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SUBDIVIDING OF HOLDER'S INEQUALITIES ON TIME SCALES AND SOME INTEGRAL INEQUALITIES VIA THE THEORY OF ISOTONIC LINEAR FUNCTIONALS

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ABSTRACT. The aim of this paper is to establish some extensions of subdividing of Holder's inequality, Minkowski's inequality and Qi's inequality to isotonic linear functionals taking into account that the time scale Cauchy delta, Cauchy nabla, α-diamond, multiple Riemann, and multiple Lebesque integrals all are isotonic linear functionals.

1. **Introduction**

In this paper we adopt the notations from the monograph [3] of Bohner and Peterson. For further information concerning time scales, see [3]. The following results will be useful below in order to establish the main results of this paper, and can be found in [3], in [12] and in [2]. The following two results present two important properties of the time scale Cauchy delta integrals.

Lemma 1. *([6], Corollary 3.3) If f is Δ- integrable on $[a, b)$ then for an arbitrary positive number α the function $|f|^\alpha$ is Δ-integrable on $[a, b)$.*

Lemma 2. *([6], Theorem 3.6) Let f and g be Δ-integrable functions on $[a, b)$. then their product fg is Δ-integrable on $[a, b)$.*

In the following we need to recall Holder's inequality on time scales and two refinements of them which will be used below.

Lemma 3. *([3],p. 259, Theorem 6.13) Let $a, b \in \mathbb{T}$. If $f, g \in C_{rd}(\mathbb{T}, \mathbb{R})$, then*

$$\int_a^b |f(x)g(x)|\Delta x \leq \left[\int_a^b |f(x)|^p \Delta x\right]^{\frac{1}{p}} \left[\int_a^b |g(x)|^q \Delta x\right]^{\frac{1}{q}}, \tag{1}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

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Lemma 4. *([12], Theorem 5) Let $f, g, h \in C_{rd}([a,b],\mathbb{R})$ and $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$; then*

$$\left(\int_a^b |h(x)||f(x)|^p\Delta x\right)^{\frac{1}{p}}\left(\int_a^b |h(x)||g(x)|^q\Delta x\right)^{\frac{1}{q}} \geq \int_a^b |h(x)||f(x)g(x)|\Delta x. \tag{2}$$

Lemma 5. *([12], Theorem 6) Let $f, g, h \in C_{rd}([a,b],\mathbb{R})$ and $\frac{1}{p}+\frac{1}{q}=1$ with $p<0$ or $q<0$; then*

$$\left(\int_a^b |h(x)||f(x)|^p\Delta x\right)^{\frac{1}{p}}\left(\int_a^b |h(x)||g(x)|^q\Delta x\right)^{\frac{1}{q}} \leq \int_a^b |h(x)||f(x)g(x)|\Delta x. \tag{3}$$

The following definition is given in [2], [5] and it is necessary to recall it here.

Definition 1. *Let E be a nonempty set and L be a class of real-valued functions $f : E \to \mathbb{R}$ having the following properties:*

(L1) If $f, g \in L$ and $a, b \in \mathbb{R}$, then $(af + bg) \in L$.

(L2) If $f(t) = 1$ for all $t \in E$, then $f \in L$.

An isotonic linear functional is a functional $A : L \to \mathbb{R}$ having the following properties:

(A1) If $f, g \in L$ and $a, b \in \mathbb{R}$, then $A(af + bg) = aA(f) + bA(g)$.

(A2) If $f \in L$ and $f(t) \geq 0$ for all $t \in E$, then $A(f) \geq 0$.

Now we will recall Holder's inequality for isotonic linear functionals as it appears in [8].

Theorem 1. *([2]) Let $E, L,$ and A such that (L1), (L2), (A1) and (A2) are satisfied. For $p \neq 1$, define $q = \frac{p}{p-1}$. Assume $|w||f|^p$, $|w||g|^q$, $|wfg| \in L$. If $p > 1$, then*

$$A(|wfg|) \leq A^{\frac{1}{p}}(|w||f|^p)A^{\frac{1}{q}}(|w||g|^q). \tag{4}$$

Then inequality is reversed if $0 < p < 1$ and $A(|w||g|^q) > 0$, and it is also reversed if $p < 0$ and $A(|w||f|^p) > 0$.

2. Subdividing of Holder's inequalities on time scales

The following result is a subdividing of Holder's inequality given on time scales as an analogue of Theorem 1.2 from [11].

Theorem 2. *Let $s, t \in \mathbb{R}$, $p = \frac{s-t}{1-t}$, and $q = \frac{s-t}{s-1}$. We consider $a, b \in \mathbb{T}$ and $f, g, h \in C_{rd}([a,b],\mathbb{R})$.*

(i) If $s < 1 < t$ or $s > 1 > t$, then

$$\int_a^b |h(x)||f(x)g(x)|\Delta x \leq \left(\int_a^b |h(x)||f(x)|^{sp}\Delta x\right)^{\frac{1}{p^2}} \cdot \left(\int_a^b |h(x)||g(x)|^{tq}\Delta x\right)^{\frac{1}{q^2}} \cdot$$

$$\cdot \left(\int_a^b |h(x)||f(x)|^{tp}\Delta x \cdot \int_a^b |h(x)||g(x)|^{sq}\Delta x\right)^{\frac{1}{pq}}$$

(ii) If $s>t>1$ *or* $<1<t;\ t>s>1$ *or* $t<s<1$, *then*

$$\int_a^b |h(x)||f(x)g(x)|\Delta x \geq \left(\int_a^b |h(x)||f(x)|^{sp}\Delta x\right)^{\frac{1}{p^2}} \cdot \left(\int_a^b |h(x)||g(x)|^{tq}\Delta x\right)^{\frac{1}{q^2}} \cdot$$

$$\cdot \left(\int_a^b |h(x)||f(x)|^{tp}\Delta x \cdot \int_a^b |h(x)||g(x)|^{sq}\Delta x\right)^{\frac{1}{pq}}$$

Proof. (i) Taking into account that hypothesis $s<1<t$ or $s>1>t$ implies $p=\frac{s-t}{1-t}>1,\ q=\frac{s-t}{s-1}$ and Lemma 1, by using Holder's inequality on time scales we have

$$\int_a^b |h(x)||f(x)g(x)|\Delta x = \int_a^b \left[|h(x)||(fg)(x)|^s\right]^{\frac{1-t}{s-t}} \left[|(fg)(x)|^t\right]^{\frac{s-1}{s-t}} \Delta x \leq$$

$$\leq \left[\int_a^b |h(x)||(fg)(x)|^s\Delta x\right]^{\frac{1-t}{s-t}} \left[\int_a^b |h(x)||(fg)(x)|^t\Delta x\right]^{\frac{s-1}{s-t}}.$$

As in [11], using again Holder's inequality from Lemma 4 for $\frac{s-t}{1-t}>1$ and Lemma 1 we get,

$$\int_a^b |h(x)||(fg)(x)|^s\Delta x \leq \left(\int_a^b |h(x)||f(x)|^{s\frac{s-t}{1-t}}\Delta x\right)^{\frac{1-t}{s-t}} \left(\int_a^b |h(x)||g(x)|^{s\frac{s-t}{s-1}}\Delta x\right)^{\frac{s-1}{s-t}}$$

and

$$\int_a^b |h(x)||(fg)(x)|^t\Delta x \leq \left(\int_a^b |h(x)||f(x)|^{t\frac{s-t}{1-t}}\Delta x\right)^{\frac{1-t}{s-t}} \left(\int_a^b |h(x)||g(x)|^{t\frac{s-t}{s-1}}\Delta x\right)^{\frac{s-1}{s-t}}.$$

Conclusion from (i) holds from last three inequalities.

(ii) From $s>t>1$ or $s<t<1$ we have $\frac{s-t}{1-t}<0$ and $t>s>1$ or $t<s<1$ involves $0<\frac{s-t}{1-t}<1$. Using now Holder's inequality from Lemma 5 for $0<\frac{s-t}{1-t}<1$ or $\frac{s-t}{1-t}<0$, we find

$$\int_a^b |h(x)||f(x)g(x)|\Delta x \geq$$

$$\geq \left[\left(\int_a^b |h(x)||f(x)|^{s\frac{s-t}{1-t}}\Delta x\right)^{\frac{1-t}{s-t}} \left(\int_a^b |h(x)||g(x)|^{s\frac{s-t}{s-1}}\Delta x\right)^{\frac{s-1}{s-t}}\right]^{\frac{1-t}{s-t}} \cdot$$

$$\cdot \left[\left(\int_a^b |h(x)||f(x)|^{t\frac{s-t}{1-t}}\Delta x\right)^{\frac{1-t}{s-t}} \left(\int_a^b |h(x)||g(x)|^{t\frac{s-t}{s-1}}\Delta x\right)^{\frac{s-1}{s-t}}\right]^{\frac{s-1}{s-t}}.$$

∎

3. Subdividing of Holder's inequalities for isotonic linear functionals

Starting from results given in [4], in [11], in [10] and in [1] we can state the following inequalities for isotonic linear functionals.

Theorem 3. *Let $s,t \in \mathbb{R}$, $p = \frac{s-t}{1-t}$, and $q = \frac{s-t}{s-1}$. Let L satisfying the conditions L1, L2 and A satisfying the conditions A1, A2 on the set E. We assume $|w||f|^{tp}$, $|w||f|^{sp}$, $|w||g|^{tq}$, $|w||g|^{sq}$, $|wfg|$, $|wfg|^s$, $|wfg|^t \in L$.*

(i) If $s<1<t$ or $s>1>t$, then

$$A(|wfg|) \le A^{\frac{1}{p^2}}(|w||f^{sp})A^{\frac{1}{q^2}}(|w||g|^{tq}) \cdot [A(|w||f|^{tp})A(|w||g|^{sq})]^{\frac{1}{pq}}.$$

(ii) If $s>t>1$ or $s<1<t$; $t>s>1$ or $t<s<1$, then

$$A(|wfg|) \ge A^{\frac{1}{p^2}}(|w||f|^{sp})A^{\frac{1}{q^2}}(|w||g|^{tq}) \cdot [A(|w||f|^{tp})A(|w||g|^{sq})]^{\frac{1}{pq}},$$

when $A(|w||fg|^s) > 0$, $A(|w||fg|^t) > 0$, $A(|w||f|^{tp}) > 0$, $A(|w||f|^{sp}) > 0$, $A(|w||g|^{tq}) > 0$, $A(|w||g|^{sq}) > 0$.

Proof. (i) By inequality (4) from Theorem 1, applied for $p = \frac{s-t}{1-t} > 1$, $q = \frac{s-t}{s-1}$ we have

$$A(|wfg|) = A\left([|wfg|^s]^{\frac{1-t}{s-t}}[|wfg|^t]^{\frac{s-1}{s-t}}\right) \le A^{\frac{1-t}{s-t}}(|w||fg|^s) \cdot A^{\frac{s-1}{s-t}}(|w||fg|^t).$$

Applying again Theorem 1 for $\frac{s-t}{1-t} > 1$ we get

$$A(|w||fg|^s) \le A^{\frac{1-t}{s-t}}(|w||f|^{s\frac{s-t}{1-t}})A^{\frac{s-1}{s-t}}(|w||g|^{s\frac{s-t}{s-1}})$$

and

$$A(|w||fg|^t) \le A^{\frac{1-t}{s-t}}(|w||f|^{t\frac{s-t}{1-t}})A^{\frac{s-1}{s-t}}(|w||g|^{t\frac{s-t}{s-1}}).$$

Taking into account these three inequalities we obtain the desired inequality.

For (ii) we use a similary motivation and the reverse inequality from Theorem 1.

∎

Taking into account Remark 2.5 from [4], we can state the following improvements of Minkowski's inequality for isotonic linear functionals.

Theorem 4. *(i) Let $p>0$, $s,t \in \mathbb{R}-\{0\}$, and $s \ne t$. We consider $p,s,t \in \mathbb{R}$ different numbers, such that $s,t>1$, $\frac{s-t}{p-t} > 1$, L satisfy conditions L1, L2 and A satisfy A1, A2 on the set E . If $w,f,g \ge 0$ on E with $w(f+g)^p$, $w(f+g)^s$, $w(f+g)^t$, wf^s, wg^s, wf^t, $wg^t \in L$ then*

$$A(w(f+g)^p) \le [A^{\frac{1}{s}}(wf^s) + A^{\frac{1}{s}}(wg^s)]^{\frac{s(p-t)}{s-t}} \cdot [A^{\frac{1}{t}}(wf^t) + A^{\frac{1}{t}}(wg^t)]^{\frac{t(s-p)}{s-t}}.$$

(ii) Let $p>0$, $s,t \in \mathbb{R}-\{0\}$, and $s \ne t$.. If we consider now $p,s,t \in \mathbb{R}$ different numbers, such that $s,t<1$, $\frac{s-t}{p-t} < 1$, L satisfy conditions L1, L2 and A satisfy conditions A1, A2 on the set E and if $w,f,g \ge 0$ on E with $w(f+g)^p$, $w(f+g)^s$, $w(f+g)^t$, wf^s, wg^s, wf^t, $wg^t \in L$ then

$$A(w(f+g)^p) \ge [A^{\frac{1}{s}}(wf^s) + A^{\frac{1}{s}}(wg^s)]^{\frac{s(p-t)}{s-t}} \cdot [A^{\frac{1}{t}}(wf^t) + A^{\frac{1}{t}}(wg^t)]^{\frac{t(s-p)}{s-t}}.$$

In this case we need the additional conditions $A(w(f+g)^s) > 0$, $A(w(f+g)^t) > 0$, $A(wf^s) > 0$, $A(wg^s) > 0$, $A(wf^t) > 0$, $A(wg^t) > 0$.

Proof. (i) We will use first Holder's inequality, Theorem 1, page 136 and then Minkowski's inequality, Theorem 2, on the same page, from [7]. For Holder's inequality we use indices $\frac{s-t}{p-t}$ and $\frac{s-t}{s-p}$ obtaining:

$$A(w(f+g)^p) = A(w(f+g)^{s\frac{p-t}{s-t}}(f+g)^{t\frac{s-p}{s-t}}) \leq A^{\frac{p-t}{s-t}}(w(f+g)^s)A^{\frac{s-p}{s-t}}(w(f+g)^t).$$

We use first time $s > 1$ for Minkowski's inequality and the second time $t > 1$, obtaining:

$$A^{\frac{p-t}{s-t}}(w(f+g)^s)A^{\frac{s-p}{s-t}}(w(f+g)^t) \leq$$

$$\leq \left[A^{\frac{1}{s}}(wf^s) + A^{\frac{1}{s}}(wg^s)\right]^{s\frac{p-t}{s-t}} \left[A^{\frac{1}{t}}(wf^t) + A^{\frac{1}{t}}(wg^t)\right]^{t\frac{s-p}{s-t}}.$$

(ii) We will use the same reason like before.

∎

We continue by giving a refinement of the subdividing of Holder's inequality from Theorm 3 for isotonic linear functionals, but first enunciate Theorem 2.2 fom [1] in the case of these functionals.

Theorem 5. *Let* $1 < p < \infty$ *and let* $q = \frac{p}{p-1}$ *be its conjugate exponent, L satisfy conditions L1, L2 and A satisfy A1, A2 on the set E. If* $|f|^p$, $|g|^q$, $|fg|$, $|f|^{\frac{p}{2}}|g|^{\frac{q}{2}} \in L$

and if $1 < p \leq 2$, *then*

$$A^{\frac{1}{p}}(|f|^p)A^{\frac{1}{q}}(|g|^q)\left(1 - \frac{1}{p}A\left[\left(\frac{|f|^{\frac{p}{2}}}{A^{\frac{1}{2}}(|f|^p)} - \frac{|g|^{\frac{q}{2}}}{A^{\frac{1}{2}}(|g|^q)}\right)^2\right]\right)_+ \leq A(|fg|) \leq$$

$$\leq A^{\frac{1}{p}}(|f|^p)A^{\frac{1}{q}}(|g|^q)\left(1 - \frac{1}{q}A\left[\left(\frac{|f|^{\frac{p}{2}}}{A^{\frac{1}{2}}(|f|^p)} - \frac{|g|^{\frac{q}{2}}}{A^{\frac{1}{2}}(|g|^q)}\right)^2\right]\right)$$

while if $2 \leq p < \infty$, *the terms* $\frac{1}{p}$ *and* $\frac{1}{q}$ *exchange their positions in the preceeding inequalities.*

Proof. We take in Lemma 2.1([1]) $u = \frac{|f|}{A^{\frac{1}{p}}(|f|^p)}$ and $v = \frac{|g|}{A^{\frac{1}{q}}(|g|^q)}$ and by replacing in inequality $\frac{1}{q}(u^{\frac{p}{2}} - v^{\frac{q}{2}})^2 \leq \frac{u^p}{p} + \frac{v^q}{q} - uv \leq \frac{1}{p}(u^{\frac{p}{2}} - v^{\frac{q}{2}})^2$ we obtain

$$\frac{1}{q}\left(\frac{|f|^{\frac{p}{2}}}{A^{\frac{1}{2}}(|f|^p)} - \frac{|g|^{\frac{q}{2}}}{A^{\frac{1}{2}}(|g|^q)}\right)^2 \leq \frac{|f|^p}{pA(|f|^p)} + \frac{|g|^q}{qA(|g|^q)} - \frac{|f||g|}{A^{\frac{1}{p}}(|f|^p)A^{\frac{1}{q}}(|g|^q)} \leq$$

$$\leq \frac{1}{p}\left(\frac{|f|^{\frac{p}{2}}}{A^{\frac{1}{2}}(|f|^p)} - \frac{|g|^{\frac{q}{2}}}{A^{\frac{1}{2}}(|g|^q)}\right)^2.$$

Using hypothesis and condition A2 we have:

$$\frac{1}{q}A\left[\left(\frac{|f|^{\frac{p}{2}}}{A^{\frac{1}{2}}(|f|^p)} - \frac{|g|^{\frac{q}{2}}}{A^{\frac{1}{2}}(|g|^q)}\right)^2\right] \leq \frac{A(|f|^p)}{pA(|f|^p)} + \frac{A(|g|^q)}{qA(|g|^q)} - \frac{A(|f||g|)}{A^{\frac{1}{p}}(|f|^p)A^{\frac{1}{q}}(|g|^q)} \leq$$

$$\leq \frac{1}{p}A\left[\left(\frac{|f|^{\frac{p}{2}}}{A^{\frac{1}{2}}(|f|^p)} - \frac{|g|^{\frac{q}{2}}}{A^{\frac{1}{2}}(|g|^q)}\right)^2\right]$$

or by calculus,

$$\frac{1}{q}A\left[\left(\frac{|f|^{\frac{p}{2}}}{A^{\frac{1}{2}}(|f|^p)}-\frac{|g|^{\frac{q}{2}}}{A^{\frac{1}{2}}(|g|^q)}\right)^2\right]\leq 1-\frac{A(|f||g|)}{A^{\frac{1}{p}}(|f|^p)A^{\frac{1}{q}}(|g|^q)}\leq$$

$$\leq\frac{1}{p}A\left[\left(\frac{|f|^{\frac{p}{2}}}{A^{\frac{1}{2}}(|f|^p)}-\frac{|g|^{\frac{q}{2}}}{A^{\frac{1}{2}}(|g|^q)}\right)^2\right]$$

which leads to our conclusion.

∎

Theorem 6. *Let $s,t\in\mathbb{R}$, $p=\frac{s-t}{1-t}$, and $q=\frac{s-t}{s-1}$ such that $s<1<t$ or $s>1>t$, and L satisfy conditions L1, L2 and A satisfy conditions A1, A2 on the set E. If f^{sp}, f^{tp}, g^{sq}, g^{tq}, $(fg)^t$, $(fg)^s$, $f^{\frac{sp}{2}}g^{\frac{sq}{2}}$ $f^{\frac{tp}{2}}g^{\frac{tq}{2}}$, $(fg)^{\frac{s+t}{2}}\in L$ and f,g are positive functions then*

$$A(fg)\leq A^{\frac{1}{p^2}}(f^{sp})A^{\frac{1}{q^2}}(g^{tq})\left[A(f^{tp})A(g^{sq})\right]^{\frac{1}{pq}}\cdot\left[1-\min\{\frac{1}{q},\frac{1}{p}\}A\left[\left(\frac{f^{\frac{sp}{2}}}{A^{\frac{1}{2}}(f^{sp})}-\frac{g^{\frac{sq}{2}}}{A^{\frac{1}{2}}(g^{sq})}\right)^2\right]\right]^{\frac{1}{p}}\cdot$$

$$\cdot\left[1-\min\{\frac{1}{q},\frac{1}{p}\}A\left[\left(\frac{f^{\frac{tp}{2}}}{A^{\frac{1}{2}}(f^{tp})}-\frac{g^{\frac{tq}{2}}}{A^{\frac{1}{2}}(g^{tq})}\right)^2\right]\right]^{\frac{1}{q}}\cdot\left[1-\min\{\frac{1}{q},\frac{1}{p}\}A\left[\left(\frac{(fg)^{\frac{t}{2}}}{A^{\frac{1}{2}}((fg)^t)}-\frac{(fg)^{\frac{s}{2}}}{A^{\frac{1}{2}}((fg)^s)}\right)^2\right]\right],$$

and

$$A(fg)\geq A^{\frac{1}{p^2}}(f^{sp})A^{\frac{1}{q^2}}(g^{tq})\left[A(f^{tp})A(g^{sq})\right]^{\frac{1}{pq}}\cdot\left[1-\max\{\frac{1}{q},\frac{1}{p}\}A\left[\left(\frac{f^{\frac{sp}{2}}}{A^{\frac{1}{2}}(f^{sp})}-\frac{g^{\frac{sq}{2}}}{A^{\frac{1}{2}}(g^{sq})}\right)^2\right]\right]_+^{\frac{1}{p}}\cdot$$

$$\cdot\left[1-\max\{\frac{1}{q},\frac{1}{p}\}A\left[\left(\frac{f^{\frac{tp}{2}}}{A^{\frac{1}{2}}(f^{tp})}-\frac{g^{\frac{tq}{2}}}{A^{\frac{1}{2}}(g^{tq})}\right)^2\right]\right]_+^{\frac{1}{q}}\cdot\left[1-\max\{\frac{1}{q},\frac{1}{p}\}A\left[\left(\frac{(fg)^{\frac{t}{2}}}{A^{\frac{1}{2}}((fg)^t)}-\frac{(fg)^{\frac{s}{2}}}{A^{\frac{1}{2}}((fg)^s)}\right)^2\right]\right]_+.$$

Proof. By inequality given in Theorem 5, applied for $p=\frac{s-t}{1-t}>1$, $q=\frac{s-t}{s-1}$ we have

$$A(fg)=A\left([(fg)^s]^{\frac{1-t}{s-t}}[(fg)^t]^{\frac{s-1}{s-t}}\right)\leq$$

$$\leq A^{\frac{1-t}{s-t}}((fg)^s)\cdot A^{\frac{s-1}{s-t}}((fg)^t)\left(1-\min\{\frac{1}{q},\frac{1}{p}\}A\left[\left(\frac{(fg)^{\frac{t}{2}}}{A^{\frac{1}{2}}((fg)^t)}-\frac{(fg)^{\frac{s}{2}}}{A^{\frac{1}{2}}((fg)^s)}\right)^2\right]\right).$$

Applying again Theorem 5 for $\frac{s-t}{1-t}>1$ we get

$$A((fg)^s)\leq A^{\frac{1-t}{s-t}}(f^{s\frac{s-t}{1-t}})A^{\frac{s-1}{s-t}}(g^{s\frac{s-t}{s-1}})\left(1-\min\{\frac{1}{q},\frac{1}{p}\}A\left[\left(\frac{f^{\frac{sp}{2}}}{A^{\frac{1}{2}}(f^{sp})}-\frac{g^{\frac{sq}{2}}}{A^{\frac{1}{2}}(g^{sq})}\right)^2\right]\right)$$

and

$$A((fg)^t)\leq A^{\frac{1-t}{s-t}}(f^{t\frac{s-t}{1-t}})A^{\frac{s-1}{s-t}}(g^{t\frac{s-t}{s-1}})\left(1-\min\{\frac{1}{q},\frac{1}{p}\}A\left[\left(\frac{f^{\frac{tp}{2}}}{A^{\frac{1}{2}}(f^{tp})}-\frac{g^{\frac{tq}{2}}}{A^{\frac{1}{2}}(g^{tq})}\right)^2\right]\right).$$

Taking into account these three inequalities we obtain the desired inequality.

For second inequality we taking into account the first inequality from Theorem 5 and use the same reason like before. ∎

Remark 1. *(i) It is known that the time-scale integral is an isotonic linear functional as is given in Definition 1. Multiple Riemann delta time-scale integral is also an isotonic linear functional, see Theorem 3.6,* [2].

(ii) The multiple Lebesque delta time-scale integral is also an isotonic linear functional, see Theorem 3.7, [2].

(iii) The Cauchy nabla time-scales integral is an isotonic linear functional, see Theorem 3.4, [2].

(iv) The Cauchy α-diamond time scale integral is an isotonic linear functional, see Theorem 3.5, [2].

Therefore these inequalities from Theorem 3, Theorem 4, Theorem 5 and Theorem can be rewritten for these kind of specific isotonic linear functionals.

We can give below an improvement of Holder's inequality like in [1] for integral n time scales and then a refinement of Theorem 3 , the subdividing of Holder's inequality for integrals on time scales.

Remark 2. *(i) Let $a, b \in \mathbb{T}$. If $f, g \in C_{rd}(\mathbb{T}, \mathbb{R})$, are two psitive functions then*

$$\left[\int_a^b f(x)^p \Delta x\right]^{\frac{1}{p}} \left[\int_a^b g(x)^q \Delta x\right]^{\frac{1}{q}} \cdot$$

$$\cdot \left[1 - \frac{2}{\min\{p,q\}}\left(1 - \frac{\int_a^b f(x)^{\frac{p}{2}} g(x)^{\frac{q}{2}} \Delta x}{(\int_a^b f(x)^p \Delta x \int_a^b g(x)^q \Delta x)^{\frac{1}{2}}}\right)\right]_+ \leq$$

$$\leq \int_a^b f(x)g(x)\Delta x \leq \left[\int_a^b f(x)^p \Delta x\right]^{\frac{1}{p}} \left[\int_a^b g(x)^q \Delta x\right]^{\frac{1}{q}} \cdot$$

$$\cdot \left[1 - \frac{2}{\max\{p,q\}}\left(1 - \frac{\int_a^b f(x)^{\frac{p}{2}} g(x)^{\frac{q}{2}} \Delta x}{(\int_a^b f(x)^p \Delta x \int_a^b g(x)^q \Delta x)^{\frac{1}{2}}}\right)\right]$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

(ii) Let $s, t \in \mathbb{R}$, $p = \frac{s-t}{1-t}$, and $q = \frac{s-t}{s-1}$. We consider $a, b \in \mathbb{T}$ and $f, g \in C_{rd}([a,b], \mathbb{R})$ two positive functions such that if $s < 1 < t$ or $s > 1 > t$, then

$$\int_a^b f(x)g(x)\Delta x \leq \left(\int_a^b f(x)^{sp}\Delta x\right)^{\frac{1}{p^2}} \cdot \left(\int_a^b g(x)^{tq}\Delta x\right)^{\frac{1}{q^2}} \cdot$$

$$\cdot \left(\int_a^b f(x)^{tp}\Delta x \cdot \int_a^b g(x)^{sq}\Delta x\right)^{\frac{1}{pq}} \left[1 - 2\min\{\frac{1}{p}, \frac{1}{q}\}\left(1 - \frac{\int_a^b f^{\frac{sp}{2}}(x) g^{\frac{sq}{2}}(x)\Delta x}{(\int_a^b f^{sp}(x)\Delta x \int_a^b g^{sq}(x)\Delta x)^{\frac{1}{2}}}\right)\right]^{\frac{1}{p}} \cdot$$

$$\cdot \left[1 - 2\min\{\frac{1}{p}, \frac{1}{q}\}\left(1 - \frac{\int_a^b f^{\frac{tp}{2}}(x) g^{\frac{tq}{2}}(x)\Delta x}{(\int_a^b f^{tp}(x)\Delta x \int_a^b g^{tq}(x)\Delta x)^{\frac{1}{2}}}\right)\right]^{\frac{1}{q}} \cdot$$

$$\cdot \left[1 - 2\min\{\frac{1}{p}, \frac{1}{q}\}\left(1 - \frac{\int_a^b (f(x)g(x))^{\frac{s+t}{2}}\Delta x}{(\int_a^b (f(x)g(x))^s \Delta x \int_a^b (f(x)g(x))^t \Delta x)^{\frac{1}{2}}}\right)\right],$$

and

$$\int_a^b f(x)g(x)\Delta x \geq \left(\int_a^b f(x)^{sp}\Delta x\right)^{\frac{1}{p^2}} \cdot \left(\int_a^b g(x)^{tq}\Delta x\right)^{\frac{1}{q^2}} \cdot$$

$$\cdot\left(\int_a^b f(x)^{tp}\Delta x\cdot\int_a^b g(x)^{sq}\Delta x\right)^{\frac{1}{pq}}\left[1-2\max\{\frac{1}{p},\frac{1}{q}\}\left(1-\frac{\int_a^b f^{\frac{sp}{2}}(x)g^{\frac{sq}{2}}(x)\Delta x}{(\int_a^b f^{sp}(x)\Delta x\int_a^b g^{sq}(x)\Delta x)^{\frac{1}{2}}}\right)\right]_+^{\frac{1}{p}}\cdot$$

$$\cdot\left[1-2\max\{\frac{1}{p},\frac{1}{q}\}\left(1-\frac{\int_a^b f^{\frac{tp}{2}}(x)g^{\frac{tq}{2}}(x)\Delta x}{(\int_a^b f^{tp}(x)\Delta x\int_a^b g^{tq}(x)\Delta x)^{\frac{1}{2}}}\right)\right]_+^{\frac{1}{q}}\cdot$$

$$\cdot\left[1-2\max\{\frac{1}{p},\frac{1}{q}\}\left(1-\frac{\int_a^b(f(x)g(x))^{\frac{s+t}{2}}\Delta x}{(\int_a^b(f(x)g(x))^s\Delta x\int_a^b(f(x)g(x))^t\Delta x)^{\frac{1}{2}}}\right)\right]_+.$$

Remark 3. *Using previous Remark 1, (iii) we can give the following subdividing of Holder's inequalities for the Cauchy nabla time-scales integral :*

Let $s,t\in\mathbb{R}$, $p=\frac{s-t}{1-t}$, *and* $q=\frac{s-t}{s-1}$. *We consider* $a,b\in\mathbb{T}$ *and* $f,g,h\in C_{ld}([a,b],\mathbb{R})$.

(a) If $s<1<t$ *or* $s>1>t$, *then*

$$\int_a^b|h(x)||f(x)g(x)|\nabla x\le\left(\int_a^b|h(x)||f(x)|^{sp}\nabla x\right)^{\frac{1}{p^2}}\cdot\left(\int_a^b|h(x)||g(x)|^{tq}\nabla x\right)^{\frac{1}{q^2}}\cdot$$

$$\cdot\left(\int_a^b|h(x)||f(x)|^{tp}\nabla x\cdot\int_a^b|h(x)||g(x)|^{sq}\nabla x\right)^{\frac{1}{pq}}.$$

(b) If $s>t>1$ *or* $<1<t$; $t>s>1$ *or* $t<s<1$, *then*

$$\int_a^b|h(x)||f(x)g(x)|\nabla x\ge\left(\int_a^b|h(x)||f(x)|^{sp}\nabla x\right)^{\frac{1}{p^2}}\cdot\left(\int_a^b|h(x)||g(x)|^{tq}\nabla x\right)^{\frac{1}{q^2}}\cdot$$

$$\cdot\left(\int_a^b|h(x)||f(x)|^{tp}\nabla x\cdot\int_a^b|h(x)||g(x)|^{sq}\nabla x\right)^{\frac{1}{pq}}.$$

(c) If $s<1<t$ *or* $s>1>t$, *and* f,g *are two positive functions then*

$$\int_a^b f(x)g(x)\nabla x\le\left(\int_a^b f(x)^{sp}\nabla x\right)^{\frac{1}{p^2}}\cdot\left(\int_a^b g(x)^{tq}\nabla x\right)^{\frac{1}{q^2}}\cdot$$

$$\cdot\left(\int_a^b f(x)^{tp}\nabla x\cdot\int_a^b g(x)^{sq}\nabla x\right)^{\frac{1}{pq}}\left[1-2\min\{\frac{1}{p},\frac{1}{q}\}\left(1-\frac{\int_a^b f^{\frac{sp}{2}}(x)g^{\frac{sq}{2}}(x)\nabla x}{(\int_a^b f^{sp}(x)\nabla x\int_a^b g^{sq}(x)\nabla x)^{\frac{1}{2}}}\right)\right]^{\frac{1}{p}}\cdot$$

$$\cdot\left[1-2\min\{\frac{1}{p},\frac{1}{q}\}\left(1-\frac{\int_a^b f^{\frac{tp}{2}}(x)g^{\frac{tq}{2}}(x)\nabla x}{(\int_a^b f^{tp}(x)\nabla x\int_a^b g^{tq}(x)\nabla x)^{\frac{1}{2}}}\right)\right]^{\frac{1}{q}}\cdot$$

$$\cdot\left[1-2\min\{\frac{1}{p},\frac{1}{q}\}\left(1-\frac{\int_a^b(f(x)g(x))^{\frac{s+t}{2}}\nabla x}{(\int_a^b(f(x)g(x))^s\nabla x\int_a^b(f(x)g(x))^t\nabla x)^{\frac{1}{2}}}\right)\right],$$

and

$$\int_a^b f(x)g(x)\nabla x\ge\left(\int_a^b f(x)^{sp}\nabla x\right)^{\frac{1}{p^2}}\cdot\left(\int_a^b g(x)^{tq}\nabla x\right)^{\frac{1}{q^2}}\cdot$$

$$\cdot\left(\int_a^b f(x)^{tp}\nabla x\cdot\int_a^b g(x)^{sq}\nabla x\right)^{\frac{1}{pq}}\left[1-2\max\{\frac{1}{p},\frac{1}{q}\}\left(1-\frac{\int_a^b f^{\frac{sp}{2}}(x)g^{\frac{sq}{2}}(x)\nabla x}{(\int_a^b f^{sp}(x)\nabla x\int_a^b g^{sq}(x)\nabla x)^{\frac{1}{2}}}\right)\right]_+^{\frac{1}{p}}\cdot$$

$$\cdot\left[1-2\max\{\frac{1}{p},\frac{1}{q}\}\left(1-\frac{\int_a^b f^{\frac{tp}{2}}(x)g^{\frac{tq}{2}}(x)\nabla x}{(\int_a^b f^{tp}(x)\nabla x\int_a^b g^{tq}(x)\nabla x)^{\frac{1}{2}}}\right)\right]_+^{\frac{1}{q}}\cdot$$

$$\cdot\left[1-2\max\{\frac{1}{p},\frac{1}{q}\}\left(1-\frac{\int_a^b (f(x)g(x))^{\frac{s+t}{2}}\nabla x}{(\int_a^b (f(x)g(x))^s\nabla x\int_a^b (f(x)g(x))^t\nabla x)^{\frac{1}{2}}}\right)\right]_+.$$

Remark 4. *Using previous Remark 1, (iv) we can give the following subdividing of Holder's inequalities for the Cauchy α-diamond time-scales integral :*

Let $s,t\in\mathbb{R}$, $p=\frac{s-t}{1-t}$, and $q=\frac{s-t}{s-1}$. We consider $a,b\in\mathbb{T}$ and $f,g,h:[a,b]\to\mathbb{R}$ be three $\diamond_\alpha$-integrable functions.

(i) If $s<1<t$ or $s>1>t$, then

$$\int_a^b |h(x)||f(x)g(x)|\diamond_\alpha x\le\left(\int_a^b |h(x)||f(x)|^{sp}\diamond_\alpha x\right)^{\frac{1}{p^2}}\cdot\left(\int_a^b |h(x)||g(x)|^{tq}\diamond_\alpha x\right)^{\frac{1}{q^2}}\cdot$$

$$\cdot\left(\int_a^b |h(x)||f(x)|^{tp}\diamond_\alpha x\cdot\int_a^b |h(x)||g(x)|^{sq}\diamond_\alpha x\right)^{\frac{1}{pq}}$$

(ii) If $s>t>1$ or $<1<t$; $t>s>1$ or $t<s<1$, then

$$\int_a^b |h(x)||f(x)g(x)|\diamond_\alpha x\ge\left(\int_a^b |h(x)||f(x)|^{sp}\diamond_\alpha x\right)^{\frac{1}{p^2}}\cdot\left(\int_a^b |h(x)||g(x)|^{tq}\diamond_\alpha x\right)^{\frac{1}{q^2}}\cdot$$

$$\cdot\left(\int_a^b |h(x)||f(x)|^{tp}\diamond_\alpha x\cdot\int_a^b |h(x)||g(x)|^{sq}\diamond_\alpha x\right)^{\frac{1}{pq}}$$

(iii) If $s<1<t$ or $s>1>t$, and f,g are two positive functions then

$$\int_a^b f(x)g(x)\diamond_\alpha x\le\left(\int_a^b f(x)^{sp}\diamond_\alpha x\right)^{\frac{1}{p^2}}\cdot\left(\int_a^b g(x)^{tq}\diamond_\alpha x\right)^{\frac{1}{q^2}}\cdot$$

$$\cdot\left(\int_a^b f(x)^{tp}\diamond_\alpha x\cdot\int_a^b g(x)^{sq}\diamond_\alpha x\right)^{\frac{1}{pq}}\left[1-2\min\{\frac{1}{p},\frac{1}{q}\}\left(1-\frac{\int_a^b f^{\frac{sp}{2}}(x)g^{\frac{sq}{2}}(x)\diamond_\alpha x}{(\int_a^b f^{sp}(x)\diamond_\alpha x\int_a^b g^{sq}(x)\diamond_\alpha x)^{\frac{1}{2}}}\right)\right]^{\frac{1}{p}}\cdot$$

$$\cdot\left[1-2\min\{\frac{1}{p},\frac{1}{q}\}\left(1-\frac{\int_a^b f^{\frac{tp}{2}}(x)g^{\frac{tq}{2}}(x)\diamond_\alpha x}{(\int_a^b f^{tp}(x)\diamond_\alpha x\int_a^b g^{tq}(x)\diamond_\alpha x)^{\frac{1}{2}}}\right)\right]^{\frac{1}{q}}\cdot$$

$$\cdot\left[1-2\min\{\frac{1}{p},\frac{1}{q}\}\left(1-\frac{\int_a^b (f(x)g(x))^{\frac{s+t}{2}}\diamond_\alpha x}{(\int_a^b (f(x)g(x))^s\diamond_\alpha x\int_a^b (f(x)g(x))^t\nabla x)^{\frac{1}{2}}}\right)\right],$$

and

$$\int_a^b f(x)g(x)\diamond_\alpha x\ge\left(\int_a^b f(x)^{sp}\diamond_\alpha x\right)^{\frac{1}{p^2}}\cdot\left(\int_a^b g(x)^{tq}\diamond_\alpha x\right)^{\frac{1}{q^2}}\cdot$$

$$\cdot\left(\int_a^b f(x)^{tp}\diamond_\alpha x\cdot\int_a^b g(x)^{sq}\diamond_\alpha x\right)^{\frac{1}{pq}}\left[1-2\max\{\frac{1}{p},\frac{1}{q}\}\left(1-\frac{\int_a^b f^{\frac{sp}{2}}(x)g^{\frac{sq}{2}}(x)\diamond_\alpha x}{(\int_a^b f^{sp}(x)\diamond_\alpha x\int_a^b g^{sq}(x)\diamond_\alpha x)^{\frac{1}{2}}}\right)\right]_+^{\frac{1}{p}}\cdot$$

$$\cdot\left[1-2\max\{\frac{1}{p},\frac{1}{q}\}\left(1-\frac{\int_a^b f^{\frac{tp}{2}}(x)g^{\frac{tq}{2}}(x)\diamond_\alpha x}{(\int_a^b f^{tp}(x)\diamond_\alpha x\int_a^b g^{tq}(x)\diamond_\alpha x)^{\frac{1}{2}}}\right)\right]_+^{\frac{1}{q}}\cdot$$

$$\cdot\left[1-2\max\{\frac{1}{p},\frac{1}{q}\}\left(1-\frac{\int_a^b (f(x)g(x))^{\frac{s+t}{2}}\diamond_\alpha x}{(\int_a^b (f(x)g(x))^{s}\diamond_\alpha x\int_a^b (f(x)g(x))^{t}\diamond_\alpha x)^{\frac{1}{2}}}\right)\right]_+ .$$

Remark 5. *We can also consider Example 3.3, from* [2] *when* $\mathbb{T}=\mathbb{R}$, *when* $\mathbb{T}=\mathbb{Z}$, *and when* $\mathbb{T}=h\mathbb{Z}$, $h>0$ *in Theorem 3.2, and to rewrite inequalities from Theorem 3 and 4.*

According to [2], when the time scale is the set of all real numbers the time-scale integral is an ordinary integral, when the time-scale is the set of all integers the time-scale integral is a sum, when the time scale is the set of all integer powers of a fixed number the time-scale integral is a Jackson integral.

4. Several inequalities and Qi's inequalities for isotonic linear functionals

Next results reprezent variants of several results given in [9] in Lemma 2.5, Lemma 2.6, Lemma 2.7, Theorem 3.1 and Theorem 3.3 in the case of isotonic linear functionals.

Lemma 6. *Let* E, L *and* A *be such that L1, L2, A1, A2 are satisfied. If* $f,\ g,\ \frac{f^p}{g^{\frac{p}{q}}}\in L$ *are positive functions then*

$$A\left(\frac{f^p}{g^{\frac{p}{q}}}\right)\geq\frac{A^p(f)}{A^{\frac{p}{q}}(g)},$$

where $p>1$ *or* $p<0$ *while* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. We apply Holder' s inequality from Theorem 1 when $p>1$ and $f,\ g,\ \frac{f^p}{g^{\frac{p}{q}}}\in L$ are positive functions, obtaining:

$$A(f)=A\left(\frac{f}{g^{\frac{1}{q}}}g^{\frac{1}{q}}\right)\leq A^{\frac{1}{p}}\left(\frac{f^p}{g^{\frac{p}{q}}}\right)A^{\frac{1}{q}}(g).$$

Then we take the p-th power on both sides of the inequalities and have:

$$A^p(f)\leq A\left(\frac{f^p}{g^{\frac{p}{q}}}\right)A^{\frac{p}{q}}(g).$$

When $p<0$ we take into account the inverse of Holder's inequality, Theorem 1. ∎

Lemma 7. *Let E, L and A be such that L1, L2, A1, A2 are satisfied. If f, g, $f^{\frac{1}{p}}g^{\frac{1}{q}} \in L$ are positive on E such that $m \leq \frac{f(x)}{g(x)} \leq M$ on E, where $m > 0$ and $M < \infty$ then we have*

$$A^{\frac{1}{p}}(f)A^{\frac{1}{q}}(g) \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} A(f^{\frac{1}{p}}g^{\frac{1}{q}}),$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using the hypothesis $\frac{f(x)}{g(x)} \leq M$, as in Lemma 2.6, [9], we find

$$f^{\frac{1}{p}}(x)g^{\frac{1}{q}}(x) \geq M^{-\frac{1}{q}}f^{\frac{1}{q}}(x)f^{\frac{1}{p}}(x) = M^{-\frac{1}{q}}f(x),$$

on E . Therefore by Definition 1, A2 and L1 we have

$$A(f^{\frac{1}{p}}g^{\frac{1}{q}}) \geq M^{-\frac{1}{q}}A(f) \geq 0 \text{ or } A^{\frac{1}{p}}(f^{\frac{1}{p}}g^{\frac{1}{q}}) \geq M^{-\frac{1}{pq}}A^{\frac{1}{p}}(f) \geq 0. \tag{5}$$

If we consider now $\frac{f(x)}{g(x)} \geq m$ by the same reason we find that

$$A(f^{\frac{1}{p}}g^{\frac{1}{q}}) \geq m^{\frac{1}{p}}A(g) \geq 0 \text{ or } A^{\frac{1}{q}}(f^{\frac{1}{p}}g^{\frac{1}{q}}) \geq m^{\frac{1}{pq}}A^{\frac{1}{q}}(g) \geq 0. \tag{6}$$

From inequalities (5) and (6) we obtain

$$A(f^{\frac{1}{p}}g^{\frac{1}{q}}) \geq \left(\frac{m}{M}\right)^{\frac{1}{pq}} A^{\frac{1}{p}}(f)A^{\frac{1}{q}}(g).$$

∎

Theorem 7. *Let E, L and A be such that L1, L2, A1, A2 are satisfied. If f, $f^p \in L$, f is positive and $A(f) \geq A^{p-1}(1)$ then*

$$A(f^p) \geq A^{p-1}(f).$$

Proof. By Lemma 6 and hypothesis we have,

$$A(f^p) = A\left(\frac{f^p}{1^{\frac{p}{q}}}\right) \geq \frac{A^p(f)}{A^{\frac{p}{q}}(1)} = \frac{A^{p-1}(f)A(f)}{A^{p-1}(1)} \geq A^{p-1}(f).$$

∎

Remark 6. *Let E, L and A be such that L1, L2, A1, A2 are satisfied. If f^p, g^q, $fq \in L$ are positive on E such that $m \leq \frac{f^p(x)}{g^q(x)} \leq M$ on E, where $m > 0$ and $M < \infty$ then we have*

$$A^{\frac{1}{p}}(f^p)A^{\frac{1}{q}}(g^q) \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} A(fg),$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 8. *Let E, L and A be such that L1, L2, A1, A2 are satisfied. If f, $f^{\frac{1}{p}}$, $f^p \in L$ are such that $m \leq f^p(x) \leq M$ on E, where $m > 0$ and $M < \infty$ then we have*

$$A^{\frac{1}{p}}(f^p) \leq \left(\frac{M}{m}\right)^{\frac{2}{pq}} A^{-\frac{p+1}{q}}(1)A^p(f^{\frac{1}{p}}),$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Taking $g(x) = 1$ on E in Remark 6 we get

$$A^{\frac{1}{p}}(f^p)A^{\frac{1}{q}}(1) \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} A(f) \text{ or } A^{\frac{1}{p}}(f^p) \leq \left(\frac{M}{m}\right)^{\frac{1}{pq}} A^{-\frac{1}{q}}(1)A(f). \tag{6}$$

Now putting $g(x) = 1$ in Lemma 7 we will obtain

$$A^{\frac{1}{p}}(f) \leq A^{-\frac{1}{q}}(1)\left(\frac{M}{m}\right)^{\frac{1}{p^2q}} A(f^{\frac{1}{p}}) \text{ or } A(f) \leq A^{-\frac{p}{q}}(1)\left(\frac{M}{m}\right)^{\frac{1}{pq}} A^p(f^{\frac{1}{p}}) \tag{7}$$

using the hypothesis that $m^{\frac{1}{p}} \leq f(x) \leq M^{\frac{1}{p}}$. By using inequalities (6) and (7) we find

$$A^{\frac{1}{p}}(f^p) \leq \left(\frac{M}{m}\right)^{\frac{2}{pq}} A^{-\frac{1}{q}-\frac{p}{q}}(1)A^p(f^{\frac{1}{p}}).$$

∎

Remark 7. *The multiple Lebesque delta time-scale integral, the Cauchy nabla time-scales integral and the Cauchy α-diamond time scale integral are also isotonic linear functionals, therefore these inequalities from Lemma 6, Lemma 7, Theorem 7 and Theorem 8 can be rewritten for these kind of special isotonic linear functionals.*

Remark 8. *(i) Let $f, g \in C_{ld}([a,b],\mathbb{R})$ be two positive functions. Then the following inequality holds:*

$$\int_a^b \frac{f^p(x)}{g^{\frac{p}{q}}(x)}\nabla x \geq \frac{[\int_a^b(x)\nabla x]^p}{[\int_a^b g(x)\nabla x]^{\frac{p}{q}}},$$

where $p > 1$ or $p < 0$ while $\frac{1}{p} + \frac{1}{q} = 1$.

(ii) Let $a, b \in \mathbb{T}$ and $f, g : [a,b] \to \mathbb{R}$ be two $\diamond_\alpha$-integrable functions. Then we have

$$\int_a^b \frac{f^p(x)}{g^{\frac{p}{q}}(x)} \diamond_\alpha x \geq \frac{[\int_a^b(x) \diamond_\alpha x]^p}{[\int_a^b g(x) \diamond_\alpha x]^{\frac{p}{q}}},$$

where $p > 1$ or $p < 0$ while $\frac{1}{p} + \frac{1}{q} = 1$.

The following two results will help us to present a refimement of inequality from Theorem 7.

Lemma 8. *Let E, L and A be such that L1, L2, A1, A2 are satisfied. If f, g, $\frac{f^p}{g^{\frac{p}{q}}}$, $\frac{f^{\frac{p}{2}}}{g^{\frac{p}{2q}}}g^{\frac{1}{2}} \in L$ are positive functions then*

$$\left[1 - \min\{\frac{1}{p}, \frac{1}{q}\}A\left[\left(\frac{\left(\frac{f^p}{g^{\frac{p}{q}}}\right)^{\frac{1}{2}}}{A^{\frac{1}{2}}\left(\frac{f^p}{g^{\frac{p}{q}}}\right)} - \frac{g^{\frac{1}{2}}}{A^{\frac{1}{2}}(g)}\right)^2\right]\right]^p A\left(\frac{f^p}{g^{\frac{p}{q}}}\right) \geq \frac{A^p(f)}{A^{\frac{p}{q}}(g)},$$

where $p > 1$ or $p < 0$ while $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. We apply Holder' s inequality from Theorem 5 when $p > 1$ and $f,\ g,\ \frac{f^p}{g^{\frac{p}{q}}},\ \frac{f^{\frac{p}{2}}}{g^{\frac{p}{2q}}} g^{\frac{1}{2}} \in L$ are positive functions, obtaining:

$$A(f) = A\left(\frac{f}{g^{\frac{1}{q}}} g^{\frac{1}{q}}\right) \leq A^{\frac{1}{p}}\left(\frac{f^p}{g^{\frac{p}{q}}}\right) A^{\frac{1}{q}}(g)\left[1-\min\{\frac{1}{p},\frac{1}{q}\} A\left[\left(\frac{\left(\frac{f^p}{g^{\frac{p}{q}}}\right)^{\frac{1}{2}}}{A^{\frac{1}{2}}\left(\frac{f^p}{g^{\frac{p}{q}}}\right)} - \frac{g^{\frac{1}{2}}}{A^{\frac{1}{2}}(g)}\right)^2\right]\right].$$

Then we take the p-th power on both sides of the inequalities and have:

$$A^p(f) \leq A\left(\frac{f^p}{g^{\frac{p}{q}}}\right) A^{\frac{p}{q}}(g)\left[1-\min\{\frac{1}{p},\frac{1}{q}\} A\left[\left(\frac{\left(\frac{f^p}{g^{\frac{p}{q}}}\right)^{\frac{1}{2}}}{A^{\frac{1}{2}}\left(\frac{f^p}{g^{\frac{p}{q}}}\right)} - \frac{g^{\frac{1}{2}}}{A^{\frac{1}{2}}(g)}\right)^2\right]\right]^p.$$

∎

Theorem 9. *Let E, L and A be such that L1, L2, A1, A2 are satisfied. If $f,\ f^p,\ f^{\frac{p}{2}} \in L$, f is positive and $A(f) \geq A^{p-1}(\mathbf{1})$ then*

$$A(f^p)\left[1-\min\{\frac{1}{p},\frac{1}{q}\} A\left[\left(\frac{f^{\frac{p}{2}}}{A^{\frac{1}{2}}(f^p)} - \frac{\mathbf{1}}{A^{\frac{1}{2}}(\mathbf{1})}\right)^2\right]\right]^p \geq A^{p-1}(f).$$

Proof. By Lemma 8 and hypothesis we have,

$$A(f^p)\left[1-\min\{\frac{1}{p},\frac{1}{q}\} A\left[\left(\frac{f^{\frac{p}{2}}}{A^{\frac{1}{2}}(f^p)} - \frac{\mathbf{1}}{A^{\frac{1}{2}}(\mathbf{1})}\right)^2\right]\right]^p =$$

$$= A\left(\frac{f^p}{1^{\frac{p}{q}}}\right)\left[1-\min\{\frac{1}{p},\frac{1}{q}\} A\left[\left(\frac{f^{\frac{p}{2}}}{A^{\frac{1}{2}}(f^p)} - \frac{\mathbf{1}}{A^{\frac{1}{2}}(\mathbf{1})}\right)^2\right]\right]^p \geq$$

$$\geq \frac{A^p(f)}{A^{\frac{p}{q}}(\mathbf{1})} = \frac{A^{p-1}(f)A(f)}{A^{p-1}(\mathbf{1})} \geq A^{p-1}(f).$$

∎

Next results present some improvements of some integral inequalities given by Qi and Yin, [9], in the cases of delta time-scale integral, the Cauchy nabla time-scales integrals and the Cauchy α-diamond time scale integrals.

Remark 9. *(i) Let $a, b \in \mathbb{T}$. If $f \in C_{rd}(\mathbb{T}, \mathbb{R})$ is positive and*

$$\int_a^b f(x)\Delta x \geq (b-a)^{p-1}$$

then

$$\int_a^b f^p(x)\Delta x\left[1-\frac{2}{\max\{p,q\}}\left(1-\frac{\int_a^b f(x)^{\frac{p}{2}}\Delta x}{(b-a)^{\frac{1}{2}}(\int_a^b f(x)^p \Delta x)^{\frac{1}{2}}}\right)\right] \geq \left[\int_a^b f(x)\Delta x\right]^{p-1},$$

where $p > 1$.

(ii) In the case of the Cauchy nabla time-scales integrals and the Cauchy α-diamond time scale integrals similary inequalities can be stated as above.

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