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SOME INEQUALITIES FOR POWER SERIES WITH NONNEGATIVE COEFFICIENTS VIA A GLOBAL REVERSE OF JENSEN INEQUALITY

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Abstract. Some inequalities for power series with nonnegative coefficients via a new global reverse of Jensen inequality are given. Applications for some fundamental functions defined by power series are also provided.

1. Introduction

On utilizing some reverses of Jensen discrete inequality for convex functions, we obtained in [5] the following result for functions defined by power series with nonnegative coefficients:

Theorem 1. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$. If $p\geq 1$, $0<\alpha<R$ and $x>0$ with $\alpha x^p, \alpha x^{p-1}<R$, then*

$$0\leq \frac{f(\alpha x^p)}{f(\alpha)}-\left[\frac{f(\alpha x)}{f(\alpha)}\right]^p\leq p\left[\frac{f(\alpha x^p)}{f(\alpha)}-\frac{f\left(\alpha x^{p-1}\right)}{f(\alpha)}\frac{f(\alpha x)}{f(\alpha)}\right]. \tag{1.1}$$

Moreover, if $0<x\leq 1$, then

$$\begin{aligned}0&\leq \frac{f(\alpha x^p)}{f(\alpha)}-\left[\frac{f(\alpha x)}{f(\alpha)}\right]^p\leq p\left[\frac{f(\alpha x^p)}{f(\alpha)}-\frac{f\left(\alpha x^{p-1}\right)}{f(\alpha)}\frac{f(\alpha x)}{f(\alpha)}\right]\\&\leq \frac{1}{2}p\left(\frac{f\left(\alpha x^{2(p-1)}\right)}{f(\alpha)}-\left[\frac{f\left(\alpha x^{p-1}\right)}{f(\alpha)}\right]^2\right)^{1/2}\leq\frac{1}{4}p\end{aligned} \tag{1.2}$$

and

$$\begin{aligned}0&\leq \frac{f(\alpha x^p)}{f(\alpha)}-\left[\frac{f(\alpha x)}{f(\alpha)}\right]^p\leq p\left[\frac{f(\alpha x^p)}{f(\alpha)}-\frac{f\left(\alpha x^{p-1}\right)}{f(\alpha)}\frac{f(\alpha x)}{f(\alpha)}\right]\\&\leq \frac{1}{2}p\left(\frac{f\left(\alpha x^{2}\right)}{f(\alpha)}-\left[\frac{f(\alpha x)}{f(\alpha)}\right]^2\right)^{1/2}\leq\frac{1}{4}p.\end{aligned} \tag{1.3}$$

Corollary 1. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$. If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $u,v>0$ with $v^p\leq u^q<R$, then*

$$\left[\frac{f(uv)}{f(u^q)}\right]^p\leq\frac{f(v^p)}{f(u^q)}\leq\frac{1}{4}p+\left[\frac{f(uv)}{f(u^q)}\right]^p \tag{1.4}$$

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and

$$0 \leq [f(v^p)]^{1/p} [f(u^q)]^{1/q} - f(uv) \leq \frac{1}{4^{1/p}} p^{1/p} f(u^q). \tag{1.5}$$

Utilising a different approach in [6] we obtained the following results as well:

Theorem 2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R > 0$ or $R = \infty$. If $p > 1$, $0 < \alpha < R$ and $0 < x \leq 1$, then*

$$0 \leq \frac{f(\alpha x^p)}{f(\alpha)} - \left(\frac{f(\alpha x)}{f(\alpha)}\right)^p \leq M_p \left(1 - \frac{f(\alpha x)}{f(\alpha)}\right) \frac{f(\alpha x)}{f(\alpha)} \leq \frac{1}{4} M_p \tag{1.6}$$

and

$$0 \leq 0 \leq \frac{f(\alpha x^p)}{f(\alpha)} - \left(\frac{f(\alpha x)}{f(\alpha)}\right)^p \leq \frac{1}{4} \cdot \frac{1 - \left(\frac{f(\alpha x)}{f(\alpha)}\right)^{p-1}}{1 - \frac{f(\alpha x)}{f(\alpha)}} \leq \frac{1}{4} M_p, \tag{1.7}$$

where

$$M_p := \begin{cases} 1 \text{ if } p \in (1,2], \\ \\ p - 1 \text{ if } p \in (2, \infty). \end{cases}$$

Corollary 2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R > 0$ or $R = \infty$. If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $u, v > 0$ with $v^p \leq u^q < R$, then*

$$0 \leq \frac{f(v^p)}{f(u^q)} - \left(\frac{f(uv)}{f(u^q)}\right)^p \leq M_p \left(1 - \frac{f(uv)}{f(u^q)}\right) \frac{f(uv)}{f(u^q)} \leq \frac{1}{4} M_p \tag{1.8}$$

and

$$0 \leq \frac{f(v^p)}{f(u^q)} - \left(\frac{f(uv)}{f(u^q)}\right)^p \leq \frac{1}{4} \cdot \frac{1 - \left(\frac{f(uv)}{f(u^q)}\right)^{p-1}}{1 - \frac{f(uv)}{f(u^q)}} \leq \frac{1}{4} M_p. \tag{1.9}$$

For some similar exponential and logarithmic inequalities see [5] and [6] where further applications for some fundamental functions were provided .

For other recent results for power series with nonnegative coefficients, see [2], [8], [12] and [13]. For more results on power series inequalities, see [2] and [8]-[11].

The most important power series with nonnegative coefficients that can be used to illustrate the above results are:

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \; z \in \mathbb{C}, \; \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \; z \in D(0,1), \tag{1.10}$$

$$\ln \frac{1}{1-z} = \sum_{n=1}^{\infty} \frac{1}{n} z^n, \; z \in D(0,1), \; \cosh z = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}, \; z \in \mathbb{C},$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1}, \; z \in \mathbb{C}.$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\ z\in D(0,1), \tag{1.11}$$

$$\sin^{-1}(z)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}z^{2n+1},\ z\in D(0,1),$$

$$\tanh^{-1}(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\ z\in D(0,1),$$

$${}_2F_1(\alpha,\beta,\gamma,z):=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}z^n,\alpha,\beta,\gamma>0$$
$$z\in D(0,1),$$

where Γ is *Gamma function.*

Motivated by the above results and utilizing a global reverse of Jensen's inequality, we provide in this paper other inequalities for power series with nonnegative coefficients. Applications for some fundamental functions are given as well.

2. A Global Reverse of Jensen's Inequality

The following result holds:

Theorem 3. *Let $f : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, $\mathring{I}$ is the interior of I. If $x_i \in [m, M]$ and $w_i \geq 0$ $(i = 1, \dots, n)$ with $W_n := \sum_{i=1}^{n} w_i = 1$, then we have the inequalities*

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} w_i f(x_i) - f\left(\sum_{i=1}^{n} w_i x_i\right) \\ &\leq 2\max\left\{\frac{M-\sum_{i=1}^{n} w_i x_i}{M-m}, \frac{\sum_{i=1}^{n} w_i x_i - m}{M-m}\right\} \\ &\times\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]. \end{aligned} \tag{2.1}$$

Proof. First of all, we recall the following result obtained by the author in [**?**] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} &n\min_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right] \\ &\leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \\ &\leq n\max_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right], \end{aligned} \tag{2.2}$$

where $f : C \to \mathbb{R}$ is a convex function defined on the convex subset C of the linear space X, $\{x_i\}_{i\in\{1,\dots,n\}} \subset C$ are vectors and $\{p_i\}_{i\in\{1,\dots,n\}}$ are nonnegative numbers with $P_n := \sum_{i=1}^{n} p_i > 0$.

For $n=2$ we deduce from (2.2) that

$$\begin{aligned}
&2\min\{t,1-t\}\left[\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right] \\
&\le tf(x)+(1-t)f(y)-f(tx+(1-t)y) \\
&\le 2\max\{t,1-t\}\left[\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right]
\end{aligned} \tag{2.3}$$

for any $x,y\in C$ and $t\in[0,1]$.

If we use the second inequality in (2.3) for the convex function $f:I\to\mathbb{R}$ where $m,M\in\mathbb{R}$, $m<M$ with $[m,M]\subset\mathring{I}$, we have for $t=\frac{M-\sum_{i=1}^n w_ix_i}{M-m}$ that

$$\begin{aligned}
&\frac{\left(M-\sum_{i=1}^n w_ix_i\right)f(m)+\left(\sum_{i=1}^n w_ix_i-m\right)f(M)}{M-m} \\
&-f\left(\frac{m\left(M-\sum_{i=1}^n w_ix_i\right)+M\left(\sum_{i=1}^n w_ix_i-m\right)}{M-m}\right) \\
&\le 2\max\left\{\frac{M-\sum_{i=1}^n w_ix_i}{M-m},\frac{\sum_{i=1}^n w_ix_i-m}{M-m}\right\} \\
&\times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right].
\end{aligned} \tag{2.4}$$

By the convexity of f we have that

$$\begin{aligned}
&\sum_{i=1}^n w_if(x_i)-f\left(\sum_{i=1}^n w_ix_i\right) \\
&=\sum_{i=1}^n w_if\left[\frac{m(M-x_i)+M(x_i-m)}{M-m}\right] \\
&-f\left(\sum_{i=1}^n w_i\left[\frac{m(M-x_i)+M(x_i-m)}{M-m}\right]\right) \\
&\le\sum_{i=1}^n w_i\frac{(M-x_i)f(m)+(x_i-m)f(M)}{M-m} \\
&-f\left(\frac{m\left(M-\sum_{i=1}^n w_ix_i\right)+M\left(\sum_{i=1}^n w_ix_i-m\right)}{M-m}\right) \\
&=\frac{\left(M-\sum_{i=1}^n w_ix_i\right)f(m)+\left(\sum_{i=1}^n w_ix_i-m\right)f(M)}{M-m} \\
&-f\left(\frac{m\left(M-\sum_{i=1}^n w_ix_i\right)+M\left(\sum_{i=1}^n w_ix_i-m\right)}{M-m}\right).
\end{aligned} \tag{2.5}$$

Utilizing the inequality (2.5) and (2.4) we deduce the desired inequality in (2.1). □

For some related integral versions, see [4].

Remark 1. *Since, obviously,*

$$\frac{M-\sum_{i=1}^n w_ix_i}{M-m},\frac{\sum_{i=1}^n w_ix_i-m}{M-m}\le 1,$$

then we obtain from the first inequality in (2.1) the simpler, however coarser inequality, namely

$$0 \le \sum_{i=1}^{n} w_i f(x_i) - f\left(\sum_{i=1}^{n} w_i x_i\right) \le 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right], \tag{2.6}$$

provided that $x_i \in [m,M]$ *and* $w_i \ge 0$ $(i=1,\dots,n)$ *with* $W_n := \sum_{i=1}^{n} w_i = 1$.

This inequality was obtained in 2008 by S. Simić in [14].

Example 1. *a) If we write the inequality (2.1) for the convex function* $f : [m,M] \subset [0,\infty) \to [0,\infty)$, $f(t) = t^p$, $p \ge 1$, *then we have*

$$\begin{aligned} 0 &\le \sum_{i=1}^{n} w_i x_i^p - \left(\sum_{i=1}^{n} w_i x_i\right)^p \\ &\le 2\max\left\{\frac{M - \sum_{i=1}^{n} w_i x_i}{M-m}, \frac{\sum_{i=1}^{n} w_i x_i - m}{M-m}\right\}\left[\frac{m^p+M^p}{2} - \left(\frac{m+M}{2}\right)^p\right] \\ &\le 2\left[\frac{m^p+M^p}{2} - \left(\frac{m+M}{2}\right)^p\right], \end{aligned} \tag{2.7}$$

for any $x_i \in [m,M]$ *and* $w_i \ge 0$ $(i=1,\dots,n)$ *with* $W_n := \sum_{i=1}^{n} w_i = 1$.

b) If we apply the inequality (2.1) for the convex function $f : [m,M] \subset [0,\infty) \to [0,\infty)$, $f(t) = -\ln t$, *then we have*

$$\begin{aligned} 0 &\le \ln\left(\sum_{i=1}^{n} w_i x_i\right) - \sum_{i=1}^{n} w_i \ln x_i \\ &\le 2\max\left\{\frac{M - \sum_{i=1}^{n} w_i x_i}{M-m}, \frac{\sum_{i=1}^{n} w_i x_i - m}{M-m}\right\}\ln\left(\frac{\frac{m+M}{2}}{\sqrt{mM}}\right) \\ &\le \ln\left(\frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^2 \end{aligned} \tag{2.8}$$

for any $x_i \in [m,M]$ *and* $w_i \ge 0$ $(i=1,\dots,n)$ *with* $W_n := \sum_{i=1}^{n} w_i = 1$.

This inequality is equivalent to

$$\begin{aligned} 1 &\le \frac{\sum_{i=1}^{n} w_i x_i}{\prod_{i=1}^{n} x_i^{w_i}} \le \left(\frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^{2\max\left\{\frac{M - \sum_{i=1}^{n} w_i x_i}{M-m}, \frac{\sum_{i=1}^{n} w_i x_i - m}{M-m}\right\}} \\ &\le \frac{(m+M)^2}{4mM} \end{aligned} \tag{2.9}$$

for any $x_i \in [m,M]$ *and* $w_i \ge 0$ $(i=1,\dots,n)$ *with* $W_n := \sum_{i=1}^{n} w_i = 1$.

We can state the following result connected to *Hölder's inequality*:

Proposition 1. *If* $x_i \ge 0$, $y_i > 0$ *for* $i \in \{1,...,n\}$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and such that*

$$0 \le k \le \frac{x_i}{y_i^{q-1}} \le K \text{ for } i \in \{1,...,n\}, \tag{2.10}$$

then we have

$$0 \leq \frac{\sum_{i=1}^{n} x_i^p}{\sum_{i=1}^{n} y_i^q} - \left(\frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} y_i^q}\right)^p \tag{2.11}$$
$$\leq 2\max\left\{\frac{K - \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} y_i^q}}{K-k}, \frac{\frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} y_i^q} - k}{K-k}\right\}\left[\frac{k^p + K^p}{2} - \left(\frac{k+K}{2}\right)^p\right]$$
$$\leq 2\left[\frac{k^p + K^p}{2} - \left(\frac{k+K}{2}\right)^p\right].$$

Proof. The inequalities (2.11) follow from (2.7) by choosing

$$z_i = \frac{x_i}{y_i^{q-1}} \text{ and } w_i = \frac{y_i^q}{\sum_{j=1}^{n} y_j^q}, i \in \{1, ..., n\}.$$

The details are omitted. □

Remark 2. *Let* $p > 1, \frac{1}{p} + \frac{1}{q} = 1$. *Assume that*

$$0 \leq k \leq \frac{a_i}{b_i^{q-1}} \leq K, \text{ for } i \in \{1, ..., n\}. \tag{2.12}$$

If $p_i > 0$ *for* $i \in \{1, ..., n\}$, *then for* $x_i := p_i^{1/p} a_i$ *and* $y_i := p_i^{1/q} b_i$ *we have*

$$\frac{x_i}{y_i^{q-1}} = \frac{p_i^{1/p} a_i}{\left(p_i^{1/q} b_i\right)^{q-1}} = \frac{p_i^{1/p} a_i}{p_i^{(q-1)/q} b_i^{q-1}} = \frac{p_i^{1/p} a_i}{p_i^{1/p} b_i^{q-1}} = \frac{a_i}{b_i^{q-1}} \in [k, K]$$

for $i \in \{1, ..., n\}$.

If we write the inequality (2.11) for these choices, we get the weighted inequalities

$$0 \leq \frac{\sum_{i=1}^{n} p_i a_i^p}{\sum_{i=1}^{n} p_i b_i^q} - \left(\frac{\sum_{i=1}^{n} p_i a_i b_i}{\sum_{i=1}^{n} p_i b_i^q}\right)^p \tag{2.13}$$
$$\leq 2\max\left\{\frac{K - \frac{\sum_{i=1}^{n} p_i a_i b_i}{\sum_{i=1}^{n} p_i b_i^q}}{K-k}, \frac{\frac{\sum_{i=1}^{n} p_i a_i b_i}{\sum_{i=1}^{n} p_i b_i^q} - k}{K-k}\right\}\left[\frac{k^p + K^p}{2} - \left(\frac{k+K}{2}\right)^p\right]$$
$$\leq 2\left[\frac{k^p + K^p}{2} - \left(\frac{k+K}{2}\right)^p\right].$$

From this inequality we have:

$$\left(\frac{\sum_{i=1}^{n} p_i a_i b_i}{\sum_{i=1}^{n} p_i b_i^q}\right)^p \leq \frac{\sum_{i=1}^{n} p_i a_i^p}{\sum_{i=1}^{n} p_i b_i^q} \tag{2.14}$$
$$\leq \left(\frac{\sum_{i=1}^{n} p_i a_i b_i}{\sum_{i=1}^{n} p_i b_i^q}\right)^p + 2\left[\frac{k^p + K^p}{2} - \left(\frac{k+K}{2}\right)^p\right].$$

Taking into the second inequality of (2.14) the power $1/p$ *and utilizing the elementary inequality*

$$(\alpha + \beta)^{1/p} \leq \alpha^{1/p} + \beta^{1/p}, \alpha, \beta \geq 0 \text{ and } p > 1,$$

then we get the following additive reverse of Hölder inequality

$$\left(\sum_{i=1}^{n} p_i a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} p_i b_i^q\right)^{1/q} \leq \sum_{i=1}^{n} p_i a_i b_i + 2^{1/p}\left[\frac{k^p+K^p}{2} - \left(\frac{k+K}{2}\right)^p\right]^{1/p} \sum_{i=1}^{n} p_i b_i^q, \tag{2.15}$$

provided

$$0 \leq k \leq \frac{a_i}{b_i^{q-1}} \leq K, \text{ for } i \in \{1, ..., n\}$$

and $p_i > 0$ *for* $i \in \{1, ..., n\}$.

3. Power Inequalities

We can state the following result for powers:

Theorem 4. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a power series with nonnegative coefficients and convergent on the open disk* $D(0,R)$ *with* $R > 0$ *or* $R = \infty$. *If* $p > 1$, $0 < \alpha < R$ *and* $0 < x \leq 1$, *then*

$$0 \leq \frac{f(\alpha x^p)}{f(\alpha)} - \left(\frac{f(\alpha x)}{f(\alpha)}\right)^p \leq \frac{2^{p-1}-1}{2^{p-1}} \max\left\{1 - \frac{f(\alpha x)}{f(\alpha)}, \frac{f(\alpha x)}{f(\alpha)}\right\} \leq \frac{2^{p-1}-1}{2^{p-1}}. \tag{3.1}$$

Proof. Let $m \geq 1$ and $0 < \alpha < R$, $0 < x \leq 1$. If we write the inequality (2.7) for

$$w_j = \frac{a_j \alpha^j}{\sum_{k=0}^{m} a_k \alpha^k} \text{ and } z_j := x^j \in [0,1], \ j \in \{0, ..., m\},$$

then we get

$$0 \leq \frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j x^{pj} - \left(\frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j x^j\right)^p \leq \frac{2^{p-1}-1}{2^{p-1}} \max\left\{1 - \frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j x^j, \frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j x^j\right\} \leq \frac{2^{p-1}-1}{2^{p-1}}. \tag{3.2}$$

Since all series whose partial sums involved in the inequality (3.2) are convergent, then by letting $m \to \infty$ in (3.2) we deduce (2.15). □

We have:

Corollary 3. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a power series with nonnegative coefficients and convergent on the open disk* $D(0,R)$ *with* $R > 0$ *or* $R = \infty$. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $u, v > 0$ *with* $v^p \leq u^q < R$, *then*

$$0 \leq \frac{f(v^p)}{f(u^q)} - \left(\frac{f(uv)}{f(u^q)}\right)^p \leq \frac{2^{p-1}-1}{2^{p-1}} \max\left\{1 - \frac{f(uv)}{f(u^q)}, \frac{f(uv)}{f(u^q)}\right\} \tag{3.3}$$

and

$$0 \le [f(v^p)]^{1/p} [f(u^q)]^{1/q} - f(uv) \le \left(\frac{2^{p-1}-1}{2^{p-1}}\right)^{1/p} f(u^q). \tag{3.4}$$

Proof. The inequality (3.3) follows by taking into (3.1) $\alpha = u^q$ and $x = \frac{v}{u^{q/p}}$. The details are omitted.

Taking the power $1/p$ and using the inequality $(a+b)^{1/p} \le a^{1/p} + b^{1/p}$, $p \ge 1$ we get from

$$\frac{f(v^p)}{f(u^q)} \le \left(\frac{f(uv)}{f(u^q)}\right)^p + \frac{2^{p-1}-1}{2^{p-1}}$$

the desired inequality (3.4). □

Example 2. *a) If we write the inequality (3.1) for the function* $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $z \in D(0,1)$, *then we have*

$$0 \le \frac{1-\alpha}{1-\alpha x^p} - \left(\frac{1-\alpha}{1-\alpha x}\right)^p \le \frac{2^{p-1}-1}{2^{p-1}} \max\left\{\frac{\alpha(1-x)}{1-\alpha x}, \frac{1-\alpha}{1-\alpha x}\right\} \tag{3.5}$$

for any $\alpha, x \in (0,1)$ *and* $p \ge 1$.

b) If we write the inequality (3.1)) for the function $\exp z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$, $z \in \mathbb{C}$, *then we have*

$$\begin{aligned} 0 &\le \exp[\alpha(x^p - 1)] - \exp[p\alpha(x-1)] \\ &\le \frac{2^{p-1}-1}{2^{p-1}} \max\{1 - \exp[\alpha(x-1)], \exp[\alpha(x-1)]\} \end{aligned} \tag{3.6}$$

for any $\alpha, p > 0$ *and* $x \in (0,1)$.

4. Logarithmic Inequalities

If we write the inequality (2.1) for the convex function $f : [m, M] \subset (0, \infty) \to \mathbb{R}$, $f(t) = t \ln t$, then we have

$$\begin{aligned} 0 &\le \sum_{i=1}^n w_i x_i \ln x_i - \left(\sum_{i=1}^n w_i x_i\right) \ln\left(\sum_{i=1}^n w_i x_i\right) \\ &\le 2 \max\left\{\frac{M - \sum_{i=1}^n w_i x_i}{M-m}, \frac{\sum_{i=1}^n w_i x_i - m}{M-m}\right\} \\ &\times \left[\frac{m \ln m + M \ln M}{2} - \left(\frac{m+M}{2}\right) \ln\left(\frac{m+M}{2}\right)\right] \end{aligned} \tag{4.1}$$

for any $x_i \in [m, M]$ and $w_i \ge 0$ $(i = 1, \dots, n)$ with $W_n := \sum_{i=1}^n w_i = 1$.

This is equivalent to

$$\begin{aligned} 1 &\le \frac{\prod_{i=1}^n x_i^{w_i x_i}}{\left(\sum_{i=1}^n w_i x_i\right)^{\left(\sum_{i=1}^n w_i x_i\right)}} \\ &\le \left[\frac{m^m M^M}{\left(\frac{m+M}{2}\right)^{m+M}}\right]^{\max\left\{\frac{M - \sum_{i=1}^n w_i x_i}{M-m}, \frac{\sum_{i=1}^n w_i x_i - m}{M-m}\right\}} \end{aligned} \tag{4.2}$$

for any $x_i \in [m, M]$ and $w_i \ge 0$ $(i = 1, \dots, n)$ with $W_n := \sum_{i=1}^n w_i = 1$.

If we take $M = 1$ and let $m \to 0+$ in the inequality (4.1), we have

$$0 \le \sum_{i=1}^{n} w_i x_i \ln x_i - \left(\sum_{i=1}^{n} w_i x_i\right) \ln \left(\sum_{i=1}^{n} w_i x_i\right) \le \max\left\{1 - \sum_{i=1}^{n} w_i x_i, \sum_{i=1}^{n} w_i x_i\right\} \ln 2, \tag{4.3}$$

for any $x_i \in (0,1]$ and $w_i \ge 0 \ \ (i = 1, \dots, n)$ with $W_n := \sum_{i=1}^{n} w_i = 1$.

This is equivalent to

$$1 \le \frac{\prod_{i=1}^{n} x_i^{w_i x_i}}{\left(\sum_{i=1}^{n} w_i x_i\right)^{\left(\sum_{i=1}^{n} w_i x_i\right)}} \le 2^{\max\left\{1 - \sum_{i=1}^{n} w_i x_i, \sum_{i=1}^{n} w_i x_i\right\}}, \tag{4.4}$$

for any $x_i \in (0,1]$ and $w_i \ge 0 \ \ (i = 1, \dots, n)$ with $W_n := \sum_{i=1}^{n} w_i = 1$.

Theorem 5. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R > 0$ or $R = \infty$. If $0 < \alpha < R$, $p > 0$ and $x \in (0,1)$, then*

$$0 \le \frac{p\alpha x^p f'(\alpha x^p)}{f(\alpha)} \ln x - \frac{f(\alpha x^p)}{f(\alpha)} \ln \left(\frac{f(\alpha x^p)}{f(\alpha)}\right) \le \max\left\{1 - \frac{f(\alpha x^p)}{f(\alpha)}, \frac{f(\alpha x^p)}{f(\alpha)}\right\} \ln 2. \tag{4.5}$$

Proof. If $0 < \alpha < R$ and $m \ge 1$, then by (4.3) for $x_j = (x^p)^j$, we have

$$\begin{aligned} 0 &\le \frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j (x^p)^j \ln (x^p)^j \\ &- \left(\frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j (x^p)^j\right) \ln \left(\frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j (x^p)^j\right) \\ &\le \max\left\{1 - \frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j (x^p)^j, \frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j (x^p)^j\right\} \ln 2, \end{aligned} \tag{4.6}$$

for $p > 0$ and $x \in (0,1)$.

This is equivalent to:

$$\begin{aligned} 0 &\le \frac{p \ln (x)}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} j a_j \alpha^j (x^p)^j \\ &- \left(\frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j (x^p)^j\right) \ln \left(\frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j (x^p)^j\right) \\ &\le \max\left\{1 - \frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j (x^p)^j, \frac{1}{\sum_{k=0}^{m} a_k \alpha^k} \sum_{j=0}^{m} a_j \alpha^j (x^p)^j\right\} \ln 2, \end{aligned} \tag{4.7}$$

for $p > 0$ and $x \in (0,1)$.

Since all series whose partial sums involved in the inequality (4.7) are convergent, then by letting $m \to \infty$ in (4.7) we deduce (4.5). □

Example 3. *a) If we write the inequality (4.5) for the function* $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $z \in D(0,1)$, *then we have for* $\alpha, x \in (0,1)$ *and* $p > 0$ *that*

$$0 \le \frac{p\alpha x^p (1-\alpha)}{(1-\alpha x^p)^2} \ln x - \frac{1-\alpha}{(1-\alpha x^p)} \ln\left(\frac{1-\alpha}{1-\alpha x^p}\right) \le \max\left\{\frac{\alpha(1-x^p)}{1-\alpha x^p}, \frac{1-\alpha}{1-\alpha x^p}\right\} \ln 2 \tag{4.8}$$

b) If we write the inequality (4.5) for the function $\exp z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$, $z \in \mathbb{C}$, *then we have*

$$\begin{aligned} 0 &\le [p\alpha x^p \ln x - \alpha (x^p - 1)] \exp[\alpha (x^p - 1)] \\ &\le \max\{1 - \exp[\alpha (x^p - 1)], \exp[\alpha (x^p - 1)]\} \ln 2 \end{aligned} \tag{4.9}$$

for $x \in (0,1)$ *and* $\alpha, p > 0$.

5. Exponential Inequalities

If we consider the exponential function $f : \mathbb{R} \to (0, \infty)$, $f(t) = \exp(\beta t)$ with $\beta > 0$, then from (2.1) we have the inequalities

$$\begin{aligned} 0 &\le \sum_{i=1}^{n} w_i \exp(\beta x_i) - \exp\left(\beta \sum_{i=1}^{n} w_i x_i\right) \\ &\le 2 \max\left\{\frac{M - \sum_{i=1}^{n} w_i x_i}{M-m}, \frac{\sum_{i=1}^{n} w_i x_i - m}{M-m}\right\} \\ &\times \left[\frac{\exp(\beta m) + \exp(\beta M)}{2} - \exp\left[\beta\left(\frac{m+M}{2}\right)\right]\right] \end{aligned} \tag{5.1}$$

if $x_i \in [m, M]$ and $w_i \ge 0 \quad (i = 1, \dots, n)$ with $W_n := \sum_{i=1}^{n} w_i = 1$.

If we take in (5.1) $M = 0$ and let $m \to -\infty$, then we get

$$0 \le \sum_{i=1}^{n} w_i \exp(\beta x_i) - \exp\left(\beta \sum_{i=1}^{n} w_i x_i\right) \le 1 \tag{5.2}$$

for $x_i \le 0$ and $w_i \ge 0$ $(i = 1, \dots, n)$ with $W_n := \sum_{i=1}^{n} w_i = 1$.

Theorem 6. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a power series with nonnegative coefficients and convergent on the open disk* $D(0,R)$ *with* $R > 0$ *or* $R = \infty$. *If* $x \le 0$, $\beta > 0$ *with* $\exp(\beta x) < R$ *and* $0 < \alpha < R$, *then*

$$0 \le \frac{f(\alpha \exp(\beta x))}{f(\alpha)} - \exp\left[\frac{\alpha \beta x f'(\alpha)}{f(\alpha)}\right] \le 1. \tag{5.3}$$

Proof. If $0 < \alpha < R$ and $m \ge 1$, then by (5.2) for $x_j = jx$, we have

$$0 \le \frac{1}{\sum_{j=0}^{m} a_j \alpha^j} \sum_{j=0}^{m} a_j \alpha^j [\exp(\beta x)]^j - \exp\left(\frac{\beta x}{\sum_{j=0}^{m} a_j \alpha^j} \sum_{j=0}^{m} j a_j \alpha^j\right) \le 1 \tag{5.4}$$

for $x \in (-\infty, 0)$.

Since all series whose partial sums involved in the inequality (5.4) are convergent, then by letting $m \to \infty$ in (5.4) we deduce (5.3). □

Example 4. *a) If we write the inequality (5.3) for the function* $\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n$, $z\in D(0,1)$, *then we have for* $x\leq 0$, $\beta>0$ *and* $0<\alpha<1$, *that*

$$0\leq \frac{1-\alpha}{1-\alpha\exp(\beta x)}-\exp\left(\frac{\alpha\beta x}{1-\alpha}\right)\leq 1. \tag{5.5}$$

b) If we write the inequality (5.3) for the function $\exp z=\sum_{n=0}^{\infty}\frac{1}{n!}z^n$, $z\in\mathbb{C}$, *then we have*

$$0\leq \exp(\alpha[\exp(\beta x)-1])-\exp(\alpha\beta x)\leq 1 \tag{5.6}$$

for any $\alpha>0$ *and* $x\leq 0$, $\beta>0$.

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