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SOME INEQUALITIES FOR POWER SERIES WITH NONNEGATIVE COEFFICIENTS VIA A DIVIDED DIFFERENCE REVERSE OF JENSEN INEQUALITY

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Abstract. Some inequalities for power series with nonnegative coefficients via a divided difference reverse of Jensen inequality are given. Applications for some fundamental functions defined by power series are also provided.

1. Introduction

On utilizing some reverses of Jensen discrete inequality for convex functions, we obtained in [5] the following result for functions defined by power series with nonnegative coefficients:

Theorem 1. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$. If $p\geq 1$, $0<\alpha<R$ and $x>0$ with $\alpha x^p, \alpha x^{p-1}<R$, then*

$$0\leq \frac{f(\alpha x^p)}{f(\alpha)}-\left[\frac{f(\alpha x)}{f(\alpha)}\right]^p\leq p\left[\frac{f(\alpha x^p)}{f(\alpha)}-\frac{f\left(\alpha x^{p-1}\right)}{f(\alpha)}\frac{f(\alpha x)}{f(\alpha)}\right]. \tag{1.1}$$

Moreover, if $0<x\leq 1$, then

$$\begin{aligned}0&\leq \frac{f(\alpha x^p)}{f(\alpha)}-\left[\frac{f(\alpha x)}{f(\alpha)}\right]^p\leq p\left[\frac{f(\alpha x^p)}{f(\alpha)}-\frac{f\left(\alpha x^{p-1}\right)}{f(\alpha)}\frac{f(\alpha x)}{f(\alpha)}\right]\\&\leq \frac{1}{2}p\left(\frac{f\left(\alpha x^{2(p-1)}\right)}{f(\alpha)}-\left[\frac{f\left(\alpha x^{p-1}\right)}{f(\alpha)}\right]^2\right)^{1/2}\leq \frac{1}{4}p\end{aligned} \tag{1.2}$$

and

$$\begin{aligned}0&\leq \frac{f(\alpha x^p)}{f(\alpha)}-\left[\frac{f(\alpha x)}{f(\alpha)}\right]^p\leq p\left[\frac{f(\alpha x^p)}{f(\alpha)}-\frac{f\left(\alpha x^{p-1}\right)}{f(\alpha)}\frac{f(\alpha x)}{f(\alpha)}\right]\\&\leq \frac{1}{2}p\left(\frac{f\left(\alpha x^{2}\right)}{f(\alpha)}-\left[\frac{f(\alpha x)}{f(\alpha)}\right]^2\right)^{1/2}\leq \frac{1}{4}p.\end{aligned} \tag{1.3}$$

Corollary 1. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$. If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $u,v>0$ with $v^p\leq u^q<R$, then*

$$\left[\frac{f(uv)}{f(u^q)}\right]^p\leq \frac{f(v^p)}{f(u^q)}\leq \frac{1}{4}p+\left[\frac{f(uv)}{f(u^q)}\right]^p \tag{1.4}$$

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and

$$0 \le [f(v^p)]^{1/p} [f(u^q)]^{1/q} - f(uv) \le \frac{1}{4^{1/p}} p^{1/p} f(u^q). \tag{1.5}$$

Utilising a different approach in [6] we obtained the following results as well:

Theorem 2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R > 0$ or $R = \infty$. If $p > 1$, $0 < \alpha < R$ and $0 < x \le 1$, then*

$$0 \le \frac{f(\alpha x^p)}{f(\alpha)} - \left(\frac{f(\alpha x)}{f(\alpha)}\right)^p \le M_p \left(1 - \frac{f(\alpha x)}{f(\alpha)}\right) \frac{f(\alpha x)}{f(\alpha)} \le \frac{1}{4} M_p \tag{1.6}$$

and

$$0 \le \frac{f(\alpha x^p)}{f(\alpha)} - \left(\frac{f(\alpha x)}{f(\alpha)}\right)^p \le \frac{1}{4} \cdot \frac{1 - \left(\frac{f(\alpha x)}{f(\alpha)}\right)^{p-1}}{1 - \frac{f(\alpha x)}{f(\alpha)}} \le \frac{1}{4} M_p, \tag{1.7}$$

where

$$M_p := \begin{cases} 1 \text{ if } p \in (1,2], \\ \\ p-1 \text{ if } p \in (2,\infty). \end{cases}$$

Corollary 2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R > 0$ or $R = \infty$. If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $u, v > 0$ with $v^p \le u^q < R$, then*

$$0 \le \frac{f(v^p)}{f(u^q)} - \left(\frac{f(uv)}{f(u^q)}\right)^p \le M_p \left(1 - \frac{f(uv)}{f(u^q)}\right) \frac{f(uv)}{f(u^q)} \le \frac{1}{4} M_p \tag{1.8}$$

and

$$0 \le \frac{f(v^p)}{f(u^q)} - \left(\frac{f(uv)}{f(u^q)}\right)^p \le \frac{1}{4} \cdot \frac{1 - \left(\frac{f(uv)}{f(u^q)}\right)^{p-1}}{1 - \frac{f(uv)}{f(u^q)}} \le \frac{1}{4} M_p. \tag{1.9}$$

For some similar exponential and logarithmic inequalities see [5] and [6] where further applications for some fundamental functions were provided .

For other recent results for power series with nonnegative coefficients, see [2], [9], [13] and [14]. For more results on power series inequalities, see [2] and [9]-[12].

The most important power series with nonnegative coefficients that can be used to illustrate the above results are:

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \ z \in \mathbb{C}, \ \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \ z \in D(0,1), \tag{1.10}$$

$$\ln \frac{1}{1-z} = \sum_{n=1}^{\infty} \frac{1}{n} z^n, \ z \in D(0,1), \ \cosh z = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}, \ z \in \mathbb{C},$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1}, \ z \in \mathbb{C}.$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\ z\in D(0,1), \tag{1.11}$$

$$\sin^{-1}(z)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}z^{2n+1},\ z\in D(0,1),$$

$$\tanh^{-1}(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1},\ z\in D(0,1),$$

$$ {}_2F_1(\alpha,\beta,\gamma,z):=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}z^n,\alpha,\beta,\gamma>0$$
$$z\in D(0,1),$$

where Γ is *Gamma function.*

Motivated by the above results and utilizing a divided difference reverse of Jensen's inequality, we provide in this paper other inequalities for power series with nonnegative coefficients. Applications for some fundamental functions are given as well.

2. A Refinement and a New Reverse

For a real function $g:[m,M]\to\mathbb{R}$ and two distinct points $\alpha,\beta\in[m,M]$ we recall that the *divided difference* of g in these points is defined by

$$[\alpha,\beta;g]:=\frac{g(\beta)-g(\alpha)}{\beta-\alpha}.$$

The following result holds:

Theorem 3. *Let $f:I\to\mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m,M\in\mathbb{R}$, $m<M$ with $[m,M]\subset\mathring{I}$, $\mathring{I}$ the interior of I. Let $\bar{\mathbf{a}}=(a_1,\dots,a_n)$, $\bar{\mathbf{p}}=(p_1,\dots,p_n)$ be n-tuples of real numbers with $p_i\geq 0$ $(i\in\{1,\dots,n\})$ and $\sum_{i=1}^n p_i=1$. If $m\leq a_i\leq M$, $i\in\{1,\dots,n\}$, with $\sum_{i=1}^n p_ia_i\neq m,M$, then*

$$\left|\sum_{i=1}^n p_i\left|f(a_i)-f\left(\sum_{j=1}^n p_ja_j\right)\right|sgn\left(a_i-\sum_{j=1}^n p_ja_j\right)\right| \tag{2.1}$$
$$\leq\sum_{i=1}^n p_if(a_i)-f\left(\sum_{i=1}^n p_ia_i\right)$$
$$\leq\frac{1}{2}\left(\left[\sum_{i=1}^n p_ia_i,M;f\right]-\left[m,\sum_{i=1}^n p_ia_i;f\right]\right)\sum_{i=1}^n p_i\left|a_i-\sum_{j=1}^n p_ja_j\right|$$
$$\leq\frac{1}{2}\left(\left[\sum_{i=1}^n p_ia_i,M;f\right]-\left[m,\sum_{i=1}^n p_ia_i;f\right]\right)\left[\sum_{i=1}^n p_ia_i^2-\left(\sum_{j=1}^n p_ja_j\right)^2\right]^{1/2}.$$

If the lateral derivatives $f'_+(m)$ and $f'_-(M)$ are finite, then we also have the inequalities

$$0 \le \sum_{i=1}^{n} p_i f(a_i) - f\left(\sum_{i=1}^{n} p_i a_i\right) \tag{2.2}$$
$$\le \frac{1}{2}\left(\left[\sum_{i=1}^{n} p_i a_i, M; f\right] - \left[m, \sum_{i=1}^{n} p_i a_i; f\right]\right) \sum_{i=1}^{n} p_i \left|a_i - \sum_{j=1}^{n} p_j a_j\right|$$
$$\le \frac{1}{2}\left[f'_-(M) - f'_+(m)\right] \sum_{i=1}^{n} p_i \left|a_i - \sum_{j=1}^{n} p_j a_j\right|$$
$$\le \frac{1}{2}\left[f'_-(M) - f'_+(m)\right] \left[\sum_{i=1}^{n} p_i a_i^2 - \left(\sum_{j=1}^{n} p_j a_j\right)^2\right]^{1/2}.$$

Proof. We recall that if $f : I \to \mathbb{R}$ is a continuous convex function on the interval of real numbers I and $\alpha \in I$, then the *divided difference function* $f_\alpha : I \setminus \{\alpha\} \to \mathbb{R}$,

$$f_\alpha(t) := [\alpha, t; f] := \frac{f(t) - f(\alpha)}{t - \alpha}$$

is *monotonic nondecreasing* on $I \setminus \{\alpha\}$.

For $\bar{a}_p := \sum_{j=1}^{n} p_j a_j \in (m, M)$, we consider now the sequence

$$f_{\bar{a}_p}(i) := \frac{f(a_i) - f(\bar{a}_p)}{a_i - \bar{a}_p}.$$

We will show that $f_{\bar{a}_p}(i)$ and $h_i := a_i - \bar{a}_p$, $i \in \{1, \ldots, n\}$ are synchronous.

Let $i, j \in \{1, \ldots, n\}$ with $a_i, a_j \neq \bar{a}_p$. Assume that $a_i \ge a_j$, then by the monotonicity of f_α we have

$$f_{\bar{a}_p}(i) = \frac{f(a_i) - f(\bar{a}_p)}{a_i - \bar{a}_p} \tag{2.3}$$
$$\ge \frac{f(a_j) - f(\bar{a}_p)}{a_j - \bar{a}_p} = f_{\bar{a}_p}(j)$$

and

$$h_i \ge h_j \tag{2.4}$$

which shows that

$$\left[f_{\bar{a}_p}(i) - f_{\bar{a}_p}(j)\right](h_i - h_j) \ge 0. \tag{2.5}$$

If $a_i < a_j$, then the inequalities (2.3) and (2.4) reverse but the inequality (2.5) still holds true.

Utilising the continuity property of the modulus we have

$$\left|\left[\left|f_{\bar{a}_p}(i)\right| - \left|f_{\bar{a}_p}(j)\right|\right](h_i - h_j)\right| \le \left|\left[f_{\bar{a}_p}(i) - f_{\bar{a}_p}(j)\right](h_i - h_j)\right|$$
$$= \left[f_{\bar{a}_p}(i) - f_{\bar{a}_p}(j)\right](h_i - h_j)$$

for any $i, j \in \{1, \ldots, n\}$.

Multiplying with $p_i, p_j \geq 0$ and summing over i and j from 1 to n we have

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j \left[\left|f_{\bar{a}_p}(i)\right| - \left|f_{\bar{a}_p}(j)\right|\right](h_i - h_j)\right| \leq \sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j \left[f_{\bar{a}_p}(i) - f_{\bar{a}_p}(j)\right](h_i - h_j). \tag{2.6}$$

A simple calculation shows that

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j \left[\left|f_{\bar{a}_p}(i)\right| - \left|f_{\bar{a}_p}(j)\right|\right](h_i - h_j) \\
&= \sum_{i=1}^{n} p_i \left|f_{\bar{a}_p}(i)\right| h_i - \sum_{i=1}^{n} p_i \left|f_{\bar{a}_p}(i)\right| \sum_{i=1}^{n} p_i h_i \\
&= \sum_{i=1}^{n} p_i \left|\frac{f(a_i) - f(\bar{a}_p)}{a_i - \bar{a}_p}\right|(a_i - \bar{a}_p) \\
&- \sum_{i=1}^{n} p_i \left|\frac{f(a_i) - f(\bar{a}_p)}{a_i - \bar{a}_p}\right| \sum_{i=1}^{n} p_i (a_i - \bar{a}_p) \\
&= \sum_{i=1}^{n} p_i \left|\frac{f(a_i) - f(\bar{a}_p)}{a_i - \bar{a}_p}\right|(a_i - \bar{a}_p) \\
&= \sum_{i=1}^{n} p_i \left|f(a_i) - f(\bar{a}_p)\right| sgn(a_i - \bar{a}_p)
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j \left[f_{\bar{a}_p}(i) - f_{\bar{a}_p}(j)\right](h_i - h_j) \\
&= \sum_{i=1}^{n} p_i f_{\bar{a}_p}(i) h_i - \sum_{i=1}^{n} p_i f_{\bar{a}_p}(i) \sum_{i=1}^{n} p_i h_i \\
&= \sum_{i=1}^{n} p_i \left(\frac{f(a_i) - f(\bar{a}_p)}{a_i - \bar{a}_p}\right)(a_i - \bar{a}_p) \\
&- \sum_{i=1}^{n} p_i \left(\frac{f(a_i) - f(\bar{a}_p)}{a_i - \bar{a}_p}\right) \sum_{i=1}^{n} p_i (a_i - \bar{a}_p) \\
&= \sum_{i=1}^{n} p_i \left(\frac{f(a_i) - f(\bar{a}_p)}{a_i - \bar{a}_p}\right)(a_i - \bar{a}_p) \\
&= \sum_{i=1}^{n} p_i f(a_i) - f\left(\sum_{i=1}^{n} p_i a_i\right).
\end{aligned} \tag{2.8}$$

On making use of the identities (2.7) and (2.8) we obtain from (2.6) the first inequality in (2.1).

Now, since $\bar{a}_p := \sum_{j=1}^{n} p_j a_j \in (m, M)$, then we have. by the monotonicity of $f_{\bar{a}_p}(i)$, that

$$[m, \bar{a}_p; f] = \frac{f(\bar{a}_p) - f(m)}{\bar{a}_p - m} \leq f_{\bar{a}_p}(i) \leq \frac{f(M) - f(\bar{a}_p)}{M - \bar{a}_p} = [\bar{a}_p, M; f] \tag{2.9}$$

for any $i \in \{1, \ldots, n\}$.

Applying now the *Grüss' type inequality* obtained by Cerone & Dragomir in [1]

$$\left| \sum_{i=1}^{n} w_i x_i y_i - \sum_{i=1}^{n} w_i x_i \sum_{i=1}^{n} w_i y_i \right| \leq \frac{1}{2} (\Gamma - \gamma) \sum_{i=1}^{n} w_i \left| x_i - \sum_{j=1}^{n} w_j x_j \right|$$

provided

$$-\infty < \delta \leq y_i \leq \Delta < \infty \tag{2.10}$$

for $i = 1, \ldots, n$, we have that

$$\sum_{i=1}^{n} p_i f(a_i) - f\left(\sum_{i=1}^{n} p_i a_i \right) \leq \frac{1}{2} ([\bar{a}_p, M; f] - [m, \bar{a}_p; f]) \sum_{i=1}^{n} p_i \left| a_i - \sum_{j=1}^{n} p_j a_j \right|,$$

which proves the second inequality in (2.1).

The last bound in (2.1) is obvious by Cauchy-Bunyakovsky-Schwarz discrete inequality.

If the lateral derivatives $f'_+(m)$ and $f'_-(M)$ are finite, then by the convexity of f we have the *gradient inequalities*

$$\frac{f(M) - f(\bar{a}_p)}{M - \bar{a}_p} \leq f'_-(M)$$

and

$$\frac{f(\bar{a}_p) - f(m)}{\bar{a}_p - m} \geq f'_+(m),$$

where $\bar{a}_p \in (m, M)$. These imply that

$$[\bar{a}_p, M; f] - [m, \bar{a}_p; f] \leq f'_-(M) - f'_+(m)$$

and the proof of the third inequality in (2.2) is concluded.

The rest is obvious. □

For an integral version see [7].

Remark 1. *Define the weighted arithmetic mean of the positive n-tuple* $x = (x_1, ..., x_n)$ *with the nonnegative weights* $w = (w_1, ..., w_n)$ *by*

$$A_n(w, x) := \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i$$

where $W_n := \sum_{i=1}^{n} w_i > 0$ *and the weighted geometric mean of the same n-tuple, by*

$$G_n(w,x) := \left(\prod_{i=1}^{n} x_i^{w_i}\right)^{1/W_n}.$$

It is well know that the following arithmetic mean-geometric mean inequality holds

$$A_n(w,x) \geq G_n(w,x).$$

Applying the inequality (2.2) for the convex function $f(t) = -\ln t, t > 0$ *we have the following reverse of the arithmetic mean-geometric mean inequality*

$$\begin{aligned} 1 &\leq \frac{A_n(w,x)}{G_n(w,x)} \\ &\leq \left[\frac{\left(\frac{A_n(w,x)}{m}\right)^{A_n(w,x)-m}}{\left(\frac{M}{A_n(w,x)}\right)^{M-A_n(w,x)}}\right]^{\frac{1}{2}A_n(w,|x-A_n(w,x)|)} \\ &\leq \exp\left[\frac{1}{2}\frac{M-m}{mM}A_n(w,|x-A_n(w,x)|)\right], \end{aligned} \tag{2.11}$$

provided that $0 < m \leq x_i \leq M < \infty$ *for* $i \in \{1,...,n\}$.

3. Applications for the Hölder Inequality

If $x_i, y_i \geq 0$ for $i \in \{1,...,n\}$, then the *Hölder inequality* holds true

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^p\right)^{1/p}\left(\sum_{i=1}^{n} y_i^q\right)^{1/q},$$

where $p > 1, \frac{1}{p} + \frac{1}{q} = 1$.

Assume that $p > 1$. If $z_i \in \mathbb{R}$ for $i \in \{1,...,n\}$, satisfies the bounds

$$0 < m \leq z_i \leq M < \infty \text{ for } i \in \{1,...,n\}$$

and $w_i \geq 0 \ (i = 1,\dots,n)$ with $W_n := \sum_{i=1}^{n} w_i > 0$, then from Theorem 3 we have amongst other the following inequality

$$\begin{aligned} &\left|\frac{1}{W_n}\sum_{i=1}^{n}\left|z_i^p - \left(\frac{\sum_{i=1}^{n} w_i z_i}{W_n}\right)^p\right| w_i sgn\left[z_i - \frac{\sum_{i=1}^{n} w_i z_i}{W_n}\right]\right| \\ &\leq \frac{\sum_{i=1}^{n} w_i z_i^p}{W_n} - \left(\frac{\sum_{i=1}^{n} w_i z_i}{W_n}\right)^p \\ &\leq \frac{1}{2}\left(\left[\frac{\sum_{i=1}^{n} w_i z_i}{W_n}, M; (\cdot)^p\right] - \left[m, \frac{\sum_{i=1}^{n} w_i z_i}{W_n}; (\cdot)^p\right]\right)\tilde{D}_w(z) \\ &\leq \frac{1}{2}\left(\left[\frac{\sum_{i=1}^{n} w_i z_i}{W_n}, M; (\cdot)^p\right] - \left[m, \frac{\sum_{i=1}^{n} w_i z_i}{W_n}; (\cdot)^p\right]\right)\tilde{D}_{w,2}(z) \\ &\leq \frac{1}{4}\left(\left[\frac{\sum_{i=1}^{n} w_i z_i}{W_n}, M; (\cdot)^p\right] - \left[m, \frac{\sum_{i=1}^{n} w_i z_i}{W_n}; (\cdot)^p\right]\right)(M-m), \end{aligned} \tag{3.1}$$

where $\frac{\sum_{i=1}^{n} w_i z_i}{W_n} \in (m, M)$ and

$$\tilde{D}_w(z) := \frac{1}{W_n}\sum_{i=1}^{n} w_i \left| z_i - \frac{\sum_{j=1}^{n} w_j z_j}{W_n} \right|$$

while

$$\tilde{D}_{w,2}(z) = \left[\frac{\sum_{i=1}^{n} w_i z_i^2}{W_n} - \left(\frac{\sum_{i=1}^{n} w_i z_i}{W_n} \right)^2 \right]^{\frac{1}{2}}.$$

The following result related to the *Hölder inequality* holds:

Proposition 1. *If $x_i \geq 0$, $y_i > 0$ for $i \in \{1, ..., n\}$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and there exists the constants $\gamma, \Gamma > 0$ such that*

$$\gamma \leq \frac{x_i}{y_i^{q-1}} \leq \Gamma \text{ for } i \in \{1, ..., n\},$$

then we have

$$\left| \sum_{i=1}^{n} \left| \frac{x_i^p}{y_i^q} - \left(\frac{\sum_{j=1}^{n} x_j y_j}{\sum_{j=1}^{n} y_j^q} \right)^p \right| y_i^q \, sgn \left[\frac{x_i}{y_i^{q-1}} - \frac{\sum_{j=1}^{n} x_j y_j}{\sum_{j=1}^{n} y_j^q} \right] \right| \tag{3.2}$$
$$\leq \frac{\sum_{i=1}^{n} x_i^p}{\sum_{i=1}^{n} y_i^q} - \left(\frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} y_i^q} \right)^p$$
$$\leq \frac{1}{2} \left(\left[\frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} y_i^q}, \Gamma; (\cdot)^p \right] - \left[\gamma, \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} y_i^q}; (\cdot)^p \right] \right) \tilde{D}_{y^q} \left(\frac{x}{y^{q-1}} \right)$$
$$\leq \frac{1}{2} \left(\left[\frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} y_i^q}, \Gamma; (\cdot)^p \right] - \left[\gamma, \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} y_i^q}; (\cdot)^p \right] \right) \tilde{D}_{y^q,2} \left(\frac{x}{y^{q-1}} \right)$$
$$\leq \frac{1}{4} \left(\left[\frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} y_i^q}, \Gamma; (\cdot)^p \right] - \left[\gamma, \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} y_i^q}; (\cdot)^p \right] \right) (\Gamma - \gamma),$$

where

$$\tilde{D}_{y^q} \left(\frac{x}{y^{q-1}} \right) = \frac{1}{\sum_{i=1}^{n} y_i^q} \sum_{i=1}^{n} y_i^q \left| \frac{x_i}{y_i^{q-1}} - \frac{\sum_{j=1}^{n} x_j y_j}{\sum_{j=1}^{n} y_j^q} \right|$$

and

$$\tilde{D}_{y^q,2} \left(\frac{x}{y^{q-1}} \right) = \left[\frac{1}{\sum_{i=1}^{n} y_i^q} \sum_{i=1}^{n} \frac{x_i^2}{y_i^{q-2}} - \left(\frac{\sum_{j=1}^{n} x_j y_j}{\sum_{j=1}^{n} y_j^q} \right)^2 \right]^{\frac{1}{2}}.$$

Proof. The inequalities (3.3) follow from (3.1) by choosing

$$z_i = \frac{x_i}{y_i^{q-1}} \text{ and } w_i = y_i^q, i \in \{1, ..., n\}.$$

The details are omitted. □

Remark 2. *We observe that for $p=q=2$ we have from the first inequality in (3.2) the following reverse of the Cauchy-Bunyakovsky-Schwarz inequality*

$$\left|\sum_{i=1}^{n}\left|\frac{x_i^2}{y_i^2}-\left(\frac{\sum_{j=1}^{n}x_jy_j}{\sum_{j=1}^{n}y_j^2}\right)^2\right|y_i^2 sgn\left(\frac{x_i}{y_i}-\frac{\sum_{j=1}^{n}x_jy_j}{\sum_{j=1}^{n}y_j^2}\right)\right| \tag{3.3}$$
$$\leq \frac{\sum_{i=1}^{n}x_i^2}{\sum_{i=1}^{n}y_i^2}-\left(\frac{\sum_{i=1}^{n}x_iy_i}{\sum_{i=1}^{n}y_i^2}\right)^2$$
$$\leq \frac{1}{2}(\Gamma-\gamma)\frac{1}{\sum_{i=1}^{n}y_i^2}\sum_{i=1}^{n}y_i^2\left|\frac{x_i}{y_i}-\frac{\sum_{j=1}^{n}x_jy_j}{\sum_{j=1}^{n}y_j^2}\right|$$
$$\leq \frac{1}{2}(\Gamma-\gamma)\left[\frac{1}{\sum_{i=1}^{n}y_i^2}\sum_{i=1}^{n}x_i^2-\left(\frac{\sum_{j=1}^{n}x_jy_j}{\sum_{j=1}^{n}y_j^2}\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}(\Gamma-\gamma)^2,$$

provided that there exists the constants $\gamma,\Gamma>0$ such that

$$\gamma\leq\frac{x_i}{y_i}\leq\Gamma \text{ for } i\in\{1,...,n\}.$$

4. Power Inequalities

Utilising the inequality (2.1) for the convex function $f:[m,M]\subset[0,\infty]\to\mathbb{R}$, $p\geq 1$, $f(t)=t^p$ we have

$$0\leq\sum_{i=1}^{n}p_ia_i^p-\left(\sum_{i=1}^{n}p_ia_i\right)^p \tag{4.1}$$
$$\leq\frac{1}{2}\left(\frac{M^p-\left(\sum_{i=1}^{n}p_ia_i\right)^p}{M-\sum_{i=1}^{n}p_ia_i}-\frac{\left(\sum_{i=1}^{n}p_ia_i\right)^p-m^p}{\sum_{i=1}^{n}p_ia_i-m}\right)$$
$$\times\left[\sum_{i=1}^{n}p_ia_i^2-\left(\sum_{j=1}^{n}p_ja_j\right)^2\right]^{1/2},$$

for $m\leq a_i\leq M$, and $p_i\geq 0$, $i\in\{1,\ldots,n\}$ with $\sum_{i=1}^{n}p_i=1$ and $\sum_{i=1}^{n}p_ia_i\neq m,M$.

If we write the inequality (4.1) for $m=0$ and $M=1$ we get

$$0\leq\sum_{i=1}^{n}p_ia_i^p-\left(\sum_{i=1}^{n}p_ia_i\right)^p \tag{4.2}$$
$$\leq\frac{1}{2}\cdot\frac{1-\left(\sum_{i=1}^{n}p_ia_i\right)^{p-1}}{1-\sum_{i=1}^{n}p_ia_i}\left[\sum_{i=1}^{n}p_ia_i^2-\left(\sum_{j=1}^{n}p_ja_j\right)^2\right]^{1/2},$$

for $0\leq a_i\leq 1$, and $p_i\geq 0$, $i\in\{1,\ldots,n\}$ with $\sum_{i=1}^{n}p_i=1$ and $\sum_{i=1}^{n}p_ia_i\neq 0,1$.

We can state the following result for powers:

Theorem 4. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$. If $p>1$, $0<\alpha<R$ and $0<x\leq 1$, then*

$$0\leq \frac{f(\alpha x^p)}{f(\alpha)}-\left(\frac{f(\alpha x)}{f(\alpha)}\right)^p \leq \frac{1}{2}\cdot\frac{1-\left(\frac{f(\alpha x)}{f(\alpha)}\right)^{p-1}}{1-\frac{f(\alpha x)}{f(\alpha)}}\left[\frac{f(\alpha x^2)}{f(\alpha)}-\left(\frac{f(\alpha x)}{f(\alpha)}\right)^2\right]^{1/2}. \tag{4.3}$$

Proof. Let $m\geq 1$ and $0<\alpha<R$, $0<x\leq 1$. If we write the inequality (4.1) for

$$w_j=\frac{a_j\alpha^j}{\sum_{k=0}^{m}a_k\alpha^k}\text{ and }z_j:=x^j\in[0,1],\ j\in\{0,...,m\},$$

then we get

$$\begin{aligned}0&\leq \frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^jx^{pj}-\left(\frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^jx^j\right)^p\\ &\leq \frac{1}{2}\frac{1-\left(\frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^jx^j\right)^{p-1}}{1-\frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^jx^j}\\ &\times\left[\frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^jx^{2j}-\left(\frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^jx^j\right)^2\right]^{1/2}.\end{aligned}\tag{4.4}$$

Since all series whose partial sums involved in the inequality (4.4) are convergent, then by letting $m\to\infty$ in (4.4) we deduce (4.3). □

Corollary 3. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$ with $R>0$ or $R=\infty$. If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $u,v>0$ with $v^p\leq u^q<R$, then*

$$0\leq \frac{f(v^p)}{f(u^q)}-\left(\frac{f(uv)}{f(u^q)}\right)^p \leq \frac{1}{2}\cdot\frac{1-\left(\frac{f(uv)}{f(u^q)}\right)^{p-1}}{1-\frac{f(uv)}{f(u^q)}}\left[\frac{f\left(u^{1-\frac{q}{p}}v^2\right)}{f(u^q)}-\left(\frac{f(uv)}{f(u^q)}\right)^2\right]^{1/2}. \tag{4.5}$$

Proof. Follows by taking into (4.3) $\alpha=u^q$ and $x=\frac{v}{u^{q/p}}$. The details are omitted. □

Example 1. *a) If we write the inequalities (4.3) for the function $\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n$, $z\in D(0,1)$, then we have*

$$0\leq \frac{1-\alpha}{1-\alpha x^p}-\left(\frac{1-\alpha}{1-\alpha x}\right)^p \leq \frac{1}{2}\cdot\frac{1-\left(\frac{1-\alpha}{1-\alpha x}\right)^{p-1}}{1-\frac{1-\alpha}{1-\alpha x}}\left[\frac{1-\alpha}{1-\alpha x^2}-\left(\frac{1-\alpha}{1-\alpha x}\right)^2\right]^{1/2} \tag{4.6}$$

for any $\alpha, x \in (0,1)$ *and* $p > 1$.

b) If we write the inequalities (4.3) for the function $\exp z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$, $z \in \mathbb{C}$, *then we have*

$$(4.7)\quad 0 \le \exp[\alpha(x^p - 1)] - \exp[p\alpha(x-1)] \le \frac{1}{2} \cdot \frac{1 - \exp[\alpha(p-1)(x-1)]}{1 - \exp[\alpha(x-1)]} \left\{\exp\left[\alpha\left(x^2 - 1\right)\right] - \exp[2\alpha(x-1)]\right\}^{1/2}$$

for any $\alpha > 0$, $p > 1$ *and* $x \in (0,1)$.

5. Logarithmic Inequalities

If we consider the convex function $f(t) = t \ln t$, $t > 0$, then from (2.1) we have

$$(5.1)\quad \begin{aligned} 0 &\le \sum_{i=1}^{n} p_i a_i \ln a_i - \left(\sum_{i=1}^{n} p_i a_i\right) \ln\left(\sum_{i=1}^{n} p_i a_i\right) \\ &\le \frac{1}{2}\left[\frac{M \ln M - \left(\sum_{i=1}^{n} p_i a_i\right) \ln\left(\sum_{i=1}^{n} p_i a_i\right)}{M - \sum_{i=1}^{n} p_i a_i} - \frac{\left(\sum_{i=1}^{n} p_i a_i\right) \ln\left(\sum_{i=1}^{n} p_i a_i\right) - m \ln m}{\sum_{i=1}^{n} p_i a_i - m}\right] \\ &\quad \times \left[\sum_{i=1}^{n} p_i a_i^2 - \left(\sum_{j=1}^{n} p_j a_j\right)^2\right]^{1/2} \end{aligned}$$

for $0 < m \le a_i \le M$, and $p_i \ge 0$, $i \in \{1, \dots, n\}$ with $\sum_{i=1}^{n} p_i = 1$ and $\sum_{i=1}^{n} p_i a_i \ne m, M$.

If we take in (5.1) $M = 1$ and let $m \to 0+$ we get

$$(5.2)\quad \begin{aligned} 0 &\le \sum_{i=1}^{n} p_i a_i \ln a_i - \left(\sum_{i=1}^{n} p_i a_i\right) \ln\left(\sum_{i=1}^{n} p_i a_i\right) \\ &\le \frac{1}{\left(1 - \sum_{i=1}^{n} p_i a_i\right) \sum_{i=1}^{n} p_i a_i} \left[\sum_{i=1}^{n} p_i a_i^2 - \left(\sum_{j=1}^{n} p_j a_j\right)^2\right]^{1/2} \\ &\quad \times \ln\left[\left(\sum_{i=1}^{n} p_i a_i\right)^{-\sum_{i=1}^{n} p_i a_i}\right] \end{aligned}$$

for $0 < a_i \le 1$, and $p_i \ge 0$, $i \in \{1, \dots, n\}$ with $\sum_{i=1}^{n} p_i = 1$ and $\sum_{i=1}^{n} p_i a_i \ne 0, 1$.

Theorem 5. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a power series with nonnegative coefficients and convergent on the open disk* $D(0,R)$ *with* $R > 0$ *or* $R = \infty$. *If* $0 < \alpha < R$,

$p>0$ and $x\in(0,1)$, then

$$0\leq\frac{p\alpha x^{p}f'(\alpha x^{p})}{f(\alpha)}\ln x-\frac{f(\alpha x^{p})}{f(\alpha)}\ln\left(\frac{f(\alpha x^{p})}{f(\alpha)}\right)\tag{5.3}$$
$$\leq\frac{1}{\left(1-\frac{f(\alpha x^{p})}{f(\alpha)}\right)\frac{f(\alpha x^{p})}{f(\alpha)}}\left[\frac{f(\alpha x^{2p})}{f(\alpha)}-\left(\frac{f(\alpha x^{p})}{f(\alpha)}\right)^{2}\right]^{1/2}$$
$$\times\ln\left[\left(\frac{f(\alpha x^{p})}{f(\alpha)}\right)^{-\frac{f(\alpha x^{p})}{f(\alpha)}}\right].$$

Proof. If $0<\alpha<R$ and $m\geq1$, then by (5.2) for $x_j=(x^p)^j$, we have

$$0\leq\frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^j x^{pj}\ln x^{pj}$$
$$-\frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^j x^{pj}\ln\left(\frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^j x^{pj}\right)$$
$$\leq\frac{1}{\left(1-\frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^j(x^p)^j\right)\frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^j(x^p)^j}$$
$$\times\left[\frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^j(x^p)^{2j}-\left(\frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^j(x^p)^j\right)^{2}\right]^{1/2}$$
$$\times\ln\left[\left(\frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^j(x^p)^j\right)^{-\frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^j(x^p)^j}\right]$$

where $p>0$ and $x\in(0,1)$.

This is equivalent to

$$0\leq\frac{\ln x^{p}}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}ja_j\alpha^j(x^p)^j\tag{5.4}$$
$$-\frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^j(x^p)^j\ln\left(\frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^j(x^p)^j\right)$$
$$\leq\frac{1}{\left(1-\frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^j(x^p)^j\right)\frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^j(x^p)^j}$$
$$\times\left[\frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^j(x^p)^{2j}-\left(\frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^j(x^p)^j\right)^{2}\right]^{1/2}$$
$$\times\ln\left[\left(\frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^j(x^p)^j\right)^{-\frac{1}{\sum_{k=0}^{m}a_k\alpha^k}\sum_{j=0}^{m}a_j\alpha^j(x^p)^j}\right].$$

Since all series whose partial sums involved in the inequality (5.4) are convergent, then by letting $m \to \infty$ in (5.4) we deduce (5.3). □

Example 2. *a) If we write the inequality (5.3) for the function* $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $z \in D(0,1)$, *then we have for* $\alpha, x \in (0,1)$ *and* $p > 0$ *that*

$$
\begin{aligned}
(5.5)\qquad 0 &\le \frac{p\alpha x^p (1-\alpha)}{(1-\alpha x^p)^2} \ln x - \frac{1-\alpha}{1-\alpha x^p} \ln\left(\frac{1-\alpha}{1-\alpha x^p}\right) \\
&\le \frac{1}{\left(1-\frac{1-\alpha}{1-\alpha x^p}\right)\frac{1-\alpha}{1-\alpha x^p}} \left[\frac{1-\alpha}{1-\alpha x^{2p}} - \left(\frac{1-\alpha}{1-\alpha x^p}\right)^2\right]^{1/2} \\
&\times \ln\left[\left(\frac{1-\alpha}{1-\alpha x^p}\right)^{-\frac{1-\alpha}{1-\alpha x^p}}\right].
\end{aligned}
$$

b) If we write the inequality (5.3) for the function $\exp z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$, $z \in \mathbb{C}$, *then we have*

$$
\begin{aligned}
(5.6)\qquad 0 &\le [p\alpha x^p \ln x - \alpha (x^p - 1)] \exp[\alpha (x^p - 1)] \\
&\le \frac{1}{(1-\exp[\alpha(x^p-1)])\exp[\alpha(x^p-1)]} \\
&\times \left[\exp\left[\alpha\left(x^{2p}-1\right)\right] - \exp[2\alpha(x^p-1)]\right]^{1/2} \\
&\times [\alpha(x^p-1)]^{-\exp[\alpha(x^p-1)]}
\end{aligned}
$$

for $x \in (0,1)$ *and* $\alpha, p > 0$.

6. Exponential Inequalities

If we consider the exponential function $f : \mathbb{R} \to (0,\infty)$, $f(t) = \exp(\beta t)$ with $\beta > 0$ then from (2.1) we have

$$
\begin{aligned}
(6.1)\qquad 0 &\le \sum_{i=1}^{n} p_i \exp(\beta a_i) - \exp\left(\beta \sum_{i=1}^{n} p_i a_i\right) \\
&\le \frac{1}{2}\left[\frac{\exp(\beta M) - \exp\left(\beta \sum_{i=1}^{n} p_i a_i\right)}{M - \sum_{i=1}^{n} p_i a_i} \right. \\
&\left. - \frac{\exp\left(\beta \sum_{i=1}^{n} p_i a_i\right) - \exp(\beta m)}{\sum_{i=1}^{n} p_i a_i - m}\right] \\
&\times \left[\sum_{i=1}^{n} p_i a_i^2 - \left(\sum_{j=1}^{n} p_j a_j\right)^2\right]^{1/2}.
\end{aligned}
$$

for any $a_i \in [m, M]$, $p_i \ge 0$ $(i = 1, \dots, n)$ with $\sum_{i=1}^{n} p_i = 1$.

If we take in (6.1) $M = 0$ and let $m \to -\infty$, then we get

$$0 \le \sum_{i=1}^{n} p_i \exp(\beta a_i) - \exp\left(\beta \sum_{i=1}^{n} p_i a_i\right) \le \frac{1}{2} \cdot \frac{1 - \exp\left(\beta \sum_{i=1}^{n} p_i a_i\right)}{-\sum_{i=1}^{n} p_i a_i} \left[\sum_{i=1}^{n} p_i a_i^2 - \left(\sum_{j=1}^{n} p_j a_j\right)^2\right]^{1/2} \tag{6.2}$$

for any $a_i \le 0$, $p_i \ge 0$ $(i = 1, \dots, n)$ with $P_n := \sum_{i=1}^{n} p_i = 1$.

Theorem 6. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients and convergent on the open disk $D(0, R)$ with $R > 0$ or $R = \infty$. If $x \le 0$, $\beta > 0$ with $\exp(\beta x) < R$ and $0 < \alpha < R$, then*

$$0 \le \frac{f(\alpha \exp(\beta x))}{f(\alpha)} - \exp\left[\frac{\alpha \beta x f'(\alpha)}{f(\alpha)}\right] \le \frac{1}{2} \cdot \frac{1 - \exp\left(\beta x \frac{\alpha f'(\alpha)}{f(\alpha)}\right)}{\frac{\alpha f'(\alpha)}{f(\alpha)}} \left[\frac{\alpha\left[f'(\alpha) + \alpha f''(\alpha)\right]}{f(\alpha)} - \left(\frac{\alpha f'(\alpha)}{f(\alpha)}\right)^2\right]^{1/2}. \tag{6.3}$$

Proof. If $0 < \alpha < R$ and $m \ge 1$, then by (6.2) for $x_j = jx$, we have

$$\begin{aligned} 0 &\le \frac{1}{\sum_{j=0}^{m} a_j \alpha^j} \sum_{j=0}^{m} a_j \alpha^j \left[\exp(\beta x)\right]^j - \exp\left(\frac{\beta x}{\sum_{j=0}^{m} a_j \alpha^j} \sum_{j=0}^{m} j a_j \alpha^j\right) \\ &\le \frac{1}{2} \cdot \frac{1 - \exp\left(\frac{\beta x}{\sum_{j=0}^{m} a_j \alpha^j} \sum_{j=0}^{m} j a_j \alpha^j\right)}{-\frac{x}{\sum_{j=0}^{m} a_j \alpha^j} \sum_{j=0}^{m} j a_j \alpha^j} \\ &\quad \times \left[\frac{x^2}{\sum_{j=0}^{m} a_j \alpha^j} \sum_{j=0}^{m} j^2 a_j \alpha^j - \left(\frac{x}{\sum_{j=0}^{m} a_j \alpha^j} \sum_{j=0}^{m} j a_j \alpha^j\right)^2\right]^{1/2} \\ &= \frac{1}{2} \cdot \frac{1 - \exp\left(\frac{\beta x}{\sum_{j=0}^{m} a_j \alpha^j} \sum_{j=0}^{m} j a_j \alpha^j\right)}{\frac{1}{\sum_{j=0}^{m} a_j \alpha^j} \sum_{j=0}^{m} j a_j \alpha^j} \\ &\quad \times \left[\frac{1}{\sum_{j=0}^{m} a_j \alpha^j} \sum_{j=0}^{m} j^2 a_j \alpha^j - \left(\frac{1}{\sum_{j=0}^{m} a_j \alpha^j} \sum_{j=0}^{m} j a_j \alpha^j\right)^2\right]^{1/2} \end{aligned} \tag{6.4}$$

for $x \in (-\infty, 0)$.

If we denote $g(u) := \sum_{n=0}^{\infty} \alpha_n u^n$, then for $|u| < R$, its radius of convergence, we have

$$\sum_{n=0}^{\infty} n \alpha_n u^n = u g'(u)$$

and

$$\sum_{n=0}^{\infty} n^2 \alpha_n u^n = u\left(u g'(u)\right)'.$$

However

$$u\left(ug'(u)\right)' = ug'(u) + u^2 g''(u)$$

and then

$$\sum_{n=0}^{\infty} n^2 \alpha_n u^n = ug'(u) + u^2 g''(u).$$

Since all series whose partial sums involved in the inequality (6.4) are convergent, then by letting $m \to \infty$ in (6.4) we deduce (6.3). □

Example 3. *If we write the inequality (6.3) for the function* $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $z \in D(0,1)$, *then we have for* $x \le 0$, $\beta > 0$ *and* $0 < \alpha < 1$, *that*

$$0 \le \frac{1-\alpha}{1-\alpha \exp(\beta x)} - \exp\left(\frac{\alpha\beta x}{1-\alpha}\right) \le \frac{1}{2} \cdot \frac{1-\exp\left(\frac{\alpha\beta x}{1-\alpha}\right)}{\alpha^{1/2}}. \tag{6.5}$$

The interested reader may obtain other inequalities like (6.5) by taking various examples of power series with nonnegative coefficients as mentioned above. The details are omitted.

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