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Symmetrization, convexity and applications

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Abstract

Based on permutation enumeration of the symmetric group and 'generalized' barycentric coordinates on arbitrary convex polytope, we develop a technique to obtain symmetrization procedures for functions that provide a unified framework to derive new Hermite-Hadamard type inequalities. We also present applications of our results to the Wright-convex functions with special emphasis on their key role in convexity. In one dimension, we obtain (up to a positive multiplicative constant) a method of symmetrization recently introduced by Dragomir [3], and also by El Farissi et al. [4]. So our approach can be seen as a multivariate generalization of their method.

Keywords: Barycentric coordinates-Convex functions-Permutations-Convex Polytopes-Hermite-Hadamard inequality-Inequalities-Symmetrization of functions-Wright functions.

1. Introduction and Preliminaries

Throughout Ω will always denote a convex polytope with a non-empty interior (that is, the convex hull of $(n+1)$ vertices $\{\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ in $\mathbb{R}^d$). The points $\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ may be regarded as vectors in any linear vector space, which in this paper will be taken to be Euclidean. We will refer to the vertex centroid of Ω or of $V(\Omega) := \{\boldsymbol{v}_i\}_{i=0}^n$ as the average of the vertices in $V(\Omega)$. We define the notion of (generalized) barycentric coordinates in the remainder of this paper as follows: let $\boldsymbol{x}$ be an arbitrary point of Ω. We call

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barycentric coordinates of $\boldsymbol{x}$ with respect to $V(\Omega)$ any set of real coefficients $\{\lambda_i(\boldsymbol{x})\}_{i=0}^n$ depending on the vertices of Ω and on $\boldsymbol{x}$ such that all the three following properties hold true:

$$\lambda_i(\boldsymbol{x}) \geq 0, \quad i = 0, \ldots, n, \tag{1}$$

$$\sum_{i=0}^{n} \lambda_i(\boldsymbol{x}) = 1, \tag{2}$$

$$\boldsymbol{x} = \sum_{i=0}^{n} \lambda_i(\boldsymbol{x}) \boldsymbol{v}_i. \tag{3}$$

The generalized barycentric coordinates for an arbitrary convex polytope are a key notion in formulating our method for symmetrization of functions. Recall that these coordinates exist for more general types of polytopes. The first result on their existence was due to Kalman [8, Theorem 2] (1961). Barycentric coordinates for simplices are uniquely determined, however they can lose their uniqueness for general convex polytopes. One possible natural approach to constructing an interesting class of particular barycentric coordinates would be to simply construct a triangulation of the polytope Ω - the convex hull of the data set $V(\Omega)$ - into simplices such that the vertices $\boldsymbol{v}_i$ of the triangulation coincide with $V(\Omega)$. The fact that every convex polytope Ω can be triangulated using only the vertices of Ω is proven in the appendix of [2]. After that, one can use the standard barycentric coordinates for these simplices. (The above choice of coordinates is obviously immaterial, and it is only meant to simplify computations throughout and to fix the notation.) For more details, see for instance [1]. As a result, each triangulation of $V(\Omega)$ generates a set of barycentric coordinates, that satisfy all the desirable properties (1), (2) and (3). From now on, we assume that $\{\lambda_i\}_{i=0}^n$ is generated by a triangulation of $V(\Omega)$. In this situation, we list some other properties of these functions of which the following are particularly relevant to us:

(1) They are well-defined, piecewise linear and nonnegative real-valued continuous functions.

(2) Since vertices of a convex polytope are extremal points, it is easily deduced from (3) that $\{\lambda_i\}_{i=0}^n$ satisfy the delta property

$$\lambda_i(\boldsymbol{v}_j) = \delta_{ij}, \quad (i, j \in \{0, \ldots, n\}), \tag{4}$$

where we use Kronecker's delta.

We refer to reference [6] for details.

This paper is organized as follows. In Section 2, we establish and analyze links between permutation enumeration of the symmetric group and barycentric coordinates. In Section 3, we give our precise definition of symmetrization for any function defined on an arbitrary convex polytope. We will show that this method of symmetrization has some desirable properties. Under this method and the convexity assumption on the original function, our main result is that the resulting symmetrized function satisfies some new Hermite-Hadamard type inequalities. In Section 4, we impose a convexity assumption on the symmetrized function instead of the original function and present a refined version of our main results in this setting. Section 5 develops, under some assumptions on the polytope, a weighted general version of our method of symmetrization. The symmetrization techniques that we propose in this section are applicable to wide variety of domains, including simplices and Cartesian hyperrectangles. Finally, in Section 6, we consider some applications to the class of Wright-convex functions with special emphasis on their key role in convexity. In one dimension, we obtain (up to a positive multiplicative constant) a method of symmetrization recently introduced by Dragomir [3] and also by El Farissi et al. [4]. Some discussion of this is in Section 3. So our approach can be seen as a multivariate generalization of their method. This paper is motivated in part by the results presented in [3, 4].

2. Permutations and barycentric coordinates

In this section, we establish and analyze links between permutations and barycentric coordinates on arbitrary convex polytopes. We will use them repeatedly in our further discussions.
We first start by presenting some basic notations and definitions. The set of all permutations on $\{0, 1, \ldots, n\}$ is denoted by S_n. Recall that each permutation $\sigma \in S_n$ is a 1-to-1 map:

$$\sigma : \{0, \ldots, n\} \to \{0, \ldots, n\},$$

so that $\{\sigma(0), \ldots, \sigma(n)\}$ is a re-arrangement of $\{0, \ldots, n\}$. There are $(n+1)!$ permutations in S_n, moreover, every permutation σ generates a mapping $T_\sigma : \Omega \to \mathbb{R}^d$, defined for all $\boldsymbol{x}$ in Ω by setting

$$
\begin{aligned}
T_\sigma(\boldsymbol{x}) &:= T_\sigma\left(\sum_{i=0}^{n} \lambda_i(\boldsymbol{x})\boldsymbol{v}_i\right) \\
&= \sum_{i=0}^{n} \lambda_i(\boldsymbol{x})\boldsymbol{v}_{\sigma(i)}.
\end{aligned}
$$

The reader may check that T_σ is well-defined, continuous, and satisfies the property $T_\sigma(\Omega) \subset \Omega$, for any σ of S_n. In fact, we may say more about T_σ. Let us start by the following observations, which show how Ω, $T_\sigma(\Omega)$, $V(\Omega)$ and $V(T_\sigma(\Omega))$ are related for any $\sigma \in S_n$. Here, for any polytope X, we have used the notation $V(X)$ to denote the set of vertices of X.

Proposition 2.1. *For any permutation $\sigma \in S_n$, the mapping T_σ satisfies:*

$$
T_\sigma(\boldsymbol{v}_j) = \boldsymbol{v}_{\sigma(j)}, \quad j = 0, \ldots, n, \tag{5}
$$

$$
T_\sigma(\Omega) = \Omega. \tag{6}
$$

In particular, T_σ sends vertices of Ω to vertices of $T_\sigma(\Omega)$ and the vertex centroid of Ω is also that of $V(T_\sigma(\Omega)) := \{T_\sigma(\boldsymbol{v}_i)\}_{i=0}^n$. That is

$$
\frac{1}{n+1}\sum_{i=0}^{n} \boldsymbol{v}_i = \frac{1}{n+1}\sum_{i=0}^{n} T_\sigma(\boldsymbol{v}_i). \tag{7}
$$

Proof. The first identity is simple to prove. It follows naturally from the Kronecker delta property of the barycentric coordinates. Indeed, by definition for all $j = 0, \ldots, n$, we have $T_\sigma(\boldsymbol{v}_j) = \sum_{i=0}^{n} \lambda_i(\boldsymbol{v}_j)\boldsymbol{v}_{\sigma(i)}$. Thus, the required equality follows immediately from the fact that $\lambda_i(\boldsymbol{v}_j) = \delta_{ij}$.
To prove the second equality, let us pick an element $\boldsymbol{y}$ in $T_\sigma(\Omega)$. Then, there exists an $\boldsymbol{x}$ in Ω such that $\boldsymbol{y} = T_\sigma(\boldsymbol{x})$. But, by definition, $T_\sigma(\boldsymbol{x}) = \sum_{i=0}^{n} \lambda_i(\boldsymbol{x})\boldsymbol{v}_{\sigma(i)}$. Thus, $T_\sigma(\boldsymbol{x}) \in \Omega$, and consequently $T_\sigma(\Omega) \subset \Omega$. Now, we have to prove the inverse inclusion $\Omega \subset T_\sigma(\Omega)$. Let us take $\boldsymbol{x} \in \Omega$, then $\boldsymbol{x} = \sum_{i=0}^{n} \lambda_i(\boldsymbol{x})\boldsymbol{v}_i$. Define $\tilde{\boldsymbol{x}} = \sum_{i=0}^{n} \lambda_{\sigma^{-1}(i)}(\tilde{\boldsymbol{x}})\boldsymbol{v}_{\sigma^{-1}(i)}$, where

$$
\lambda_{\sigma^{-1}(i)}(\tilde{\boldsymbol{x}}) := \lambda_i(\boldsymbol{x}), i = 0, \ldots n.
$$

Now, it is easy to verify that $\tilde{\boldsymbol{x}}$ belongs to Ω and $T_\sigma(\tilde{\boldsymbol{x}}) = \boldsymbol{x}$. Thus, $\Omega \subset T_\sigma(\Omega)$, which completes the proof of the equality of two sets. To show the last equality, it suffices to observe that

$$\begin{aligned}
T_\sigma\left(\sum_{i=0}^{n}\frac{1}{n+1}\boldsymbol{v}_i\right) &:= \sum_{i=0}^{n}\frac{1}{n+1}\boldsymbol{v}_{\sigma(i)} \\
&= \frac{1}{n+1}\sum_{i=0}^{n}\boldsymbol{v}_{\sigma(i)} \\
&= \frac{1}{n+1}\sum_{i=0}^{n}\boldsymbol{v}_i.
\end{aligned}$$

As a consequence of the definition of T_σ, some other basic facts are given in Proposition 2.2 below.

Proposition 2.2. *For any permutation $\sigma \in S_n$, the mapping T_σ is a homeomorphism of Ω onto itself. Moreover, we have*

$$T_\sigma^{-1} = T_{\sigma^{-1}}. \tag{8}$$

Proof. For any permutation $\sigma \in S_n$, surjectivity of T_σ follows from the fact that $T_\sigma(\Omega) = \Omega$. Its injectivity is an immediate consequence of the uniqueness property of the barycentric coordinates for simplices. Also, as mentioned in the introduction, the continuity of T_σ and T_σ^{-1} follows from the fact that the barycentric coordinates of a point are continuous functions of that point. Finally, we compute that $T_{\sigma^{-1}}(T_\sigma(\boldsymbol{x})) = \boldsymbol{x}$ for all $\boldsymbol{x} \in \Omega$. Indeed, we get

$$\begin{aligned}
T_{\sigma^{-1}}(T_\sigma(\boldsymbol{x})) &= T_{\sigma^{-1}}\left(\sum_{i=0}^{n}\lambda_i(\boldsymbol{x})\boldsymbol{v}_{\sigma(i)}\right) \\
&= T_{\sigma^{-1}}\left(\sum_{i=0}^{n}\lambda_{\sigma^{-1}(i)}(\boldsymbol{x})\boldsymbol{v}_i\right) \\
&:= \sum_{i=0}^{n}\lambda_{\sigma^{-1}(i)}(\boldsymbol{x})\boldsymbol{v}_{\sigma^{-1}(i)} \\
&= \boldsymbol{x}.
\end{aligned}$$

This shows that the inverse of T_σ is $T_{\sigma^{-1}}$ and completes the proof of Proposition 2.2.

As regards the composition, we may observe the following:

Remark 2.3. *A simple inspection shows that if* $\sigma, \tau \in S_n$ *then*

$$\begin{aligned} T_\sigma \circ T_\tau &= T_\sigma(T_\tau) \\ &= T_{\tau\circ\sigma}. \end{aligned}$$

This confirms equality (8) given by Proposition 2.2.

For any permutation $\sigma \in S_n$, let us define the set of functions $\left\{\tilde{\lambda}_{i,\sigma}\right\}_{i=0}^{n}$ as follows: for all $i = 0, \ldots, n$ and all $\boldsymbol{y}$ in $T_\sigma(\Omega)$, we set

$$\tilde{\lambda}_{i,\sigma}(\boldsymbol{y}) := \lambda_{\sigma(i)}(T_\sigma(\boldsymbol{x})),$$

where $\boldsymbol{x} \in \Omega$ is such that $\boldsymbol{y} = T_\sigma(\boldsymbol{x})$.

Proposition 2.4. *For any permutation* $\sigma \in S_n$, $\left\{\tilde{\lambda}_{i,\sigma}\right\}_{i=0}^{n}$ *defines a set of barycentric coordinates on* $T_\sigma(\Omega)$ *with respect to* $\left\{\boldsymbol{v}_{\sigma(i)}\right\}_{i=0}^{n}$.

PROOF. Let us fix a permutation σ in S_n. Clearly, since $\{\lambda_i\}_{i=0}^n$ defines a set of barycentric coordinates, then, for any $\boldsymbol{y}$ in $T_\sigma(\Omega)$ it holds

$$\tilde{\lambda}_{i,\sigma}(\boldsymbol{y}) \geq 0, \quad i = 0, \ldots, n, \tag{9}$$

$$\sum_{i=0}^{n} \tilde{\lambda}_{i,\sigma}(\boldsymbol{y}) = 1, \tag{10}$$

so we only need to prove that $\left\{\tilde{\lambda}_{i,\sigma}\right\}_{i=0}^{n}$ satisfies the linear precision property, that is, for all $\boldsymbol{y} \in T_\sigma(\Omega)$, we have

$$\boldsymbol{y} = \sum_{i=0}^{n} \tilde{\lambda}_{i,\sigma}(\boldsymbol{y}) \boldsymbol{v}_{\sigma(i)}. \tag{11}$$

Now, since $\boldsymbol{y} = T_\sigma(\boldsymbol{x})$ also belongs to Ω then from (3) we get

$$\begin{aligned} \boldsymbol{y} = T_\sigma(\boldsymbol{x}) &= \sum_{i=0}^{n} \lambda_i(T_\sigma(\boldsymbol{x}))\boldsymbol{v}_i \\ &= \sum_{i=0}^{n} \lambda_{\sigma(i)}(T_\sigma(\boldsymbol{x}))\boldsymbol{v}_{\sigma(i)} \\ &= \sum_{i=0}^{n} \tilde{\lambda}_{i,\sigma}(\boldsymbol{y})\boldsymbol{v}_{\sigma(i)}, \end{aligned}$$

which is exactly the required equality. This completes the proof of Proposition 2.4.

The next results show simple but important identities:

Lemma 2.5. *For any $\boldsymbol{x}$ in Ω and any permutation τ, the following identity holds*

$$\sum_{i=0}^{n} \lambda_{\tau(i)}(\boldsymbol{x}) = 1. \tag{12}$$

Moreover, we have

$$\frac{1}{(n+1)!} \sum_{\sigma \in S_n} \lambda_{\sigma(i)}(\boldsymbol{x}) = \frac{1}{n+1}, i = 0, \ldots, n. \tag{13}$$

Proof. The first identity follows immediately from the partition of unity property of the barycentric coordinates (2) and the fact that any permutation τ in S_n satisfies

$$\sum_{i=0}^{n} \lambda_{\tau(i)}(\boldsymbol{x}) = \sum_{i=0}^{n} \lambda_i(\boldsymbol{x}).$$

To prove the second identity, it is easy to see that $\sum_{\sigma \in S_n} \lambda_{\sigma(i)}(\boldsymbol{x})$ is independent of i. The result now follows from the first identity.

As a consequence, we have the following representations of the centroid of Ω. These representations will be useful in the sequel.

Proposition 2.6. *The centroid of Ω may be represented as follows:*

$$\begin{aligned} \frac{1}{n+1} \sum_{i=0}^{n} \boldsymbol{v}_i &= T_\tau \left(\frac{1}{n+1} \sum_{i=0}^{n} \boldsymbol{v}_i \right), \quad (\forall \tau \in S_n), & (14) \\ &= \frac{1}{(n+1)!} \sum_{\sigma \in S_n} \boldsymbol{v}_{\sigma(i)}, \quad (\forall i = 0, \ldots, n), & (15) \\ &= \frac{1}{(n+1)!} \sum_{\sigma \in S_n} T_\sigma(\boldsymbol{x}), \quad (\forall \boldsymbol{x} \in \Omega). & (16) \end{aligned}$$

Proof. We have already proved the first equality in Proposition 2.1. To prove the second equality, it suffices to observe that

$$\sum_{\sigma \in S_n} \boldsymbol{v}_{\sigma(i)} = n! \sum_{i=0}^{n} \boldsymbol{v}_i, i = 0, \ldots, n.$$

Now, the last equality may be verified as follows:

$$\begin{aligned}
\frac{1}{(n+1)!}\sum_{\sigma\in S_n} T_\sigma(\boldsymbol{x}) &:= \frac{1}{(n+1)!}\sum_{\sigma\in S_n}\left(\sum_{i=0}^{n}\lambda_i(\boldsymbol{x})\boldsymbol{v}_{\sigma(i)}\right)\\
&= \sum_{i=0}^{n}\lambda_i(\boldsymbol{x})\left(\frac{1}{(n+1)!}\sum_{\sigma\in S_n}\boldsymbol{v}_{\sigma(i)}\right)\\
&= \sum_{i=0}^{n}\lambda_i(\boldsymbol{x})\left(\frac{1}{n+1}\sum_{j=0}^{n}\boldsymbol{v}_j\right)\\
&= \frac{1}{n+1}\sum_{j=0}^{n}\boldsymbol{v}_j\left(\sum_{i=0}^{n}\lambda_i(\boldsymbol{x})\right)\\
&= \frac{1}{n+1}\sum_{j=0}^{n}\boldsymbol{v}_j.
\end{aligned}$$

This checks (16) and completes the proof of the theorem.

Remark 2.7. *It is remarkable that the term on the right-hand side of last equality in Proposition 2.6 is independent of $\boldsymbol{x}$. Note that since, for any permutation $\sigma \in S_n$, Proposition 2.1 shows that the mapping T_σ sends the vertex $\boldsymbol{v}_i$ to $\boldsymbol{v}_{\sigma(i)}$, then by taking $\boldsymbol{x} = \boldsymbol{v}_i$ in (16) we have again that:*

$$\frac{1}{n+1}\sum_{j=0}^{n}\boldsymbol{v}_j = \frac{1}{(n+1)!}\sum_{\sigma\in S_n} T_\sigma(\boldsymbol{v}_i) \tag{17}$$

$$= \frac{1}{(n+1)!}\sum_{\sigma\in S_n}\boldsymbol{v}_{\sigma(i)}, i = 0,\dots,n. \tag{18}$$

This is exactly the equality in (15). Also, by taking the centroid as a particular value in (16), the first equality may similarly be derived from (16).

3. Construction of the symmetrization procedure

In this section, based on permutations and barycentric coordinates on arbitrary convex polytopes, we develop a technique to obtain new symmetrization procedure of functions. We will also show that this method has some desirable properties. Up to now the symmetrization of functions has been investigated through index permutations of components $x_1, \dots, x_d$ of the vector

$\boldsymbol{x}$ in domains, that satisfy certain symmetry conditions. Techniques suitable for more general domains are largely unknown. To avoid this type of problem, our main idea was to use barycentric coordinates and permutations of the vertices in order to establish a symmetrization procedure. A general symmetrization construction will be introduced later. This is the objective of Section 5.

We will use the following symmetrization technique.

Definition 3.1. *The symmetrization of an arbitrary function* $f : \Omega \to \mathbb{R}$ *is*

$$\tilde{f}(\boldsymbol{x}) = \frac{1}{(n+1)!} \sum_{\sigma \in S_n} f(T_\sigma(\boldsymbol{x})). \tag{19}$$

We note that the symmetrized function (19) is correctly defined in our definition since for any permutation σ the map T_σ sends Ω to Ω.

In order to motivate our symmetrization approach, let us start with a very simple one-dimensional example of the computation of a symmetrized function created in this manner. Consider the case $\Omega = [a,b] \subset \mathbb{R}$; then it is easily seen that the barycentric coordinates of a point x of Ω with respect to $v_0 := a, v_1 := b$ are given respectively as follows:

$$\begin{aligned} \lambda_0(x) &= \frac{x-b}{a-b}, \\ \lambda_1(x) &= \frac{x-a}{b-a}. \end{aligned}$$

Since $n = 1$, then there are only two permutations:

$$\begin{aligned} \sigma_0(\{0,1\}) &:= \{0,1\}, \\ \sigma_1(\{0,1\}) &:= \{1,0\}, \end{aligned}$$

and therefore the symmetrized function (19) is nicely reduced to the simple form

$$\tilde{f}(x) = \frac{f(x) + f(a+b-x)}{2}. \tag{20}$$

Moreover, it is easy to verify that the function $\tilde{f}$ is symmetric with respect to $(a+b)/2$ in the sense that for all x on Ω, we have

$$\tilde{f}(a+b-x) = \tilde{f}(x).$$

Therefore, in that case the symmetrization procedure of Definition 3.1 assigns to any function f a unique *symmetric* function $\tilde{f}$.
Coincidentally, at the same time, formula (20), up to a multiplicative factor of 2, was introduced very recently as a symmetrization of functions in one dimension by Dragomir [3], and also by El Farissi et al. [4]. Hence, our proposed method of symmetrization can be viewed as a multivariate generalization of their approach in the univariate case. The word symmetrization is used since the construction in the one-dimensional case produces functions which are really symmetric in the classical sense. Finally, we might recall that the symmetrized function (20) has been studied extensively in [7] to determine approximate integrals. It should be noted that

$$\frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{b-a}\int_a^b \tilde{f}(x)dx.$$

The next result shows that the symmetrized function of an arbitrary affine functions has the following simple representations:

Lemma 3.2. *The symmetrized function $\tilde{f}$ of any affine function, $f:\Omega\to\mathbb{R}$ satisfies, for all $\boldsymbol{x}\in\Omega$,*

$$\tilde{f}(\boldsymbol{x}) = f\left(\frac{1}{n+1}\sum_{i=0}^{n}\boldsymbol{v}_i\right) \tag{21}$$

$$= \frac{1}{n+1}\sum_{i=0}^{n} f(\boldsymbol{v}_i). \tag{22}$$

Proof. We first define the operator $\mathcal{L}$ in $C(\Omega)$ the space set of real continuous functions on Ω by the formula $\mathcal{L}(f)=\tilde{f}$. It is easy to verify that constant functions are fixed points for the operator $\mathcal{L}$, in the sense that if f is a constant function then $\mathcal{L}(f)=f$. This shows that equalities (21) and (22) are satisfied for all constant functions. Now, since $\mathcal{L}$ is a linear operator from $C(\Omega)$ into $C(\Omega)$, it remains to show that (21) and (22) hold for the projection functions $e_j, j=1,\ldots,d$, defined by:

$$e_j \,:\, \boldsymbol{x}=(x_1,\ldots,x_d)\in\Omega \longmapsto x_j.$$

For any function e_j, we have by definition:

$$\tilde{e_j}(\boldsymbol{x}) := \frac{1}{(n+1)!}\sum_{\sigma\in S_n} e_j\left(T_\sigma(\boldsymbol{x})\right).$$

But, by the linearity of the $j-$th projection function e_j and the representations of centroid, given in Proposition 2.6, we already know:

$$\frac{1}{(n+1)!}\sum_{\sigma\in S_n} e_j\left(T_\sigma(\boldsymbol{x})\right) = \frac{1}{n+1}\sum_{i=0}^{n} e_j\left(\boldsymbol{v}_i\right) \tag{23}$$

$$= e_j\left(\frac{1}{n+1}\sum_{i=0}^{n}\boldsymbol{v}_i\right). \tag{24}$$

Hence, equalities (21) and (22) hold for any $\boldsymbol{x}\in\Omega$. This completes the proof of Lemma 3.2.

Remark 3.3. *The following simple proof of Lemma (3.2) was suggested to us by the referee*[1]*. Let us denote the standard inner product in $\mathbb{R}^d$ by $\langle .,.\rangle$. Since f is affine, it can be represented in the form $f(\boldsymbol{x}) = \langle \boldsymbol{a},\boldsymbol{x}\rangle + b$, for $\boldsymbol{x}\in\Omega$ and some $\boldsymbol{a}\in\mathbb{R}, b\in\mathbb{R}$. So*

$$\begin{aligned}\tilde{f}(\boldsymbol{x}) &= \tfrac{1}{(n+1)!}\textstyle\sum_{\sigma\in S_n} f\left(T_\sigma(\boldsymbol{x})\right) = \left\langle \boldsymbol{a}, \tfrac{1}{(n+1)!}\textstyle\sum_{\sigma\in S_n} T_\sigma(\boldsymbol{x})\right\rangle + b\\ &= \left\langle \boldsymbol{a}, \tfrac{1}{n+1}\textstyle\sum_{i=0}^{n}\boldsymbol{v}_i\right\rangle + b = f\left(\tfrac{1}{n+1}\textstyle\sum_{i=0}^{n}\boldsymbol{v}_i\right)\\ &= \tfrac{1}{n+1}\textstyle\sum_{i=0}^{n}\langle\boldsymbol{a},\boldsymbol{v}_i\rangle + b = \tfrac{1}{n+1}\textstyle\sum_{i=0}^{n} f(\boldsymbol{v}_i).\end{aligned}$$

The following lower and upper bounds for the symmetrized function (19) will be useful throughout the paper. This result was announced by F. Guessab who gave a first version of its proof.

Theorem 3.4. *The symmetrized function $\tilde{f}$ of any convex function, $f:\Omega\to\mathbb{R}$ satisfies for all $\boldsymbol{x}$ in Ω the following double inequality*

$$f\left(\frac{1}{n+1}\sum_{i=0}^{n}\boldsymbol{v}_i\right) \le \tilde{f}(\boldsymbol{x}) \le \frac{1}{n+1}\sum_{i=0}^{n} f(\boldsymbol{v}_i). \tag{25}$$

Equality is attained for all affine functions.

[1]We thank the referee for suggesting this simple proof of Lemma (3.2).

Proof. By recalling (16) and applying the Jensen's inequality, we get the left-hand side of (25). For the right hand side, we again use the Jensen's inequality of f and invert the order of summation to get the simple estimation:

$$\begin{aligned}
\tilde{f}(\boldsymbol{x}) &:= \frac{1}{(n+1)!}\sum_{\sigma\in S_n} f\left(T_\sigma(\boldsymbol{x})\right) \\
&= \frac{1}{(n+1)!}\sum_{\sigma\in S_n} f\left(\sum_{i=0}^{n}\lambda_i(\boldsymbol{x})\boldsymbol{v}_{\sigma(i)}\right) \\
&= \frac{1}{(n+1)!}\sum_{\sigma\in S_n} f\left(\sum_{i=0}^{n}\lambda_{\sigma^{-1}(i)}(\boldsymbol{x})\boldsymbol{v}_i\right) \\
&\leq \sum_{i=0}^{n}\left(\frac{1}{(n+1)!}\sum_{\sigma\in S_n}\lambda_{\sigma^{-1}(i)}(\boldsymbol{x})\right) f\left(\boldsymbol{v}_i\right) \\
&= \frac{1}{n+1}\sum_{i=0}^{n} f\left(\boldsymbol{v}_i\right),
\end{aligned}$$

where in the last inequality we have used the identity (13) of Lemma 2.5. This shows that the double inequality (25) holds. Finally, by Lemma 3.2, the case of equality is verified. This completes the proof of Theorem 3.4.

Remark 3.5. *By using Proposition 2.6, equality (15), we may verify that*

$$\begin{aligned}
\tilde{f}\left(\frac{1}{n+1}\sum_{i=0}^{n}\boldsymbol{v}_i\right) &:= \frac{1}{(n+1)!}\sum_{\sigma\in S_n} f\left(T_\sigma\left(\frac{1}{n+1}\sum_{i=0}^{n}\boldsymbol{v}_i\right)\right), \quad (26)\\
&= \frac{1}{(n+1)!}\sum_{\sigma\in S_n} f\left(\frac{1}{n+1}\sum_{i=0}^{n}\boldsymbol{v}_i\right) \quad (27)\\
&= f\left(\frac{1}{n+1}\sum_{i=0}^{n}\boldsymbol{v}_i\right). \quad (28)
\end{aligned}$$

This shows that the function f and its symmetrized function $\tilde{f}$ take the same value at the centroid.
A staightforward calculation also verifies the following identities:

$$\tilde{f}(\boldsymbol{v}_j) = \frac{1}{n+1}\sum_{i=0}^{n} f(\boldsymbol{v}_i), j = 0, \ldots, n. \tag{29}$$

These equalities may be interpreted as the image of any vertex by $\tilde{f}$ is the centroid of the images by f of all the vertices. It should be mentioned that equalities (28) and (29) are satisfied without any convexity assumption on f or $\tilde{f}$.
Finally, Theorem 3.4 reveals that the global minimum value of $\tilde{f}$ in Ω is attained at the centroid of Ω, while its global maximum is attained at every vertex of Ω. Recall the well-known fact that a convex function attains its maximum at a vertex of Ω, however its minimal value is not necessary attained at the centroid.

Theorem 3.4 is a key ingredient in establishing the proof of the next theorem, which tells us that the symmetrized function of any convex function necessary satisfies the Hermite-Hadamard (double) inequality:

Theorem 3.6. *Let μ be a probability measure on Ω. Then, the symmetrized function $\tilde{f}$ of any convex function $f : \Omega \to \mathbb{R}$ satisfies the following double inequality*

$$f\left(\frac{1}{n+1}\sum_{i=0}^{n}\boldsymbol{v}_i\right) \leq \int_{\Omega} \tilde{f}(\boldsymbol{x})\ d\mu(\boldsymbol{x}) \leq \frac{1}{n+1}\sum_{i=0}^{n} f(\boldsymbol{v}_i). \tag{30}$$

Equality is attained for all affine functions.

Proof. Just multiply both sides of (25) by $d\mu(\boldsymbol{x})$ and integrate the resulting inequalities with respect to $\boldsymbol{x}$. Finally, the case of equality is easily verified.

We observe that the left and the right-hand side of the double inequality (30) may respectively be written as:

$$\begin{aligned} f\left(\frac{1}{n+1}\sum_{i=0}^{n}\boldsymbol{v}_i\right) &= \tilde{f}\left(\frac{1}{n+1}\sum_{i=0}^{n}\boldsymbol{v}_i\right), \\ \frac{1}{n+1}\sum_{i=0}^{n} f(\boldsymbol{v}_i) &= \frac{1}{n+1}\sum_{i=0}^{n} \tilde{f}(\boldsymbol{v}_i). \end{aligned}$$

Hence Theorem 3.6 can be immediately reformulated as follows:

Theorem 3.7. *Let μ be a probability measure on Ω. Then, the symmetrized function $\tilde{f}$ of any convex function, $f : \Omega \to \mathbb{R}$ satisfies the following double inequality*

$$\tilde{f}\left(\frac{1}{n+1}\sum_{i=0}^{n} \boldsymbol{v}_i\right) \le \int_{\Omega} \tilde{f}(\boldsymbol{x})\ d\mu(\boldsymbol{x}) \le \frac{1}{n+1}\sum_{i=0}^{n} \tilde{f}(\boldsymbol{v}_i). \tag{31}$$

Equality is attained for all affine functions.

4. A refined version of our main result

In the last section convexity assumption on the original function f has been indispensable for establishing Theorem 3.4, 3.6 and 3.7. In this section, we extend these results to the case when this hypothesis is violated by f, but instead we impose a convexity assumption on its symmetrized function $\tilde{f}$.

In order to justify our next considerations, let us start with an easy example showing the somewhat surprising fact that a *non-convex* function may have a *convex* symmetrized function. Figure 1 shows Cartesian and barycentric coordinates for the triangle T (2-dimensional simplex) of vertices $\boldsymbol{v}_1 = (0,0), \boldsymbol{v}_2 = (2,0)$ and $\boldsymbol{v}_3 = (1,3)$. A little manipulation yields that the barycentric coordinates of a point (x,y) of T with respect to $\boldsymbol{v}_1, \boldsymbol{v}_1, \boldsymbol{v}_3$ are given respectively as follows:

$$\begin{aligned} \lambda_1(x,y) &= \frac{6-3x-y}{6}, \\ \lambda_2(x,y) &= \frac{3x-y}{6}, \\ \lambda_3(x,y) &= \frac{y}{3}. \end{aligned}$$

A long but straightforward calculation also shows that $\tilde{f}(x,y) = 0$ is the symmetrized function associated to the function $f(x,y) = (x-1)^3$. Then, it is easy to see that f is clearly not convex on T, while its symmetrized function is. This example provides a good starting point for further investigations.

Theorem 4.1. *Let μ be a probability measure on Ω. If the symmetrized function $\tilde{f}$ of f is convex, then the following double inequality holds*

$$\tilde{f}\left(\int_{\Omega} \boldsymbol{x} d\mu(\boldsymbol{x})\right) \le \int_{\Omega} \tilde{f}(\boldsymbol{x})\ d\mu(\boldsymbol{x}) \le \frac{1}{n+1}\sum_{i=0}^{n} f(\boldsymbol{v}_i). \tag{32}$$

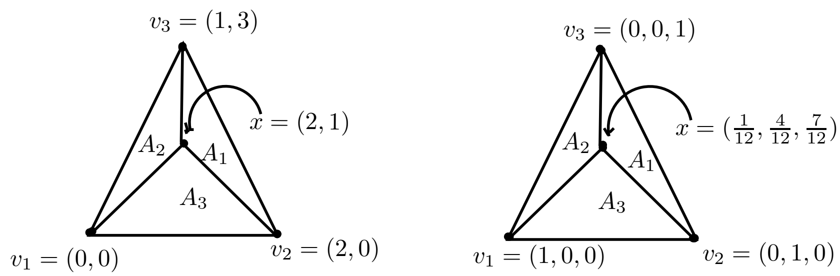

Figure 1: Cartesian (left) and barycentric coordinates (right).

Equality is attained for all affine functions.

PROOF. The left hand inequality in (32) is a simple consequence of the well known integral Jensen's inequality. For the right hand side, we use the Jensen's inequality of $\tilde{f}$ to get, for all $\boldsymbol{x}$ in Ω,

$$\begin{aligned}\tilde{f}(\boldsymbol{x}) &= \tilde{f}\left(\sum_{i=0}^{n} \lambda_i(\boldsymbol{x})\boldsymbol{v}_i\right), \\ &\leq \sum_{i=0}^{n} \lambda_i(\boldsymbol{x})\tilde{f}(\boldsymbol{v}_i).\end{aligned}$$

Multiplying the above inequality with μ and integrating over Ω yields

$$\int_{\Omega} \tilde{f}(\boldsymbol{x})d\mu(\boldsymbol{x}) \leq \sum_{i=0}^{n} \left(\int_{\Omega} \lambda_i(\boldsymbol{x})d\mu(\boldsymbol{x})\right) \tilde{f}(\boldsymbol{v}_i). \tag{33}$$

Now, by (29) it follows from (33) that

$$\int_{\Omega} \tilde{f}(\boldsymbol{x})d\mu(\boldsymbol{x}) \leq \left(\frac{1}{n+1}\sum_{j=0}^{n} f(\boldsymbol{v}_j)\right) \sum_{i=0}^{n} \int_{\Omega} \lambda_i(\boldsymbol{x})d\mu(\boldsymbol{x}).$$

Finally, since $\sum_{i=0}^{n} \lambda_i(\boldsymbol{x}) = 1$ and $\int_{\Omega} d\mu(\boldsymbol{x}) = 1$, then

$$\begin{aligned}\sum_{i=0}^{n} \int_{\Omega} \lambda_i(\boldsymbol{x})d\mu(\boldsymbol{x}) &= \int_{\Omega} \left(\sum_{i=0}^{n} \lambda_i(\boldsymbol{x})\right) d\mu(\boldsymbol{x}) \\ &= \int_{\Omega} d\mu(\boldsymbol{x}) \\ &= 1.\end{aligned}$$

Hence, the double inequality (32) is established and the proof is complete.

As a corollary, any convex symmetrized function satisfies the Hermite-Hadamard inequality on simplices. Analogous result will also be given for a more general form of symmetrization of functions.

Theorem 4.2. *Let S be a nondegenerate simplex in $\mathbb{R}^d$ with vertices $\{\boldsymbol{v}_i\}_{i=0}^d$. If the symmetrized function $\tilde{f}$ of f is convex, then the following double inequality holds*

$$f\left(\frac{1}{d+1}\sum_{i=0}^{d}\boldsymbol{v}_i\right) \leq \frac{\int_S f(\boldsymbol{x})\, d\boldsymbol{x}}{|S|} \leq \frac{1}{d+1}\sum_{i=0}^{d} f(\boldsymbol{v}_i). \tag{34}$$

Equality is attained for all affine functions.

Proof. The assertion follows directly from Theorem 4.1. Indeed, note that if S is a simplex, then the center of gravity $\frac{1}{|S|}\int_S \boldsymbol{x} d\boldsymbol{x}$ coincides with the vertex centroid $\frac{1}{d+1}\sum_{i=0}^d \boldsymbol{v}_i$. Also it is well known that any permutation defines a bijective affine function from S to S. Hence, the assertion is an immediate consequence of the invariance under the permutations, that is :

$$\int_S f(\boldsymbol{x})\, d\boldsymbol{x} = \int_S f(T_\sigma(\boldsymbol{x}))\, d\boldsymbol{x},$$

which holds for any permutation σ.

We would like to recall some works on inequalities of the type (34). The double inequality (34) is known in the literature as Hermite-Hadamard inequality on simplices. We refer the reader to [5, 6] where the latter has been extensively reviewed both from the theoretical as well as the numerical point of view.

5. General symmetrization procedures

The initial idea for our symmetrization procedure was to take a convex combination, with *constant* coefficients, of *all* the functions $f(T_\sigma)$, where σ belongs to the set of all permutations of S_n. A question that naturally arises in this context is the following: which other convex combinations with non constant coefficients could be used to establish new symmetrization procedures?

In order to extend the last result to cover a more general form of symmetrization of functions, we now concentrate on the following situation. Let Ω be an arbitrary polytope that satisfies the conditions

$$\int_{\Omega} \lambda_i(\boldsymbol{x})\ d\boldsymbol{x} = \frac{|\Omega|}{n+1}, i = 0, \ldots, n. \tag{35}$$

Simplices and Cartesian hyperrectangles belong to the simple and natural class of domains satisfying (35). The more general theorem proved here concerns the weighted version of our method of symmetrization of functions, which has the general form:

$$\tilde{f}(\boldsymbol{x}) = \sum_{\sigma \in S_n} c_\sigma f(T_\sigma(\boldsymbol{x})), \tag{36}$$

where all coefficients $c_\sigma, \sigma \in S_n$, are nonnegative and their sum equals to 1. To make the distinction between the two proposed symmetrization techniques (19) and (36), we call the latter *weighted* symmetrized function. We observe that if the values of the coefficients c_σ are all equal to $1/(n+1)!$ then the function of symmetrization given by the formula (36) coincides with (19). From now on in this work, we say that an integration formula

$$I[f] \approx \int_{\Omega} f(\boldsymbol{x}) d\boldsymbol{x} \tag{37}$$

has linear precision, if for any affine function we always have equality in (37).

We want to make the following observations concerning some implications of assumptions (35).

Theorem 5.1. *Let Ω be a polytope that satisfies conditions (35). Then, the following statements hold:*

(1) *The center of gravity $\frac{1}{|\Omega|}\int_{\Omega} \boldsymbol{x} d\boldsymbol{x}$ coincides with the centroid $\frac{1}{n+1}\sum_{i=0}^{n} \boldsymbol{v}_i$.*
(2) *The Hermite-Hadamard double inequality is satisfied for all convex functions:*

$$f\left(\frac{1}{n+1}\sum_{i=0}^{n} \boldsymbol{v}_i\right) \leq \frac{\int_{\Omega} f(\boldsymbol{x})\ d\boldsymbol{x}}{|\Omega|} \leq \frac{1}{n+1}\sum_{i=0}^{n} f(\boldsymbol{v}_i).$$

(3) *The two numerical integration formulae*

$$\int_\Omega f(\boldsymbol{x})\ d\boldsymbol{x} \ \approx \ |\Omega|\, f\left(\frac{1}{n+1}\sum_{i=0}^{n} \boldsymbol{v}_i\right)$$

$$\int_\Omega f(\boldsymbol{x})\ d\boldsymbol{x} \ \approx \ \frac{|\Omega|}{n+1}\sum_{i=0}^{n} f(\boldsymbol{v}_i)$$

have linear precision.

PROOF. As this is quite standard, we will omit the proof.

With these notations the following general result holds true for any convex symmetrized function of the form (36).

Theorem 5.2. *Let Ω be a polytope that satisfies conditions (35). If the weighted symmetrized function $\tilde{f}$ (36) is convex, then the following double inequality holds true*

$$\tilde{f}\left(\frac{1}{n+1}\sum_{i=0}^{n} \boldsymbol{v}_i\right) \leq \frac{\int_\Omega \tilde{f}(\boldsymbol{x})\ d\boldsymbol{x}}{|\Omega|} \leq \frac{1}{n+1}\sum_{i=0}^{n} f(\boldsymbol{v}_i). \tag{38}$$

PROOF. We provide details for completeness. By Theorem 5.1, we already know that the center of gravity $\frac{1}{|\Omega|}\int_\Omega \boldsymbol{x} d\boldsymbol{x}$ coincides with the vertex centroid $\frac{1}{n+1}\sum_{i=0}^{n} \boldsymbol{v}_i$. Hence, the left hand inequality in (38) is just a direct consequence of the integral Jensen's inequality. For the right hand side, we use again the Jensen's inequality of $\tilde{f}$ to get, for all $\boldsymbol{x}$ in Ω,

$$\begin{aligned} \tilde{f}(\boldsymbol{x}) \ &= \ \tilde{f}\left(\sum_{i=0}^{n} \lambda_i(\boldsymbol{x})\boldsymbol{v}_i\right), \\ &\leq \ \sum_{i=0}^{n} \lambda_i(\boldsymbol{x})\tilde{f}(\boldsymbol{v}_i). \end{aligned}$$

But since

$$\begin{aligned} \tilde{f}(\boldsymbol{v}_i) \ &= \ \sum_{\sigma\in S_n} c_\sigma f(T_\sigma(\boldsymbol{v}_i)), \\ &= \ \sum_{\sigma\in S_n} c_\sigma f(\boldsymbol{v}_{\sigma(i)}), \end{aligned}$$

we must have

$$\tilde{f}(\boldsymbol{x}) \ \leq \ \sum_{i=0}^{n} \lambda_i(\boldsymbol{x}) \left(\sum_{\sigma \in S_n} c_\sigma f(\boldsymbol{v}_{\sigma(i)}) \right).$$

We may therefore integrate the above inequality over Ω, getting now

$$\begin{aligned}
\int_\Omega \tilde{f}(\boldsymbol{x}) d\boldsymbol{x} \ &\leq \ \frac{|\Omega|}{n+1} \sum_{i=0}^{n} \left(\sum_{\sigma \in S_n} c_\sigma f(\boldsymbol{v}_{\sigma(i)}) \right) \\
&= \ \frac{|\Omega|}{n+1} \sum_{\sigma \in S_n} c_\sigma \left(\sum_{i=0}^{n} f(\boldsymbol{v}_{\sigma(i)}) \right) \\
&= \ \frac{|\Omega|}{n+1} \left(\sum_{i=0}^{n} f(\boldsymbol{v}_i) \right) \sum_{\sigma \in S_n} c_\sigma \\
&= \ \frac{|\Omega|}{n+1} \left(\sum_{i=0}^{n} f(\boldsymbol{v}_i) \right),
\end{aligned}$$

where in the last step, we have used the fact that the sum of the coefficients c_σ equals to 1. Hence, the double inequality (38) is established and the proof is complete.

Observe that in the case where all the coefficients c_σ are equal to $1/(n+1)!$ Theorem 5.2 contains, as a special case, Theorem 4.2.

We now give another representation of the centroid of Ω. The proof is the same as for identity (16) of Proposition 2.6.

Lemma 5.3. *Let $c_\sigma, \sigma \in S_n$ be any nonnegative coefficients whose sum equals* 1*. Then, the centroid of* Ω *may be represented as follows:*

$$\frac{1}{n+1} \sum_{i=0}^{n} \boldsymbol{v}_i = \sum_{\sigma \in S_n} c_\sigma T_\sigma(\boldsymbol{x}), \quad (\forall \boldsymbol{x} \in \Omega). \tag{39}$$

A generalization of Lemma 3.2 for the weighted case as defined in (36) can be stated as follows:

Lemma 5.4. *The weighted symmetrized function $\tilde{f}$ (36) of any affine function, $f : \Omega \to \mathbb{R}$ satisfies, for all $\boldsymbol{x} \in \Omega$,*

$$\begin{aligned}\tilde{f}(\boldsymbol{x}) &= f\left(\frac{1}{n+1}\sum_{i=0}^{n} \boldsymbol{v}_i\right) &(40)\\ &= \frac{1}{n+1}\sum_{i=0}^{n} f(\boldsymbol{v}_i). &(41)\end{aligned}$$

Proof. The proof is similar to the one given for Lemma 3.2, so we omit it.

With the help of Lemmas 5.3 and 5.4, Theorem 5.2 may be reformulated as follows:

Theorem 5.5. *Let Ω be a polytope that satisfies conditions (35). If the weighted symmetrized function $\tilde{f}$ (36) is convex, then the following double inequality holds true*

$$f\left(\frac{1}{n+1}\sum_{i=0}^{n} \boldsymbol{v}_i\right) \leq \frac{\int_\Omega \tilde{f}(\boldsymbol{x})\, d\boldsymbol{x}}{|\Omega|} \leq \frac{1}{n+1}\sum_{i=0}^{n} f(\boldsymbol{v}_i). \tag{42}$$

Equality is attained for all affine functions.

Remark 5.6. *The problem, which now arises, concerns the restrictive conditions (35) for domains. It should also be possible to generalize the symmetrization function (36) to domains that do not satisfy conditions (35). This would be an interesting issue.*

6. Applications to Wright-convex functions

Due to the relevance of the notion of convexity in theoretical and applied areas of mathematics, several notions of convexity have been proposed. A function $f : \Omega \to \mathbb{R}$ is called Wright-convex (W-convex, for short), if

$$f(t\boldsymbol{x} + (1-t)\boldsymbol{y}) + f(t\boldsymbol{y} + (1-t)\boldsymbol{x}) \leq f(\boldsymbol{x}) + f(\boldsymbol{y})$$

for every $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $t \in [0,1]$. It is immediate from the definition that any convex function is necessarily W-convex. But the converse is not true. W-convexity has been of fundamental importance in connection with the converse of Minkowski's inequality, for more details, see [11].

The purpose of this section is to prove that the principal results of the preceding section remain true for W-convex functions. W-convex functions have a nice characterization in terms of the classical notion of convexity. Indeed, a function f is W-convex if and only if it can be represented in the form $f = g + a$, where g is a convex function and a is an additive function, see [9]. This representation plays an important role in improving results of Theorems 3.4 and 3.6. We recall that an additive function $a : \Omega \to \mathbb{R}$ is one which satisfies the equality

$$a(\boldsymbol{x} + \boldsymbol{y}) = a(\boldsymbol{x}) + a(\boldsymbol{y})$$

for every $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x} + \boldsymbol{y} \in \Omega$. We state now an intermediate result, which allows us to write the symmetrized function of an additive function in two different ways.

Lemma 6.1. *Let $a : \Omega \to \mathbb{R}$ be an additive function. Then, for all $\boldsymbol{x}$ in Ω, the symmetrized function $\tilde{a}$ of a satisfies*

$$a\left(\frac{1}{n+1}\sum_{i=0}^{n} \boldsymbol{v}_i\right) = \tilde{a}(\boldsymbol{x}) = \frac{1}{n+1}\sum_{i=0}^{n} a(\boldsymbol{v}_i). \tag{43}$$

Proof. Indeed, using the additivity of the function a and the fact that

$$\begin{aligned} a\left(\sum_{j=0}^{n} \boldsymbol{v}_j\right) &= a\left(\frac{n+1}{n+1}\sum_{j=0}^{n} \boldsymbol{v}_j\right) \\ &= (n+1)a\left(\frac{1}{n+1}\sum_{j=0}^{n} \boldsymbol{v}_j\right), \end{aligned}$$

we immediately get for any $\boldsymbol{x}$ in Ω:

$$\begin{aligned}
\tilde{a}(\boldsymbol{x}) &:= \frac{1}{(n+1)!}\sum_{\sigma\in S_n} a(T_\sigma(\boldsymbol{x})) \\
&:= \frac{1}{(n+1)!}\sum_{\sigma\in S_n} a\left(\sum_{i=0}^{n}\lambda_i(\boldsymbol{x})\boldsymbol{v}_{\sigma(i)}\right) \\
&= \frac{1}{(n+1)!} a\left(\sum_{i=0}^{n}\lambda_i(\boldsymbol{x})\left(\sum_{\sigma\in S_n}\boldsymbol{v}_{\sigma(i)}\right)\right) \\
&= \frac{1}{(n+1)!} a\left(\sum_{i=0}^{n}\lambda_i(\boldsymbol{x})\left(n!\sum_{j=0}^{n}\boldsymbol{v}_j\right)\right) \\
&= \frac{1}{(n+1)!} a\left(n!\sum_{j=0}^{n}\boldsymbol{v}_j\right) \\
&= \frac{1}{n+1} a\left(\sum_{j=0}^{n}\boldsymbol{v}_j\right) \\
&= a\left(\frac{1}{n+1}\sum_{j=0}^{n}\boldsymbol{v}_j\right).
\end{aligned}$$

This establishes that (43) holds for any $\boldsymbol{x}$ in Ω and completes the proof of Lemma 6.1.

Our next goal is to state versions of Theorems 3.4 and 3.6 for the class of W-convex functions. Since any W-convex function is a sum of an additive function and a convex function, then, by employing Lemma 6.1, we can use the same arguments as we did in Theorems 3.4 and 3.6 to prove the following theorems.

Theorem 6.2. *Let $f : \Omega \to \mathbb{R}$ be a W-convex function. Then, for all $\boldsymbol{x}$ in Ω, the associated symmetrized function $\tilde{f}$ satisfies*

$$f\left(\frac{1}{n+1}\sum_{i=0}^{n}\boldsymbol{v}_i\right) \le \tilde{f}(\boldsymbol{x}) \le \frac{1}{n+1}\sum_{i=0}^{n} f(\boldsymbol{v}_i).$$

Equality is attained for all affine functions.

Theorem 6.3. *Let μ be a probability measure on Ω. Then, for any W-convex function $f : \Omega \to \mathbb{R}$, the symmetrized function $\tilde{f}$ of f satisfies the following double inequality*

$$f\left(\frac{1}{n+1}\sum_{i=0}^{n} \boldsymbol{v}_i\right) \leq \int_{\Omega} \tilde{f}(\boldsymbol{x})\ d\mu(\boldsymbol{x}) \leq \frac{1}{n+1}\sum_{i=0}^{n} f(\boldsymbol{v}_i). \tag{44}$$

Equality is attained for all affine functions.

Remark 6.4. *We finally observe that the reader can easily reformulate results of Theorems 6.2 and 6.3 for the general weighted symmetrization case given by formula (36). The basic ideas are the same.*

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