Received 22/05/14

# ON THE PROPERTIES OF THE $\lambda-$CONVEX FUNCTION AND $\lambda-$CONVEX FUNCTION ON CO-ORDINATES WITH ITS INEQUALITIES

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ABSTRACT. Firstly,we write a short historical background about Convex Functions, Secondly this study contain some properties of $\lambda-$convex with its inequality. Finally we extend the above study for $\lambda-$convex function on Coordinates.

this study submitted J. Numerical Analy. at date 21.05.2014

## 1.*INTRODUCTION*

Convex functions were first defined systematically studied by Jensen in 1905, who adopted $f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$ as defining inequality. We say that a function satisfying this inequality is convex in the sense of Jensen, or briefly Jensen convex. Possibility the convexity of some significant convex functions which were studied before 1905 was recognized though not mentioned. In 1889 Hölder had obtained the fundemental inequality

$$f\left(\sum_{j=1}^{m} q_j x_j\right) \leq \sum_{j=1}^{m} q_j f(x_j) \qquad \left(a < x_j < b,\ q_j > 0,\ \sum_{j=1}^{m} q_j = 1\right),$$

for the subclass of convex functions $f(x)$ for which $f''(x)$ is continuous in $(a,b)$. Also in 1893 Hadamard showed that if $f(x)$ an increasing derivative , for $a < x < b$ we have

$$f\left(\frac{a+b}{2}\right) < \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{0.1}$$

If $f(x)$ is convex , then $f(x)$ satisfies (1) as special case of the definition of convexity in interval $[a,b]$, that is , if $f(x)$ is convex in the sense of Jensen (midpoint convex), or convex $(J)$. Convesely $f(x)$ is continuous and convex $(J)$, then $f(x)$ is convex. But though a convex function must be continuous, a function which is convex $(J)$ is not necessarily continuous.

A function $f : I \to \mathbb{R}$ is called convex if

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda) f(x) + \lambda f(y) \tag{0.2}$$

for all points $x$ and $y$ in $I$ and all $\lambda \in [0,1]$. If $-f$ is convex then we say that $f$ is concave . If $f$ is both convex and concave, then $f$ is said to be affine.

The convexity of a function $f : I \to \mathbb{R}$ means geometrically that the points of the graph of $f_{[u,v]}$ are under(or on) the chord joining the endpoints $(u, f(u))$ and

1991 *Mathematics Subject Classification.* 26D15, 25D10.

*Key words and phrases.* convex, $\lambda-$convex, hadamard inequality,inequality,jensen convex,

$(v, f(v))$ ,for all $u, v \in I,\ u < v$; then

$$f(x) \leq f(u) + \frac{f(v) - f(u)}{v - u}(x - u)$$

for all $x \in [u, v]$ , and all $u, v \in I,\ u < v$.

There are two basic properties of convex functions that make them so widely used in theoretical and applied mathematics. 1) The maximum is attained at a boundary point. 2) Any local maximum is a global one.

In recent years, Autors obtained the integral inequalities Hermite-Hadamard type and the other integral inequalities for different kind of convex functions, see [[10]-[32]].

In [7], Dragomir wrote some properties of $\lambda-$convex function $f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) \leq \frac{\lambda(\alpha)f(x)+\lambda(\beta)f(y)}{\lambda(\alpha+\beta)}$ and then he obtained some inequalities of Hermite Hadamard type for function $f$.

First of all , the following definitions and notations used to the simplify the details of presentation are given.

**Definition 1.** [1] *If two functions $f$ and $g$ are either both nondecreasing, increasing, nonincreasing or decreasing, we say that they are monotone in the same sense.*

i.e.,

$$(f(x) - f(y))(g(x) - g(y)) \geq (\leq) 0.$$

Assume that $I$ and $J$ are intervals in $\mathbb{R}$, $(0,1) \subset J$ and functions $h$ and $f$ are real non-negative functions defined in $J$ and $I$,respectively.

**Definition 2.** [9] *Let $h:\ J \to [0, \infty)$ with not identical to 0. We say that $f : I \to [0, \infty)$ is an $h-$convex if for all $x, y \in I$ we have*

$$f(tx + (1-t)y) \leq h(t) f(x) + h(1-t) f(y)$$

for all $t \in (0, 1)$ .

Some results concerning this class of functions can be found in [3],[4],[5], [6].

**Definition 3.** [8] *Let $\lambda : [0, \infty) \to [0, \infty)$ be a function with the property that $\lambda(t) > 0$ for all $t > 0$. A mapping $f : C \to \mathbb{R}$ defined on a convex subset $C$ of a linear space $X$ is called $\lambda-$convex on $C$ if*

$$f\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) \leq \frac{\lambda(\alpha) f(x) + \lambda(\beta) f(y)}{\lambda(\alpha + \beta)}$$

*for all $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ and $x, y \in C$. We observe that if $f : C \to \mathbb{R}$ is $\lambda-$convex on $C$, then $f$ is $h-$convex on $C$ with $h(t) = \frac{\lambda(t)}{\lambda(1)}$, $t \in [0, 1]$ .*

**Definition 4.** [8] *A function $h : J \to \mathbb{R}$ is said to be supermultiplicative if*

$$h(ts) \geq h(t)h(s) \quad \text{for any } t, s \in J. \tag{0.3}$$

*If the inequality (0.3) is reversed, then $h$ is said to be submultiplicative.*

If $f : C \to [0, \infty)$ is $h-$ convex function with $h$ supermultiplicative on $[0, \infty)$ ,then $f$ is $\lambda$ convex with $\lambda = h$.

Indeed, if $\alpha, \beta \geq 0$ with $\alpha+\beta>0$ and $x, y \in C$ then

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) \leq h\left(\frac{\alpha}{\alpha+\beta}\right) f(x)+h\left(\frac{\beta}{\alpha+\beta}\right) f(y) \leq \frac{h(\alpha) f(x)+h(\beta) f(y)}{h(\alpha+\beta)}. \tag{0.4}$$

In [7], if we have

$$\lambda(\alpha+\beta)=\lambda(\alpha)+\lambda(\beta)$$

for all $\alpha, \beta>0$. Then $\lambda$ is additive on $(0, \infty)$.

In [33], Dragomir established the following inequalities:

Suppose that $f\colon=[a, b] \times[c ; d] \rightarrow \mathrm{R}$ is convex on the co-ordinates on $\Delta$. Then one has the inequalities:

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{(b-a)(d-c)} \int_{a}^{b} \int_{c}^{d} f(x, y)\, d y d x \leq \frac{f(a, c)+f(a, d)+f(b, c)+f(b, d)}{4} \tag{0.5}$$

## 1. 2. MAIN RESULTS

**Theorem 1.** *If $f$ and $g$ oppositely ordered and $f, g$ are $\lambda-$convex functions and if the function $\lambda$ is additive on $(0, \infty)$, then the product $fg$ is also $\lambda-$convex on $(0, \infty)$.*

*Proof.* Since $f$ and $g$ are $\lambda-$convex on $(0, \infty)$, for all $x, y \in C$ with $\alpha, \beta \geq 0$ and $\alpha+\beta>0$, we can write the following inequality

$$\begin{aligned} fg\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) & \leq \frac{\lambda(\alpha) f(x)+\lambda(\beta) f(y)}{\lambda(\alpha+\beta)} . \frac{\lambda(\alpha) g(x)+\lambda(\beta) g(y)}{\lambda(\alpha+\beta)} \\ & = \frac{\lambda^{2}(\alpha) f(x) g(x)+\lambda(\alpha) \lambda(\beta) f(x) g(y)}{\lambda^{2}(\alpha+\beta)} \\ & \quad +\frac{\lambda^{2}(\beta) f(y) g(y)+\lambda(\alpha) \lambda(\beta) f(y) g(x)}{\lambda^{2}(\alpha+\beta)} . \end{aligned}$$

On the other hand Since $f$ and $g$ are oppositely ordered and $\lambda$ is additive on $(0, \infty)$, we have,

$$f(x) g(x)+f(y) g(y) \geq f(x) g(y)+f(y) g(x)$$

and

$$\lambda(\alpha+\beta)=\lambda(\alpha)+\lambda(\beta) .$$

Combining these we get

$$\begin{aligned} fg\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) & \leq \frac{[\lambda(\alpha)+\lambda(\beta)][\lambda(\alpha) f(x) g(x)+\lambda(\beta) f(y) g(y)]}{\lambda^{2}(\alpha+\beta)} \\ & = \frac{\lambda(\alpha)+\lambda(\beta)}{\lambda(\alpha+\beta)} . \frac{\lambda(\alpha) f(x) g(x)+\lambda(\beta) f(y) g(y)}{\lambda(\alpha+\beta)} \\ & = \frac{\lambda(\alpha) f(x) g(x)+\lambda(\beta) f(y) g(y)}{\lambda(\alpha+\beta)} \end{aligned}$$

which is the required result. □

**Theorem 2.** *Under the conditions of Theorem 1, If $\lambda(\alpha)=\lambda(\beta)$, we obtain the following integral inequality*

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b fg\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)d\alpha d\beta \le \frac{f(x)g(x)+f(y)g(y)}{2}.$$

*Proof.* Since

$$fg\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)\le \frac{\lambda(\alpha)f(x)g(x)+\lambda(\beta)f(y)g(y)}{\lambda(\alpha+\beta)}$$

by integrating on the square $[a,b]^2$, we get

$$\begin{aligned}&\int_a^b\int_a^b fg\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)d\alpha d\beta\\ \le\;& f(x)g(x)\int_a^b\int_a^b \frac{\lambda(\alpha)}{\lambda(\alpha+\beta)}d\alpha d\beta + f(y)g(y)\int_a^b\int_a^b \frac{\lambda(\beta)}{\lambda(\alpha+\beta)}d\alpha d\beta\\ =\;& \frac{f(x)g(x)}{2}(b-a)^2+\frac{f(y)g(y)}{2}(b-a)^2.\end{aligned}$$

This completes the proof. □

**Theorem 3.** *Let the $f: C\to\mathbb{R}$ be a nondecreasing $\lambda-$convex function and $g$ a $h-$convex function such that $h(t)=t$. Then the composition of $f$ with $g(fog)$ is also $\lambda-$convex function on a convex subset $C$ of a linear space $X$.*

*Proof.* We set $k=fog$, then we have

$$k\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)=f\left(g\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)\right)$$

for all $t\in(0,1)$ and $x,y\in C,\ \alpha,\beta\ge 0$, with $\alpha+\beta>0,\ t=\frac{\alpha}{\alpha+\beta}$:

Since $g$ is a $h-$convex function and $f$ is $\lambda-$convex, we get

$$\begin{aligned}f\left(g\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)\right) &\le f\left(\frac{\alpha}{\alpha+\beta}g(x)+\frac{\beta}{\alpha+\beta}g(y)\right)\\ &\le \frac{\alpha f(g(x))+\beta f(g(y))}{\lambda(\alpha+\beta)}\\ &= \frac{\lambda(\alpha)k(x)+\lambda(\beta)k(y)}{\lambda(\alpha+\beta)}\end{aligned}$$

which completes the desired result. □

**Theorem 4.** *If $f: C\to[0,\infty)$ is $h-$convex mapping with $h$ supermultiplicative on $[0,\infty)$, $f$ is a linear mapping, then*

$$f\left(\sum_{i=1}^n \frac{\alpha_i x_i+\beta_i y_i}{\alpha_i+\beta_i}\right)\le\sum_{i=1}^n\frac{h(\alpha_i)f(x_i)+h(\beta_i)f(y_i)}{h(\alpha_i+\beta_i)} \tag{1.1}$$

*for all $\alpha_i,\beta_i\ge 0$ with $\alpha_i+\beta_i>0,\ \lambda=h$ and $x_i,y_i\in C$.*

If $n=1$,with $\alpha_1=\alpha,\ \beta_1=\beta$, then 1.1 is equivalent to 0.4 with $\alpha_1,\beta_1\ge 0$ with $\alpha_1+\beta_1>0$ ,$x_1,x_2\in C$.Suppose that inequality 1.1 holds for $n-1$. Then for $\alpha_i,\beta_i\ge 0,\ \alpha_i+\beta_i>0, x_i,y_i\in C\,(i=1,...,n)$ ,we have

$$
\begin{aligned}
f\left(\sum_{i=1}^{n}\frac{\alpha_i x_i+\beta_i y_i}{\alpha_i+\beta_i}\right) &= f\left(\frac{\alpha_n x_n+\beta_n y_n}{\alpha_n+\beta_n}+\sum_{i=1}^{n-1}\frac{\alpha_i x_i+\beta_i y_i}{\alpha_i+\beta_i}\right)\\
&\leq h\left(\frac{\alpha_n}{\alpha_n+\beta_n}\right)f(x_n)+h\left(\frac{\beta_n}{\alpha_n+\beta_n}\right)f(y_n)\\
&\quad+\sum_{i=1}^{n}\left[h\left(\frac{\alpha_i}{\alpha_i+\beta_i}\right)f(x_i)+h\left(\frac{\beta_i}{\alpha_i+\beta_i}\right)f(y_i)\right]\\
&= \sum_{i=1}^{n}\left[\frac{h(\alpha_i)}{h(\alpha_i+\beta_i)}f(x_i)+\frac{h(\beta_i)}{h(\alpha_i+\beta_i)}f(y_i)\right]
\end{aligned}
$$

**Theorem 5.** *Let $f : C \to \mathbb{R}$ be a $\lambda-$convex function on $C$, If $\lambda$ is a additive and $\lambda(\alpha) = \lambda(\beta)$ for all $x, y \in C$ with $\alpha, \beta \geq 0$ and $\alpha + \beta > 0$, then we have the following inequality*

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) \leq \frac{1}{2}\left[f(x)+f(y)\right] \tag{1.2}$$

*Proof.* Since $f$ is a $\lambda-$convex and $\lambda$ is additive and for all $x, y \in C$ with $\alpha, \beta \geq 0$ and $\alpha + \beta > 0$,we have

$$
\begin{aligned}
f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right) &\leq \frac{\lambda(\alpha)f(x)+\lambda(\beta)f(y)}{\lambda(\alpha+\beta)}\\
&= \frac{\lambda(\alpha)}{\lambda(\alpha+\beta)}\left[f(x)+f(y)\right]\\
&= \frac{\lambda(\alpha)}{\lambda(\alpha)+\lambda(\beta)}\left[f(x)+f(y)\right]\\
&= \frac{\lambda(\alpha)}{\lambda(\alpha)+\lambda(\alpha)}\left[f(x)+f(y)\right]\\
&= \frac{1}{2}\left[f(x)+f(y)\right]
\end{aligned}
$$

which completes the proof. □

**Remark 1.** *A function $f$ satisfying inequality (2.2) is said to be $\lambda-$Jensen convex function. Also note that for $\alpha = \beta = \frac{1}{2}$ in (2.2), we have Jensen convex function in the literature.*

**Proposition 1.** *Let $f : C \to \mathbb{R}$ be a $\lambda-$convex function on $C$.*

(*i*) If $f(0) > 0$ , then we have

$$\lambda(\alpha+\beta) \leq \lambda(\alpha)+\lambda(\beta)$$

(*ii*) If $f(0) < 0$ , then we have

$$\lambda(\alpha+\beta) \geq \lambda(\alpha)+\lambda(\beta)$$

(*iii*) If $f(0) > 0$ for all $\alpha \in (0, \infty)$ , then we have

$$\lambda(2\alpha) \leq 2\lambda(\alpha)$$

*Proof.* $(i)$ Putting $x = y = 0$ in definition 3 we have

$$f\left(\frac{\alpha 0+\beta 0}{\alpha+\beta}\right)=f(0)\leq\frac{\lambda(\alpha)f(0)+\lambda(\beta)f(0)}{\lambda(\alpha+\beta)}=f(0)\frac{\lambda(\alpha)+\lambda(\beta)}{\lambda(\alpha+\beta)}$$

and since $f(0)>0$ we get that $\lambda(\alpha+\beta)\leq\lambda(\alpha)+\lambda(\beta)$.

$(ii)$ If we take $f(0)<0$ in (2.3) we get that $\lambda(\alpha+\beta)\geq\lambda(\alpha)+\lambda(\beta)$.

$(iii)$ let us suppose that $f(0)>0$. Putting $x=y=0$ in definitin 3 we get

$$f(0)\leq\frac{\lambda(\alpha)f(0)+\lambda(\beta)f(0)}{\lambda(\alpha+\beta)}$$

Now putting $\alpha=\beta$ and dividing by $f(0)$ we get

$$\lambda(2\alpha)\leq 2\lambda(\alpha)$$

for all $\alpha\in(0,\infty)$. □

**Remark 2.** *In* [7]*, Dragomir obtained the inequalities* $(i)-(iii)$ *,but for either* $\lambda$ *is special case or subadditive.*

In [7], the second inequality (4.1) should be as the inequality (1.3).

**Theorem 6.** *Let* $f: C\to\mathbb{R}$ *be a* $\lambda-$*convex function on* $C$*, If* $\lambda$ *is a additive and* $\lambda(\alpha)=\lambda(\beta)$ *for all* $x,y\in C$ *with* $\alpha,\beta\geq 0$ *and* $\alpha+\beta>0$, $0\leq a<b$, $f\in L[a,b]$*, then we have the following Hermite Hadamard type inequality*

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b\left[f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)+f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)\right]d\alpha d\beta\leq[f(x)+f(y)] \tag{1.3}$$

*Proof.* With the properties of $f$ and $\lambda$, for all $\alpha,\beta\geq 0$ and $\alpha+\beta>0$,we have

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)\leq\frac{\lambda(\alpha)}{\lambda(\alpha+\beta)}[f(x)+f(y)]$$

and

$$f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)\leq\frac{\lambda(\beta)}{\lambda(\alpha+\beta)}[f(x)+f(y)]$$

By adding these inequalities we obtain

$$f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)+f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)\leq[f(x)+f(y)] \tag{1.4}$$

by integrating the inequality(1.4) on the square $[a,b]^2$ we get

$$\int_a^b\int_a^b\left[f\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)+f\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)\right]d\alpha d\beta\leq[f(x)+f(y)](b-a)^2$$

which completes the proof. □

## 2. 3.THE λ-CONVEX FUNCTIONS ON CO-ORDİNATES

**Definition 5.** *Let us* $\lambda:[0,\infty)\times[0,\infty)\to[0,\infty)$, $\lambda(t_1,t_2)>0$, $t_1,t_2>0$. *Consider the bidimensional interval* $\Delta: C\times C$. *The mapping* $f:\Delta\to\mathbb{R}$ *is* $\lambda-$*convex on* $\Delta$ *if*

$$f\left(\frac{\alpha x+\beta z}{\alpha+\beta},\frac{\delta y+\mu w}{\delta+\mu}\right)\leq\frac{\lambda(\alpha,\delta)f(x,y)+\lambda(\beta,\mu)f(z,w)}{\lambda(\alpha+\beta,\delta+\mu)} \tag{2.1}$$

holds for all $(x,y),(z,w)\in\Delta$ with $\alpha,\beta,\delta,\mu\geq 0$, $\alpha+\beta,\delta+\mu>0,(\alpha+\beta,\delta+\mu)\in[0,\infty)\times[0,\infty)$, $C$ is a convex subset of $X$ linear space.

**Proposition 2.** *Let $f:\Delta\to\mathbb{R}$ be a $\lambda-$convex on $\Delta$.*

$(i)$ If $\lambda(0,0)>0$, then $f(z,w)\geq 0$ *and* $f(x,y)\geq 0$
$(ii)$ If $f(0,0)>0$ for $(0,0)\in\Delta$, then

$$\lambda((\alpha,\delta)+(\beta,\mu))\leq\lambda(\alpha,\delta)+\lambda(\beta,\mu)$$

i.e., the mapping $\lambda:[0,\infty)\times[0,\infty)\to[0,\infty)$ is subadditive on $[0,\infty)\times[0,\infty)$.
$(iii)$ If $f(0,0)<0$ for $(0,0)\in\Delta$,

$$\lambda((\alpha,\delta)+(\beta,\mu))\geq\lambda(\alpha,\delta)+\lambda(\beta,\mu)$$

*Proof.* $(i)$ For every $\alpha,\delta>0$ we have

$$\begin{aligned}
f\left(\frac{\alpha x+0z}{\alpha+0},\frac{\delta y+0w}{\delta+0}\right) &\leq \frac{\lambda(\alpha,\delta)f(x,y)+\lambda(0,0)f(z,w)}{\lambda(\alpha+0,\delta+0)}\\
f(x,y) &\leq \frac{\lambda(\alpha,\delta)f(x,y)+\lambda(0,0)f(z,w)}{\lambda(\alpha,\delta)}\\
&= f(x,y)+\frac{\lambda(0,0)f(z,w)}{\lambda(\alpha,\delta)}\\
0 &\leq \frac{\lambda(0,0)f(z,w)}{\lambda(\alpha,\delta)}
\end{aligned}$$

and Since $\lambda(0,0)>0$ we get that $f(z,w)\geq 0$ for all $(z,w)\in\Delta$.
$(ii)$ If $f(0,0)>0$ for $(0,0)\in\Delta$, If we choose $x=z=y=w=0$ *in* (2.1) for all $\alpha,\beta,\delta,\mu>0$, then

$$\begin{aligned}
f\left(\frac{\alpha 0+\beta 0}{\alpha+\beta},\frac{\delta 0+\mu 0}{\delta+\mu}\right) &\leq \frac{\lambda(\alpha,\delta)f(0,0)+\lambda(\beta,\mu)f(0,0)}{\lambda(\alpha+\beta,\delta+\mu)}\\
f(0,0) &\leq f(0,0)\frac{\lambda(\alpha,\delta)+\lambda(\beta,\mu)}{\lambda(\alpha+\beta,\delta+\mu)}
\end{aligned}$$

and since $f(0,0)>0$,then we get that

$$\lambda((\alpha,\delta)+(\beta,\mu))=\lambda(\alpha+\beta,\delta+\mu)\leq\lambda(\alpha,\delta)+\lambda(\beta,\mu)$$

$(iii)$ Since

$$f(0,0)\leq f(0,0)\frac{\lambda(\alpha,\delta)+\lambda(\beta,\mu)}{\lambda(\alpha+\beta,\delta+\mu)}$$

the proof is clearly. □

**Theorem 7.** *Let $f:\Delta\rightarrow\mathbb{R}$ be a $\lambda-convex$ on $\Delta$. Then we have the Hermite-Hadamard type inequalitiy for $\lambda-$ convex function on $\Delta$.*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leq\frac{\lambda\left(1,1\right)}{\lambda\left(2,2\right)}\frac{2}{\left(b-a\right)\left(d-c\right)}\int_{a}^{b}\int_{c}^{d}f\left(x,y\right)dydx$$

*for all $(x,y)\in\Delta,\ a<b,c<d$.*

*Proof.* Since $f$ is a $\lambda-convex$ on $\Delta$ we have

$$f\left(\frac{\alpha x+\beta z}{\alpha+\beta},\frac{\delta y+\mu w}{\delta+\mu}\right)\leq\frac{\lambda(\alpha,\delta)f(x,y)+\lambda(\beta,\mu)f(z,w)}{\lambda(\alpha+\beta,\delta+\mu)}$$

Putting $\alpha=\beta=\delta=\mu=1$ we get

$$f\left(\frac{x+z}{2},\frac{y+w}{2}\right)\leq\frac{\lambda(1,1)}{\lambda(2,2)}\left[f(x,y)+f(z,w)\right]$$

As the change of variables $x=ta+(1-t)\,b,\ \ z=(1-t)\,a++tb,\ \ y=tc+(1-t)\,d,\ w=(1-t)\,c+td$ , for $t\in[0,1]$ , give us that

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leq\frac{\lambda(1,1)}{\lambda(2,2)}\left\{f\left(ta+(1-t)\,b\right),\left(tc+(1-t)\right)d+f\left((1-t)\,a+tb,(1-t)\,c+td\right)\right\}$$

integrating the final inequality on $[0,1]^{2}$

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leq\frac{\lambda(1,1)}{\lambda(2,2)}\frac{2}{\left(b-a\right)\left(d-c\right)}\int_{a}^{b}\int_{c}^{d}f\left(x,y\right)dydx$$

which completes the proof. □

**Remark 3.** *If* $\frac{\lambda(1,1)}{\lambda(2,2)}=\frac{1}{2}$ *then we obtain the first inequality of* (0.5).

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