 \leq ... \leq m_r$, $0 < \beta < 1$; $m_i^{1-\beta} \geq 3$, $i = 1, ..., r$, $x \in Y$, and let $(L_{m_1}, ..., L_{m_r})$ as $(A_{m_1}, ..., A_{m_r})$ or $(B_{m_1}, ..., B_{m_r})$ or $(C_{m_1}, ..., C_{m_r})$ or $(D_{m_1}, ..., D_{m_r})$. Then*

$$\left|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right)(x) - f(x)\right| \leq$$

$$\left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right) - f\right\|_\infty \leq$$

$$\sum_{i=1}^{r} \left\|L_{m_i}f - f\right\|_\infty \leq$$

$$c_N \sum_{i=1}^{r} \left[\omega_1\left(f, \varphi\left(m_i\right)\right) + \frac{\|f\|_\infty}{\sqrt{\pi}\left(m_i^{1-\beta} - 2\right) e^{\left(m_i^{1-\beta}-2\right)^2}}\right] \leq$$

$$r c_N \left[\omega_1\left(f, \varphi\left(m_1\right)\right) + \frac{\|f\|_\infty}{\sqrt{\pi}\left(m_1^{1-\beta} - 2\right) e^{\left(m_1^{1-\beta}-2\right)^2}}\right]. \tag{101}$$

Clearly, we notice that the speed of convergence to the unit operator of the multiply iterated operator is not worse than the speed of L_{m_1}.

Proof. Using (100), (96), (97) and (91), (92). ■

We continue with

Theorem 18 *Let all as in Theorem 8, and $r \in \mathbb{N}$. Here K_n is as in (57). Then*

$$\left\|A_n^r f - f\right\|_\infty \leq r \left\|A_n f - f\right\|_\infty \leq r K_n. \tag{102}$$

Proof. By (94) and (57). ■

4 Complex Multivariate Neural Network Approximations

We make

Remark 19 *Let $Y = \prod_{i=1}^{n} [a_i, b_i]$ or $\mathbb{R}^N$, and $f : Y \to \mathbb{C}$ with real and imaginary parts $f_1, f_2 : f = f_1 + i f_2$, $i = \sqrt{-1}$. Clearly f is continuous iff f_1 and f_2 are continuous.*

Given that $f_1, f_2 \in C^m(Y)$, $m \in \mathbb{N}$, it holds

$$f_\alpha(x) = f_{1,\alpha}(x) + i f_{2,\alpha}(x), \tag{103}$$

where α indicates a partial derivative of any order and arrangement.

We denote by $C_B\left(\mathbb{R}^N,\mathbb{C}\right)$ the space of continuous and bounded functions $f:\mathbb{R}^N\rightarrow\mathbb{C}$. Clearly f is bounded, iff both f_1,f_2 are bounded from $\mathbb{R}^N$ into $\mathbb{R}$, where $f=f_1+if_2$.

Here L_n is any of A_n,B_n,C_n,D_n, $n\in\mathbb{N}$.

We define

$$L_n(f,x):=L_n(f_1,x)+iL_n(f_2,x),\quad\forall\ x\in Y. \tag{104}$$

We observe that

$$|L_n(f,x)-f(x)|\leq|L_n(f_1,x)-f_1(x)|+|L_n(f_2,x)-f_2(x)|, \tag{105}$$

and

$$\|L_n(f)-f\|_\infty\leq\|L_n(f_1)-f_1\|_\infty+\|L_n(f_2)-f_2\|_\infty. \tag{106}$$

We present

Theorem 20 *Let $f\in C(Y,\mathbb{C})$ which is bounded, $f=f_1+if_2$, $0<\beta<1$, $n,N\in\mathbb{N}:n^{1-\beta}\geq3,$, $x\in Y$. Then*

i)

$$|L_n(f,x)-f(x)|\leq c_N\cdot$$
$$\left[\omega_1(f_1,\varphi(n))+\omega_1(f_2,\varphi(n_2))+\frac{(\|f_1\|_\infty+\|f_2\|_\infty)}{\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}\right]=:\varepsilon, \tag{107}$$

ii)

$$\|L_n(f)-f\|_\infty\leq\varepsilon. \tag{108}$$

Proof. Use of (91). ■

In the next we discuss high order of complex approximation by using the smoothness of f.

We give

Theorem 21 *Let $f:\prod_{i=1}^n[a_i,b_i]\rightarrow\mathbb{C}$, such that $f=f_1+if_2$. Assume $f_1,f_2\in C^m\left(\prod_{i=1}^n[a_i,b_i]\right)$, $0<\beta<1$, $n,m,N\in\mathbb{N}$, $n^{1-\beta}\geq3$, $x\in\left(\prod_{i=1}^n[a_i,b_i]\right)$. Then*

i)

$$\left|A_n(f,x)-f(x)-\sum_{j=1}^m\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha(x)}{\prod_{i=1}^N\alpha_i!}\right)A_n\left(\prod_{i=1}^N(\cdot-x_i)^{\alpha_i},x\right)\right)\right|\leq \tag{109}$$
$$(4.019)^N\cdot\left\{\frac{N^m}{m!n^{m\beta}}\left(\omega_{1,m}^{\max}\left(f_{1,\alpha},\frac{1}{n^\beta}\right)+\omega_{1,m}^{\max}\left(f_{2,\alpha},\frac{1}{n^\beta}\right)\right)+\right.$$
$$\left.\left(\frac{\|b-a\|_\infty^m\left(\|f_{1,\alpha}\|_{\infty,m}^{\max}+\|f_{2,\alpha}\|_{\infty,m}^{\max}\right)N^m}{m!}\right)\frac{1}{\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}\right\},$$

ii)

$$\left|A_n(f,x)-f(x)\right| \leq (4.019)^N \cdot \tag{110}$$

$$\left\{\sum_{j=1}^{m}\left(\sum_{|\alpha|=j}\left(\frac{\left|f_{1,\alpha}(x)\right|+\left|f_{2,\alpha}(x)\right|}{\prod_{i=1}^{N}\alpha_i!}\right)\left[\frac{1}{n^{\beta j}}+\right.\right.\right.$$

$$\left.\left.\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)\cdot\frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}\right]\right)+$$

$$\frac{N^m}{m!n^{m\beta}}\left(\omega_{1,m}^{\max}\left(f_{1,\alpha},\frac{1}{n^\beta}\right)+\omega_{1,m}^{\max}\left(f_{2,\alpha},\frac{1}{n^\beta}\right)\right)+$$

$$\left.\left(\frac{\|b-a\|_\infty^m\left(\|f_{1,\alpha}\|_{\infty,m}^{\max}+\|f_{2,\alpha}\|_{\infty,m}^{\max}\right)N^m}{m!}\right)\frac{1}{\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}\right\},$$

iii)

$$\left\|A_n(f)-f\right\|_\infty \leq (4.019)^N \cdot \tag{111}$$

$$\left\{\sum_{j=1}^{m}\left(\sum_{|\alpha|=j}\left(\frac{\left\|f_{1,\alpha}(x)\right\|_\infty+\left\|f_{2,\alpha}(x)\right\|_\infty}{\prod_{i=1}^{N}\alpha_i!}\right)\left[\frac{1}{n^{\beta j}}+\right.\right.\right.$$

$$\left.\left.\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)\cdot\frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}\right]\right)+$$

$$\frac{N^m}{m!n^{m\beta}}\left(\omega_{1,m}^{\max}\left(f_{1,\alpha},\frac{1}{n^\beta}\right)+\omega_{1,m}^{\max}\left(f_{2,\alpha},\frac{1}{n^\beta}\right)\right)+$$

$$\left.+\left(\frac{\|b-a\|_\infty^m\left(\|f_{1,\alpha}\|_{\infty,m}^{\max}+\|f_{2,\alpha}\|_{\infty,m}^{\max}\right)N^m}{m!}\right)\frac{1}{\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}\right\},$$

iv) Assume $f_\alpha(x_0)=0$, *for all* $\alpha : |\alpha| = 1,...,m$; $x_0 \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$. *Then*

$$\left|A_n(f,x_0)-f(x_0)\right| \leq (4.019)^N \cdot \tag{112}$$

$$\left\{\frac{N^m}{m!n^{m\beta}}\left(\omega_{1,m}^{\max}\left(f_{1,\alpha},\frac{1}{n^\beta}\right)+\omega_{1,m}^{\max}\left(f_{2,\alpha},\frac{1}{n^\beta}\right)\right)+\right.$$

$$\left.\left(\frac{\|b-a\|_\infty^m\left(\|f_{1,\alpha}\|_{\infty,m}^{\max}+\|f_{2,\alpha}\|_{\infty,m}^{\max}\right)N^m}{m!}\right)\frac{1}{\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}\right\},$$

notice in the last the extremely high rate of convergence at $n^{-\beta(m+1)}$.

Proof. By Theorem 8 and Remark 19. ■

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