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OSTROWSKI AND TRAPEZOID TYPE INEQUALITIES RELATED TO POMPEIU'S MEAN VALUE THEOREM

PIETRO CERONE[1], SEVER S. DRAGOMIR[2,3] AND EDER KIKIANTY[4,*]

Abstract. In this paper, some new Ostrowski and trapezoid type inequalities, which are related to Pompeiu's mean value theorem, are obtained for absolutely continuous functions. Some applications to special means and inequalities for f-divergence measures are also given.

1. Introduction

In 1946, Pompeiu [29] derived a variant of Lagrange's mean value theorem, known as *Pompeiu's mean value theorem* (cf. Sahoo and Riedel [33, p. 83]).

Theorem 1. *For every real-valued function f differentiable on an interval $[a,b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a,b]$, there exists a point ξ between x_1 and x_2 such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi). \tag{1}$$

In 1938, Ostrowski [26] proved the following estimate of the integral mean:

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $|f'(t)| \leq M < \infty$ for all $t \in (a,b)$. Then, for any $x \in [a,b]$, we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] M(b-a). \tag{2}$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

Inequality (2) is referred to, in the literature, as the Ostrowski inequality.

Inequalities providing upper bounds for the quantity

$$\left| \frac{(x-a) f(a) + (b-x) f(b)}{b-a} - \frac{1}{b-a}\int_a^b f(t)\,dt \right|, \qquad x \in [a,b] \tag{3}$$

are known in the literature as the *(generalized) trapezoid inequalities*. Cerone and Dragomir [7] proved the following result:

Theorem 3. *Under the assumptions of Theorem 2, we have*

$$\begin{aligned} &\left| \frac{(x-a) f(a) + (b-x) f(b)}{b-a} - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ &\leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] M(b-a), \end{aligned} \tag{4}$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is best possible.

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It is important to note that the bounds in inequalities (2) and (4) are the same. Cerone [6, Remark 1] stated that there is a strong relationship between the Ostrowski and the trapezoidal functionals which is highlighted by the symmetric transformations amongst their kernels.

In the next theorem, Pompeiu's mean value theorem is utilised in order to provide another approximation of the integral mean. Throughout the text, we denote by ℓ, the identity function: $\ell(x) = x$, for all $x \in [a, b]$.

Theorem 4 (Dragomir, 2005 [13])**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, b)$with $[a, b]$ not containing 0. Then for any $x \in [a, b]$, we have the inequality*

$$\left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq \frac{b-a}{|x|} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \| f - \ell f' \|_\infty . \tag{5}$$

The constant $\frac{1}{4}$ is best possible.

By using a mean value theorem, Popa [30] obtained a generalization of (4).

Theorem 5. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Assume that $\alpha \notin [a, b]$. Then for any $x \in [a, b]$, we have the inequality*

$$\begin{aligned} & \left| \left(\frac{a+b}{2} - \alpha \right) f(x) + \frac{\alpha - x}{b-a} \int_a^b f(t)\,dt \right| \\ \leq & \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \| f - \ell_\alpha f' \|_\infty , \end{aligned} \tag{6}$$

where $\ell_\alpha(t) = t - \alpha$, $t \in [a, b]$.

Pečarić and Ungar [27] have proved a general estimate with the p-norms, where $1 \leq p \leq \infty$, which for $p \to \infty$ gives Theorem 4.

Theorem 6. *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $0 < a < b$. Then for $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have the inequality*

$$\left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \leq PU(x, p) \| f - \ell f' \|_p , \tag{7}$$

for $x \in [a, b]$, where

$$\begin{aligned} PU(x,p) \quad := \quad & (b-a)^{\frac{1}{p}-1} \left[\left(\frac{a^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - a^{1+q} x^{1-2q}}{(1-2q)(1+q)} \right)^{1/q} \right. \\ & \left. + \left(\frac{b^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - b^{1+q} x^{1-2q}}{(1-2q)(1+q)} \right)^{1/q} \right] . \end{aligned}$$

In the cases $(p, q) = (1, \infty), (\infty, 1)$ and $(2, 2)$, the quantity $PU(x, p)$ has to be taken as the limit as $p \to 1, \infty$ and 2, respectively.

We refer the readers to Acu and Sofonea [1] and Acu et al. [2], for other inequalities in terms of the p-norms of the quantity $f - \ell_\alpha f'$, where $\ell_\alpha(t) = t - \alpha$, $t \in [a, b]$, and $\alpha \notin [a, b]$.

Throughout the text, for any positive numbers a and b, we denote by $A := A(a, b)$ the arithmetic mean of a and b; $G := G(a, b)$ the geometric mean of a and b; and $H := H(a, b)$ the harmonic mean of a and b, given by:

$$A(a, b) = \frac{a+b}{2}; \quad G(a, b) = \sqrt{ab}, \quad H(a, b) = \frac{2ab}{a+b}.$$

Some recent inequalities of Ostrowski type related to Pompeiu's mean value theorem can be summarised in the following theorem:

Theorem 7 (Dragomir, 2013 [15]). *Let $f : [a, b] \to \mathbb{C}$ be an absolutely continuous function on the interval $[a, b]$ with $b > a > 0$. Then for any $x \in [a, b]$ we have*

$$\left|\frac{f(x)}{x} - \frac{1}{b-a}\int_a^b \frac{f(t)}{t}\,dt\right| \tag{8}$$

$$\leq \begin{cases} \dfrac{2}{b-a}\|f-\ell f'\|_\infty \left(\log\left(\dfrac{x}{G}\right) + \dfrac{A-x}{x}\right), & \text{if } f-\ell f' \in L_\infty[a,b], \\ \dfrac{1}{(2q-1)(b-a)^{\frac{1}{q}}}\|f-\ell f'\|_p C_q(a,b;x)^{\frac{1}{q}}, & \text{if } f-\ell f' \in L_p[a,b],\ p>1 \\ \dfrac{1}{b-a}\|f-\ell f'\|_1 \left(\dfrac{x^2+ab-2ax}{x^2 a}\right), & \text{if } f-\ell f' \in L_1[a,b],. \end{cases}$$

where $q > 1$ is such that $\frac{1}{p} + \frac{1}{q} = 1$ and

$$C_q(a,b;x) = \frac{1}{x^{2q-1}}(b+a-2x) + \frac{a^{2-2q}+b^{2-2q}-2x^{2-2q}}{2(q-1)},\ q>1. \tag{9}$$

We also refer the readers to the related results and their applications to the special means by Dragomir [14].

In this paper, we give refinements of the inequalities in Theorem 7 in Section 2. We also present similar results for trapezoid inequalities in Section 3. In Section 4, we apply these inequalities to compare the special means, in the same spirit to the applications given in Dragomir [14, Section 4]. Finally, in Section 5, the application for inequalities for f-divergence measures is established.

2. Ostrowski type inequalities

We start with the following refinement of Theorem 7 for the case of the ∞-norm:

Theorem 8. *Let $b > a > 0$ and $f : [a, b] \to \mathbb{C}$ an absolutely continuous function on $[a, b]$. Then for any $x \in [a, b]$ we have*

$$\left|\frac{f(x)}{x}(b-a) - \int_a^b \frac{f(t)}{t}\,dt\right| \tag{10}$$

$$\leq \left[\log\left(\frac{x}{a}\right) - \frac{x-a}{x}\right]\|f'\ell - f\|_{[a,x],\infty} + \left[\frac{b-x}{x} - \log\left(\frac{b}{x}\right)\right]\|f'\ell - f\|_{[x,b],\infty}$$

$$\leq 2\left(\log\left(\frac{x}{G}\right) + \frac{A-x}{x}\right)\|f'\ell - f\|_{[a,b],\infty}.$$

The constant 2 is best possible.

Proof. We use the Montgomery identity for the absolutely continuous function $g : [a, b] \to \mathbb{C}$:

$$g(x)(b-a) - \int_a^b g(t)\,dt = \int_a^x (t-a)g'(t)\,dt + \int_x^b (t-b)g'(t)\,dt, \tag{11}$$

where $x \in [a, b]$. If $g(t) = f(t)/t$, then $g'(t) = (f'(t)t - f(t))/t^2$; and with this choice of g, (11) becomes:

$$\begin{aligned} &\frac{f(x)}{x}(b-a) - \int_a^b \frac{f(t)}{t}\,dt \\ &= \int_a^x \frac{t-a}{t^2}\left[f'(t)t - f(t)\right]dt + \int_x^b \frac{t-b}{t^2}\left[f'(t)t - f(t)\right]dt. \end{aligned} \tag{12}$$

Taking the modulus in (12) we get

$$\left|\frac{f(x)}{x}(b-a)-\int_a^b \frac{f(t)}{t}\,dt\right| \tag{13}$$
$$\le \int_a^x \frac{t-a}{t^2}\,dt\ \|f'\ell-f\|_{[a,x],\infty}+\int_x^b \frac{b-t}{t^2}\,dt\ \|f'\ell-f\|_{[x,b],\infty}.$$

However,

$$\int_a^x \frac{t-a}{t^2}\,dt=\int_a^x \frac{1}{t}\,dt-a\int_a^x \frac{1}{t^2}\,dt=\log\left(\frac{x}{a}\right)-\frac{x-a}{x};$$

and

$$\int_x^b \frac{b-t}{t^2}\,dt=\frac{b-x}{x}-\log\left(\frac{b}{x}\right).$$

Making use of (13) we get the first inequality in (10). Furthermore,

$$\left[\log\left(\frac{x}{a}\right)-\frac{x-a}{x}\right]\|f'\ell-f\|_{[a,x],\infty}+\left[\frac{b-x}{x}-\log\left(\frac{b}{x}\right)\right]\|f'\ell-f\|_{[x,b],\infty}$$
$$\le \left[\log\left(\frac{x}{a}\right)-\frac{x-a}{x}+\frac{b-x}{x}-\log\left(\frac{b}{x}\right)\right]\|f'\ell-f\|_{[a,b],\infty}$$
$$= 2\left(\log\left(\frac{x}{G}\right)+\frac{A-x}{x}\right)\|f'\ell-f\|_{[a,b],\infty}.$$

Now we prove the sharpness of the constant. First, we assume that the inequality holds for a constant $K>0$ instead of 2, i.e.

$$\left|\frac{f(x)}{x}(b-a)-\int_a^b \frac{f(t)}{t}\,dt\right|\le K\left(\log\left(\frac{x}{G}\right)+\frac{A-x}{x}\right)\|f'\ell-f\|_{[a,b],\infty}. \tag{14}$$

Choose $f(x)=1$ in (14) and thus, $(f'\ell-f)(x)=-1$, and now we have

$$\left|\frac{1}{x}(b-a)-\log\left(\frac{b}{a}\right)\right|\le K\left(\log\left(\frac{x}{G}\right)+\frac{A-x}{x}\right). \tag{15}$$

We let $x=a$ in (15) to obtain

$$\left|\frac{1}{a}(b-a)-\log\left(\frac{b}{a}\right)\right| \le K\left(\log\left(\frac{a}{G}\right)+\frac{A-a}{a}\right)$$
$$= K\left[\frac{1}{2}\log\left(\frac{a}{b}\right)+\frac{b-a}{2a}\right]=\frac{K}{2}\left[\frac{b-a}{a}-\log\left(\frac{b}{a}\right)\right],$$

which asserts that $\frac{K}{2}\ge 1$, i.e. $K\ge 2$ as desired. □

Remark 9. The function $\phi(x)=\log(x/G)+(A-x)/x$ is minimal for $x=A$, which can be verified by the derivative tests, since ϕ is differentiable on $[a,b]$ for $b>a>0$, as follows: $\phi'(x)=1/x-A/x^2=0$ yields $x=A$ is the stationary point, and further $\phi''(A)=A^{-2}>0$ shows that it is a minimum.

Corollary 10. *If $x=A$, then we get*

$$\left|\frac{f(A)}{A}(b-a)-\int_a^b \frac{f(t)}{t}\,dt\right|\le 2\log\left(\frac{A}{G}\right)\|f'\ell-f\|_{\infty}. \tag{16}$$

If $x=G$, then we get

$$\left|\frac{f(G)}{G}(b-a)-\int_a^b \frac{f(t)}{t}\,dt\right|\le 2\left(\frac{A-G}{G}\right)\|f'\ell-f\|_{\infty}. \tag{17}$$

If $x = H$, then we get

$$\left|\frac{f(H)}{H}(b-a) - \int_a^b \frac{f(t)}{t}\,dt\right| \le 2\left[\log\left(\frac{G}{A}\right) + \frac{A-H}{H}\right]\|f'\ell - f\|_\infty. \tag{18}$$

Remark 11. If we put $f(t) = \ell(t)h(t) = th(t)$, then we get $f'(t) = h(t) + th'(t)$. Then, $(f'\ell - f)(t) = t[h(t) + th'(t)] - th(t) = t^2h'(t)$. From (10), for any $x \in [a, b]$, we get the Ostrowski inequality

$$\begin{aligned} &\left|h(x)(b-a) - \int_a^b h(x)\,dt\right| \\ \le\ & \left[\log\left(\frac{x}{a}\right) - \frac{x-a}{x}\right]\|\ell^2h'\|_{[a,x],\infty} + \left[\frac{b-x}{x} - \log\left(\frac{b}{x}\right)\right]\|\ell^2h'\|_{[x,b],\infty} \\ \le\ & 2\left(\log\left(\frac{x}{G}\right) + \frac{A-x}{x}\right)\|\ell^2h'\|_{[a,b],\infty}. \end{aligned} \tag{19}$$

We recall the definition of the incomplete beta function:

$$B(z, a, b) = \int_0^z u^{a-1}(1-u)^{b-1}\,du,$$

to obtain a refinement of Theorem 7 for the case of the p-norms $(1 < p < \infty)$, which is given in the next theorem.

Theorem 12. *Let $b > a > 0$, $f : [a, b] \to \mathbb{C}$ be an absolutely continuous function on $[a, b]$ and $1 < p < \infty$. Then for any $x \in [a, b]$ we have*

$$\begin{aligned} &\left|\frac{f(x)}{x}(b-a) - \int_a^b \frac{f(t)}{t}\,dt\right| \\ \le\ & a^{\frac{1-q}{q}}\left[B(1, q-1, q+1) - B(a/x, q-1, q+1)\right]^{\frac{1}{q}}\ \|f'\ell - f\|_{[a,x],p} \\ & + b^{\frac{1-q}{q}}\left[B(1, 1-2q, 1+q) - B(x/b, 1-2q, 1+q)\right]^{\frac{1}{q}}\ \|f'\ell - f\|_{[x,b],p} \\ \le\ & \Big[a^{\frac{1-q}{q}}\left[B(1, q-1, q+1) - B(a/x, q-1, q+1)\right]^{\frac{1}{q}} \\ & + b^{\frac{1-q}{q}}\left[B(1, 1-2q, 1+q) - B(x/b, 1-2q, 1+q)\right]^{\frac{1}{q}}\Big]\ \|f'\ell - f\|_{[a,b],p}, \end{aligned} \tag{20}$$

where q is the Hölder's conjugate of p, i.e. $1/p + 1/q = 1$.

Proof. Observe the following inequalities by taking the modulus of (12) and Hölder's inequality for $p > 1$ and its Hölder's conjugate q,

$$\begin{aligned} &\left|\frac{f(x)}{x}(b-a) - \int_a^b \frac{f(t)}{t}\,dt\right| \\ &\le \int_a^x \frac{t-a}{t^2}\,|f'(t)t - f(t)|\,dt + \int_x^b \frac{b-t}{t^2}\,|f'(t)t - f(t)|\,dt \\ &\le \left(\int_a^x \frac{(t-a)^q}{t^{2q}}\,dt\right)^{\frac{1}{q}}\ \|f'\ell - f\|_{[a,x],p} + \left(\int_x^b \frac{(b-t)^q}{t^{2q}}\,dt\right)^{\frac{1}{q}}\ \|f'\ell - f\|_{[x,b],p}. \end{aligned}$$

We evaluate the integrals $\left(\int_a^x (t-a)^q/t^{2q}\,dt\right)^{1/q}$ and $\left(\int_x^b (b-t)^q/t^{2q}\,dt\right)^{1/q}$ in the following. We have

$$\begin{aligned} \int_a^x \frac{(t-a)^q}{t^{2q}}\,dt &= \int_a^x \left(1 - \frac{a}{t}\right)^q \frac{1}{t^q}\,dt \\ &= \int_a^x \left(1 - \frac{a}{t}\right)^q \left(\frac{a}{t}\right)^{q-2} \frac{1}{a^{q-1}}\frac{a}{t^2}\,dt \\ &= a^{1-q}\int_{\frac{a}{x}}^1 u^{q-2}(1-u)^q\,du \\ &= a^{1-q}\left[B(1, q-1, q+1) - B(a/x, q-1, q+1)\right] \end{aligned} \tag{21}$$

and thus

$$\left(\int_a^x \frac{(t-a)^q}{t^{2q}}\,dt\right)^{\frac{1}{q}} = a^{\frac{1-q}{q}}\left[B(1,q-1,q+1)-B(a/x,q-1,q+1)\right]^{\frac{1}{q}}.$$

We also have

$$\begin{aligned}\int_x^b \frac{(b-t)^q}{t^{2q}}\,dt &= \int_x^b \left(1-\frac{t}{b}\right)^q\left(\frac{t}{b}\right)^{-2q} b^{-q}\,dt \\ &= b^{1-q}\int_x^b \left(1-\frac{t}{b}\right)^q\left(\frac{t}{b}\right)^{-2q}\frac{1}{b}\,dt \\ &= b^{1-q}\int_{\frac{x}{b}}^1 u^{-2q}(1-u)^q\,du \\ &= b^{1-q}\left[B(1,1-2q,1+q)-B(x/b,1-2q,1+q)\right]\end{aligned} \tag{22}$$

and thus

$$\left(\int_x^b \frac{(b-t)^q}{t^{2q}}\,dt\right)^{\frac{1}{q}} = b^{\frac{1-q}{q}}\left[B(1,1-2q,1+q)-B(x/b,1-2q,1+q)\right]^{\frac{1}{q}}.$$

Therefore,

$$\begin{aligned}&\left|\frac{f(x)}{x}(b-a)-\int_a^b\frac{f(t)}{t}\,dt\right| \\ \leq\ & \left(\int_a^x \frac{(t-a)^q}{t^{2q}}\,dt\right)^{\frac{1}{q}}\|f'\ell-f\|_{[a,x],p}+\left(\int_x^b \frac{(b-t)^q}{t^{2q}}\,dt\right)^{\frac{1}{q}}\|f'\ell-f\|_{[x,b],p} \\ \leq\ & a^{\frac{1-q}{q}}\left[B(1,q-1,q+1)-B(a/x,q-1,q+1)\right]^{\frac{1}{q}}\|f'\ell-f\|_{[a,x],p} \\ & +b^{\frac{1-q}{q}}\left[B(1,1-2q,1+q)-B(x/b,1-2q,1+q)\right]^{\frac{1}{q}}\|f'\ell-f\|_{[x,b],p} \\ \leq\ & \Big[a^{\frac{1-q}{q}}\left[B(1,q-1,q+1)-B(a/x,q-1,q+1)\right]^{\frac{1}{q}} \\ & +b^{\frac{1-q}{q}}\left[B(1,1-2q,1+q)-B(x/b,1-2q,1+q)\right]^{\frac{1}{q}}\Big]\|f'\ell-f\|_{[a,b],p}\end{aligned}$$

which completes the proof. □

Remark 13. When $p=q=2$ in Theorem 12, we have the following inequalities for all $x\in[a,b]$:

$$\begin{aligned}&\left|\frac{f(x)}{x}(b-a)-\int_a^b\frac{f(t)}{t}\,dt\right| \\ \leq\ & \left(\int_a^x \frac{(t-a)^2}{t^4}\,dt\right)^{\frac{1}{2}}\|f'\ell-f\|_{[a,x],2}+\left(\int_x^b \frac{(b-t)^2}{t^4}\,dt\right)^{\frac{1}{2}}\|f'\ell-f\|_{[x,b],2} \\ =\ & \left(\frac{-3t^2+3at-a^2}{3t^3}\Big|_a^x\right)^{\frac{1}{2}}\|f'\ell-f\|_{[a,x],2}+\left(\frac{-3t^2+3bt-b^2}{3t^3}\Big|_x^b\right)^{\frac{1}{2}}\|f'\ell-f\|_{[x,b],2} \\ =\ & \left(\frac{(x-a)^3}{3ax^3}\right)^{\frac{1}{2}}\|f'\ell-f\|_{[a,x],2}+\left(\frac{(b-x)^3}{3bx^3}\right)^{\frac{1}{2}}\|f'\ell-f\|_{[x,b],2} \\ \leq\ & \left[\left(\frac{(x-a)^3}{3ax^3}\right)^{\frac{1}{2}}+\left(\frac{(b-x)^3}{3bx^3}\right)^{\frac{1}{2}}\right]\|f'\ell-f\|_{[a,b],2}.\end{aligned}$$

Finally, a refinement of Theorem 7 for the case of the 1-norm can be stated as:

Theorem 14. *Let $b > a > 0$ and $f : [a, b] \to \mathbb{C}$ an absolutely continuous function on $[a, b]$. Then for any $x \in [a, b]$ we have*

$$\begin{aligned} &\left| \frac{f(x)}{x}(b-a) - \int_a^b \frac{f(t)}{t}\, dt \right| \\ \tag{23} \leq &\begin{cases} \dfrac{1}{4a}\|f'\ell - f\|_{[a,x],1} + \dfrac{b-x}{x^2}\|f'\ell - f\|_{[x,b],1}, & \text{when } x \geq 2a \\[2ex] \dfrac{x-a}{x^2}\|f'\ell - f\|_{[a,x],1} + \dfrac{b-x}{x^2}\|f'\ell - f\|_{[x,b],1}, & \text{when } x < 2a \end{cases} \\ \leq &\ \frac{x^2 - 4ax + 4ab}{4ax^2}\|f'\ell - f\|_{[a,b],1}. \end{aligned}$$

Proof. We have the following by Hölder's inequality

$$\begin{aligned} &\left| \frac{f(x)}{x}(b-a) - \int_a^b \frac{f(t)}{t}\, dt \right| \\ \leq &\int_a^x \frac{t-a}{t^2} \left| f'(t)t - f(t) \right| dt + \int_x^b \frac{b-t}{t^2} \left| f'(t)t - f(t) \right| dt \\ \leq &\left(\max_{t \in [a,x]} \frac{(t-a)}{t^2} \right) \|f'\ell - f\|_{[a,x],1} + \left(\max_{t \in [x,b]} \frac{(b-t)}{t^2} \right) \|f'\ell - f\|_{[x,b],1}. \end{aligned}$$

Using the derivative test, we obtain that when $x \leq 2a$, the function $t \mapsto (t-a)/t^2$ attains its maximum at $t = 2a$, i.e. $1/(4a)$; otherwise, when $x > a$, the maximum is achieved at $t = x$, i.e. $(x-a)/x^2$. The maximum of $t \mapsto (b-t)/t^2$ is achieved at $t = x$ as it is a decreasing function on $[x, b]$, thus the maximum is $(b-x)/x^2$. Therefore, we now have

$$\begin{aligned} &\left| \frac{f(x)}{x}(b-a) - \int_a^b \frac{f(t)}{t}\, dt \right| \\ \leq &\begin{cases} \dfrac{1}{4a}\|f'\ell - f\|_{[a,x],1} + \dfrac{b-x}{x^2}\|f'\ell - f\|_{[x,b],1}, & \text{when } x \geq 2a \\ \dfrac{x-a}{x^2}\|f'\ell - f\|_{[a,x],1} + \dfrac{b-x}{x^2}\|f'\ell - f\|_{[x,b],1}, & \text{when } x < 2a \end{cases} \\ \leq &\begin{cases} \dfrac{x^2 - 4ax + 4ab}{4ax^2}\|f'\ell - f\|_{[a,b],1}, & \text{when } x \geq 2a \\ \dfrac{b-a}{x^2}\|f'\ell - f\|_{[a,b],1}, & \text{when } x < 2a \end{cases} \\ \leq &\ \frac{x^2 - 4ax + 4ab}{4ax^2}\|f'\ell - f\|_{[a,b],1}, \end{aligned}$$

where the last inequality follows by the fact that

$$\frac{b-a}{x^2} \leq \frac{b-a}{x^2} + \frac{(x-2a)^2}{4ax^2} = \frac{x^2 - 4ax + 4ab}{4ax^2}$$

and this completes the proof. □

3. Trapezoid type inequalities

In this section we consider similar results (as described in Section 2) for trapezoid inequalities. We start with the inequalities in terms of the ∞-norm.

Theorem 15. *Let $b > a > 0$ and $f : [a, b] \to \mathbb{C}$ an absolutely continuous function on $[a, b]$. Then for any $x \in [a, b]$ we have*

$$
\begin{aligned}
(24)\qquad & \left| \frac{f(b)}{b}(b-x) + \frac{f(a)}{a}(x-a) - \int_a^b \frac{f(t)}{t}\,dt \right| \\
\le\ & \left[\frac{x-a}{a} - \log\left(\frac{x}{a}\right) \right] \|f'\ell - f\|_{[a,x],\infty} + \left[\log\left(\frac{b}{x}\right) - \frac{b-x}{b} \right] \|f'\ell - f\|_{[x,b],\infty} \\
\le\ & 2\left[\log\left(\frac{G}{x}\right) + \frac{x-H}{H} \right] \|f'\ell - f\|_{[a,b],\infty}.
\end{aligned}
$$

The constant 2 is best possible.

Proof. We have the trapezoid identity for absolutely continuous function $g : [a, b] \to \mathbb{C}$

$$
(25)\qquad g(b)(b-x) + g(a)(x-a) - \int_a^b g(t)\,dt = \int_a^b (t-x)g'(t)\,dt
$$

where $x \in [a, b]$.

If $g(t) = f(t)/t$, then $g'(t) = (f'(t)t - f(t))/t^2$; and with this choice of g, (25) becomes:

$$
(26)\qquad \frac{f(b)}{b}(b-x) + \frac{f(a)}{a}(x-a) - \int_a^b \frac{f(t)}{t}\,dt = \int_a^b \frac{t-x}{t^2}\left[f'(t)t - f(t)\right] dt.
$$

Taking the modulus in (26) produces

$$
\begin{aligned}
& \left| \frac{f(b)}{b}(b-x) + \frac{f(a)}{a}(x-a) - \int_a^b \frac{f(t)}{t}\,dt \right| \\
\le\ & \int_a^b \frac{|t-x|}{t^2}\left|f'(t)t - f(t)\right| dt \\
=\ & \int_a^x \frac{x-t}{t^2}\left|f'(t)t - f(t)\right| dt + \int_x^b \frac{t-x}{t^2}\left|f'(t)t - f(t)\right| dt \\
\le\ & \int_a^x \frac{x-t}{t^2}\,dt\ \|f'\ell - f\|_{[a,x],\infty} + \int_x^b \frac{t-x}{t^2}\,dt\ \|f'\ell - f\|_{[x,b],\infty} \\
=\ & \left[\frac{x-a}{a} - \log\left(\frac{x}{a}\right) \right] \|f'\ell - f\|_{[a,x],\infty} + \left[\log\left(\frac{b}{x}\right) - \frac{b-x}{x} \right] \|f'\ell - f\|_{[x,b],\infty} \\
\le\ & 2\left[x\left(\frac{a+b}{2ab}\right) - 1 + \log\left(\frac{\sqrt{ab}}{x}\right) \right] \|f'\ell - f\|_{[a,b],\infty} \\
=\ & 2\left[\log\left(\frac{G}{x}\right) + \frac{x-H}{H} \right] \|f'\ell - f\|_{[a,b],\infty}.
\end{aligned}
$$

Now we prove the sharpness of the constant. First, we assume that the inequality holds for a constant $M > 0$ instead of 2, i.e.

$$
\begin{aligned}
(27)\qquad & \left| \frac{f(b)}{b}(b-x) + \frac{f(a)}{a}(x-a) - \int_a^b \frac{f(t)}{t}\,dt \right| \\
\le\ & M\left[\log\left(\frac{G}{x}\right) + \frac{x-H}{H} \right] \|f'\ell - f\|_{[a,b],\infty}.
\end{aligned}
$$

Choose $f(x) = 1$ in (27) and thus, $(f'\ell - f)(x) = -1$, and now we have

$$
(28)\qquad \left| \frac{(b-a)x}{ab} - \log\left(\frac{b}{a}\right) \right| \le M\left(\log\left(\frac{G}{x}\right) + \frac{x-H}{H} \right).
$$

We let $x = a$ in (28) to obtain

$$\begin{aligned}\left|\frac{b-a}{b} - \log\left(\frac{b}{a}\right)\right| &\leq M\left(\log\left(\frac{G}{a}\right) + \frac{a - \frac{G^2}{A}}{\frac{G^2}{A}}\right)\\ &= M\left[\frac{1}{2}\log\left(\frac{b}{a}\right) + \frac{a(\frac{a+b}{2}) - ab}{ab}\right] = \frac{M}{2}\left[\log\left(\frac{b}{a}\right) - \frac{b-a}{b}\right],\end{aligned}$$

which asserts that $\frac{M}{2} \geq 1$, i.e. $M \geq 2$ as desired. □

Remark 16. The function $\psi(x) = (x - H)/H + \log(G/x)$ is minimal for $x = H$, which can be easily verified by the derivative tests, since ψ is differentiable on $[a, b]$ for $b > a > 0$, as follows: $\psi'(x) = 1/H - 1/x = 0$ yields $x = H$ is the stationary point, and further $\psi''(H) = H^{-2} > 0$ shows that it is a minimum.

Remark 17. Note the similarity of the bounds in Theorems 8 and 15. Observe the first upper bound of (24) and let $a = x$, $x = b$ in the first term, and $x = a$, $b = x$ in the second term, we get

$$\left[\frac{b-x}{x} - \log\left(\frac{b}{x}\right)\right]\|f'\ell - f\|_{[x,b],\infty} + \left[\log\left(\frac{x}{a}\right) - \frac{x-a}{x}\right]\|f'\ell - f\|_{[a,x],\infty},$$

which is the first upper bound in (10) of Theorem 8.

Corollary 18. *If we take $x = A$, then we get*

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(b)}{b} + \frac{f(a)}{a}\right](b-a) - \int_a^b \frac{f(t)}{t}\,dt\right|\\ \leq\ &2\left[\frac{A-H}{H} - \log\left(\frac{A}{G}\right)\right]\|f'\ell - f\|_{[a,b],\infty}.\end{aligned} \tag{29}$$

If we take $x = G$, then we get

$$\left|\frac{f(b)}{b}(b-G) + \frac{f(a)}{a}(G-a) - \int_a^b \frac{f(t)}{t}\,dt\right| \leq \frac{2(G-H)}{H}\|f'\ell - f\|_{[a,b],\infty}. \tag{30}$$

If we take $x = H$, then we get

$$\left|\frac{f(b)}{b}(b-H) + \frac{f(a)}{a}(H-a) - \int_a^b \frac{f(t)}{t}\,dt\right| \leq 2\log\left(\frac{G}{H}\right)\|f'\ell - f\|_{[a,b],\infty}. \tag{31}$$

Remark 19. If we put $f(t) = \ell(t)h(t) = th(t)$, then we get $f'(t) = h(t) + th'(t)$. Then, $(f'\ell - f)(t)t^2h'(t)$. From (24), for any $x \in [a, b]$, we get the trapezoid inequality

$$\begin{aligned}&\left|h(b)(b-x) + h(a)(x-a) - \int_a^b h(t)\,dt\right|\\ \leq\ &\left[\frac{x-a}{a} - \log\left(\frac{x}{a}\right)\right]\|\ell^2h'\|_{[a,x],\infty} + \left[\log\left(\frac{b}{x}\right) - \frac{b-x}{x}\right]\|\ell^2h'\|_{[x,b],\infty}\\ \leq\ &2\left[\log\left(\frac{G}{x}\right) + \frac{x-H}{H}\right]\|\ell^2h'\|_{[a,b],\infty}.\end{aligned} \tag{32}$$

The case for the p-norms $(1 < p < \infty)$ is as follows:

Theorem 20. *Let $b > a > 0$, $f : [a, b] \to \mathbb{C}$ be an absolutely continuous function on $[a, b]$ and $1 < p < \infty$. Then for any $x \in [a, b]$ we have*

$$\begin{aligned}
&\left|\frac{f(b)}{b}(b-x)+\frac{f(a)}{a}(x-a)-\int_a^b \frac{f(t)}{t}\,dt\right| \\
\leq\;& x^{\frac{1-q}{q}}\left[B(1,1-2q,1+q)-B(a/x,1-2q,1+q)\right]^{\frac{1}{q}}\|f'\ell-f\|_{[a,x],p} \\
&+x^{\frac{1-q}{q}}\left[B(1,q-1,q+1)-B(x/b,q-1,q+1)\right]^{\frac{1}{q}}\ \|f'\ell-f\|_{[x,b],p} \\
\leq\;& x^{\frac{1-q}{q}}\Big[\left(B(1,1-2q,1+q)-B(a/x,1-2q,1+q)\right)^{\frac{1}{q}} \\
&+\left(B(1,q-1,q+1)-B(x/b,q-1,q+1)\right)^{\frac{1}{q}}\Big]\ \|f'\ell-f\|_{[a,b],p},
\end{aligned} \tag{33}$$

where q is the Hölder's conjugate of p, i.e. $1/p + 1/q = 1$.

Proof. Observe the following inequalities by taking the modulus of (26) and Hölder's inequality for $p > 1$ and its Hölder's conjugate q,

$$\begin{aligned}
&\left|\frac{f(b)}{b}(b-x)+\frac{f(a)}{a}(x-a)-\int_a^b \frac{f(t)}{t}\,dt\right| \\
\leq\;& \int_a^b \frac{|t-x|}{t^2}\left|f'(t)t-f(t)\right|dt \\
=\;& \int_a^x \frac{(x-t)}{t^2}\left|f'(t)t-f(t)\right|dt+\int_x^b \frac{(t-x)}{t^2}\left|f'(t)t-f(t)\right|dt \\
\leq\;& \left(\int_a^x \frac{(x-t)^q}{t^{2q}}\,dt\right)^{\frac{1}{q}}\|f'\ell-f\|_{[a,x],p}+\left(\int_x^b \frac{(t-x)^q}{t^{2q}}\,dt\right)^{\frac{1}{q}}\|f'\ell-f\|_{[x,b],p} \\
=\;& x^{\frac{1-q}{q}}\left[B(1,1-2q,1+q)-B(a/x,1-2q,1+q)\right]^{\frac{1}{q}}\|f'\ell-f\|_{[a,x],p} \\
&+x^{\frac{1-q}{q}}\left[B(1,q-1,q+1)-B(x/b,q-1,q+1)\right]^{\frac{1}{q}}\ \|f'\ell-f\|_{[x,b],p} \\
\leq\;& x^{\frac{1-q}{q}}\Big[\left(B(1,1-2q,1+q)-B(a/x,1-2q,1+q)\right)^{\frac{1}{q}} \\
&+\left(B(1,q-1,q+1)-B(x/b,q-1,q+1)\right)^{\frac{1}{q}}\Big]\ \|f'\ell-f\|_{[a,b],p},
\end{aligned}$$

where the last inequalities follows similarly to the calculations in (21) and (22). □

Remark 21. Note the similarity of the bounds in Theorems 12 and 20. Observe the first upper bound of (33) and let $a = x$, $x = b$ in the first term, and $x = a$, $b = x$ in the second term, we get

$$\begin{aligned}
&b^{\frac{1-q}{q}}\left[B(1,1-2q,1+q)-B(x/b,1-2q,1+q)\right]^{\frac{1}{q}}\|f'\ell-f\|_{[x,b],p} \\
&+a^{\frac{1-q}{q}}\left[B(1,q-1,q+1)-B(a/x,q-1,q+1)\right]^{\frac{1}{q}}\ \|f'\ell-f\|_{[a,x],p}
\end{aligned}$$

which is the first upper bound in (20) of Theorem 12.

Remark 22. When $p = q = 2$ in Theorem 20, we have the following inequalities for all $x \in [a, b]$:

$$\begin{aligned}
&\left|\frac{f(b)}{b}(b-x)+\frac{f(a)}{a}(x-a)-\int_a^b \frac{f(t)}{t}\,dt\right| \\
\leq\;& \left(\int_a^x \frac{(x-t)^2}{t^4}\,dt\right)^{\frac{1}{2}}\|f'\ell-f\|_{[a,x],2}+\left(\int_x^b \frac{(t-x)^2}{t^4}\,dt\right)^{\frac{1}{2}}\|f'\ell-f\|_{[x,b],2} \\
=\;& \left(\left.\frac{-x^2+3tx-3t^2}{3t^3}\right|_a^x\right)^{\frac{1}{2}}\|f'\ell-f\|_{[a,x],2}+\left(\left.\frac{-x^2+3tx-3t^2}{3t^3}\right|_x^b\right)^{\frac{1}{2}}\|f'\ell-f\|_{[x,b],2}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{(x-a)^3}{3a^3x}\right)^{\frac{1}{2}} \|f'\ell - f\|_{[a,x],2} + \left(\frac{(b-x)^3}{3b^3x}\right)^{\frac{1}{2}} \|f'\ell - f\|_{[x,b],2} \\
&= \left[\left(\frac{(x-a)^3}{3a^3x}\right)^{\frac{1}{2}} + \left(\frac{(b-x)^3}{3b^3x}\right)^{\frac{1}{2}}\right] \|f'\ell - f\|_{[a,b],2}.
\end{aligned}
$$

The case of the 1-norm can be stated as:

Theorem 23. *Let $b > a > 0$ and $f : [a,b] \to \mathbb{C}$ an absolutely continuous function on $[a,b]$. Then for any $x \in [a,b]$ we have*

$$
\begin{aligned}
&\left|\frac{f(b)}{b}(b-x) + \frac{f(a)}{a}(x-a) - \int_a^b \frac{f(t)}{t}\,dt\right| \\
\leq &\begin{cases} \frac{x-a}{a^2}\|f'\ell - f\|_{[a,x],1} + \frac{1}{4x}\|f'\ell - f\|_{[x,b],1}, & \text{when } x \leq b/2 \\ \frac{x-a}{a^2}\|f'\ell - f\|_{[a,x],1} + \frac{b-x}{b^2}\|f'\ell - f\|_{[x,b],1}, & \text{when } x > b/2 \end{cases} \\
\leq &\frac{4x^2 - 4ax + a^2}{4a^2x}\|f'\ell - f\|_{[a,b],1}.
\end{aligned} \tag{34}
$$

Proof. We have

$$
\left|\frac{f(b)}{b}(b-x) + \frac{f(a)}{a}(x-a) - \int_a^b \frac{f(t)}{t}\,dt\right|
$$

$$
\begin{aligned}
&\leq \int_a^b \frac{|t-x|}{t^2}\left|f'(t)t - f(t)\right| dt \\
&= \int_a^x \frac{(x-t)}{t^2}\left|f'(t)t - f(t)\right| dt + \int_x^b \frac{(t-x)}{t^2}\left|f'(t)t - f(t)\right| dt, \\
&\leq \left(\max_{t\in[a,x]} \frac{x-t}{t^2}\right)\|f'\ell - f\|_{[a,x],1} + \left(\max_{t\in[x,b]} \frac{t-x}{t^2}\right)\|f'\ell - f\|_{[x,b],1}
\end{aligned}
$$

in which we have used Hölder's inequality. The function $t \mapsto (x-t)/t^2$ is strictly decreasing on $[a,x]$; thus, the maximum is achieved at $t = a$, i.e. $(x-a)/a^2$. Using the derivative test, we find that when $x \leq b/2$, the function $t \mapsto (t-x)/t^2$ attains its maximum at $t = 2x$, thus, the maximum is $1/4x$; otherwise, when $x \geq b/2$ the maximum is achieved at $t = b$, i.e. $(b-x)/b^2$. Therefore,

$$
\begin{aligned}
&\left|\frac{f(b)}{b}(b-x) + \frac{f(a)}{a}(x-a) - \int_a^b \frac{f(t)}{t}\,dt\right| \\
\leq &\begin{cases} \frac{x-a}{a^2}\|f'\ell - f\|_{[a,x],1} + \frac{1}{4x}\|f'\ell - f\|_{[x,b],1}, & \text{when } x \leq b/2 \\ \frac{x-a}{a^2}\|f'\ell - f\|_{[a,x],1} + \frac{b-x}{b^2}\|f'\ell - f\|_{[x,b],1}, & \text{when } x > b/2 \end{cases} \\
\leq &\begin{cases} \frac{4x^2 - 4ax + a^2}{4a^2x}\|f'\ell - f\|_{[a,b],1}, & \text{when } b \geq 2x \\ \frac{(b^2-a^2)x + ba^2 - ab^2}{a^2b^2}\|f'\ell - f\|_{[a,b],1}, & \text{when } b < 2x \end{cases} \\
\leq &\frac{4x^2 - 4ax + a^2}{4a^2x}\|f'\ell - f\|_{[a,b],1},
\end{aligned}
$$

where the last inequality follows by the fact that

$$\frac{4x^2-4ax+a^2}{4a^2x}=\frac{(2x-a)^2}{4a^2x}\geq\frac{(2x-a)^2}{4a^2x}-\frac{(2x-b)^2}{4b^2x}=\frac{(b^2-a^2)x+ba^2-ab^2}{a^2b^2}$$

and this completes the proof. □

Remark 24. Note the similarity of the bounds in Theorems 14 and 23. Observe the first set of upper bounds in (34) and let $a=x$, $x=b$ in the first term, and $x=a$, $b=x$ in the second term, we get

$$\begin{cases} \dfrac{b-x}{x^2}\|f'\ell-f\|_{[x,b],1}+\dfrac{1}{4a}\|f'\ell-f\|_{[a,x],1}, & \text{when } a\leq x/2 \\ \dfrac{b-x}{x^2}\|f'\ell-f\|_{[x,b],1}+\dfrac{x-a}{x^2}\|f'\ell-f\|_{[a,x],1}, & \text{when } a\geq x/2 \end{cases}$$

which is the first upper bound in (23) of Theorem 14.

4. Applications to special means

Recall the following special means:

(1) The identric mean

$$I=I(a,b):=\begin{cases} a, & \text{if } a=b, \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & \text{if } a\neq b \end{cases},\ a,b>0.$$

(2) The logarithmic mean

$$L=L(a,b):=\begin{cases} a, & \text{if } a=b, \\ \frac{b-a}{\log(b)-\log(a)}, & \text{if } a\neq b \end{cases},\ a,b>0.$$

We note that

$$L(a,b)^{-1}=\frac{1}{b-a}\int_a^b\frac{1}{t}\,dt.$$

(3) The r-logarithmic mean (or extended logarithmic mean) $(r\neq 0,-1)$ for two positive numbers:

$$L_r=L_r(a,b):=\begin{cases} a, & \text{if } a=b, \\ \left[\frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}\right]^{\frac{1}{r}}, & \text{if } a\neq b \end{cases},\ a,b>0.$$

We note that

$$\big(L_r(a,b)\big)^r=\frac{1}{b-a}\int_a^b t^r\,dt.$$

The r-logarithmic mean is monotonically increasing over $r\in\mathbb{R}$. We note that $L_1(a,b)=A(a,b)$ and $L_{-2}(a,b)=G(a,b)$. By taking the limits of $r\to 0$, we have $L_0(a,b)=I(a,b)$ and $L_{-1}(a,b)=L(a,b)$. Thus, we have the inequality

$$G\leq L\leq I\leq A.$$

The following inequality is also well-known.

$$H\leq G\leq L\leq I\leq A.$$

4.1. **Ostrowski type inequalities.** We apply Theorem 8 to obtain some inequalities involving the special means.

Let $b > a > 0$ and $r \in \mathbb{R}$, $r \neq 0, 1$. If $f(x) = x^{r+1}$ ($x \in [a,b]$), then $f'(x) = (r+1)x^r$, then $f'(x)\ell(x) - f(x) = (r+1)x^{r+1} - x^{r+1} = rx^{r+1}$. Letting $f(x) = x^{r+1}$ in (10) and multiplying the results by $\frac{1}{b-a}$, we get

$$\begin{aligned} |x^r - (L_r(a,b))^r| &= \left| x^r - \frac{1}{b-a}\int_a^b t^r\,dt \right| \\ &\leq \frac{1}{b-a}\left[\log\left(\frac{x}{a}\right) - \frac{x-a}{x}\right] \|r\ell^{r+1}\|_{[a,x],\infty} \\ &\quad + \frac{1}{b-a}\left[\frac{b-x}{x} - \log\left(\frac{b}{x}\right)\right] \|r\ell^{r+1}\|_{[x,b],\infty} \\ &\leq \frac{2}{b-a}\left(\log\left(\frac{x}{G}\right) + \frac{A-x}{x}\right) \|r\ell^{r+1}\|_{[a,b],\infty} \end{aligned} \tag{35}$$

for $x \in [a,b]$. In particular, for $r > -1, r \neq 0$, we have

$$\begin{aligned} & |x^r - (L_r(a,b))^r| \\ \leq\ & \frac{|r|x^{r+1}}{b-a}\left[\log\left(\frac{x}{a}\right) - \frac{x-a}{x}\right] + \frac{|r|b^{r+1}}{b-a}\left[\frac{b-x}{x} - \log\left(\frac{b}{x}\right)\right] \\ \leq\ & \frac{2|r|b^{r+1}}{b-a}\left(\log\left(\frac{x}{G}\right) + \frac{A-x}{x}\right), \end{aligned}$$

for $x \in [a,b]$; and when $r < -1$,

$$\begin{aligned} & |x^r - (L_r(a,b))^r| \\ \leq\ & \frac{ra^{r+1}}{b-a}\left[\frac{x-a}{x} - \log\left(\frac{x}{a}\right)\right] + \frac{rx^{r+1}}{b-a}\left[\log\left(\frac{b}{x}\right) - \frac{b-x}{x}\right] \\ \leq\ & \frac{2ra^{r+1}}{b-a}\left(\frac{x-A}{x} - \log\left(\frac{x}{G}\right)\right), \end{aligned}$$

for $x \in [a,b]$.

Let $b > a > 0$. If $f(x) = 1$ ($x \in [a,b]$), then $f'(x) = 0$ and $f'(x)\ell(x) - f(x) = -1$. Letting $f(x) = 1$ in (10) and multiplying the results by $\frac{1}{b-a}$, we get

$$\begin{aligned} \left|\frac{1}{x} - L(a,b)^{-1}\right| &= \left|\frac{1}{x} - \frac{1}{b-a}\int_a^b \frac{1}{t}\,dt\right| \\ &\leq \frac{1}{b-a}\left[\log\left(\frac{x}{a}\right) - \frac{x-a}{x}\right] + \frac{1}{b-a}\left[\frac{b-x}{x} - \log\left(\frac{b}{x}\right)\right] \\ &= \frac{2}{b-a}\left(\log\left(\frac{x}{G}\right) + \frac{A-x}{x}\right) \end{aligned} \tag{36}$$

for $x \in [a,b]$.

Let $b > a > 0$. If $f(x) = -x\log(x)$ ($x \in [a,b]$), then $f'(x) = -\log(x) - 1$, and $f'(x)\ell(x) - f(x) = -x$. We let $f(x) = -x\log(x)$ in (10) and multiplying the results by $\frac{1}{b-a}$, we get

$$\begin{aligned} \left|\frac{1}{b-a}\int_a^b \log(t)\,dt - \log(x)\right| &\leq \frac{x}{b-a}\left[\log\left(\frac{x}{a}\right) - \frac{x-a}{x}\right] + \frac{b}{b-a}\left[\frac{b-x}{x} - \log\left(\frac{b}{x}\right)\right] \\ &\leq \frac{2b}{b-a}\left(\log\left(\frac{x}{G}\right) + \frac{A-x}{x}\right) \end{aligned}$$

for $x \in [a,b]$. We note that

$$\frac{1}{b-a}\int_a^b \log(x)\,dx = \log(I(a,b)) \tag{37}$$

for $x \in [a,b]$. Thus, for any $x \in [a,b]$, we have:

$$\begin{aligned}
\log\left(\frac{I(a,b)}{x}\right) &\leq \frac{x}{b-a}\left[\log\left(\frac{x}{a}\right) - \frac{x-a}{x}\right] + \frac{b}{b-a}\left[\frac{b-x}{x} - \log\left(\frac{b}{x}\right)\right] \\
&\leq \frac{2b}{b-a}\left(\log\left(\frac{x}{G}\right) + \frac{A-x}{x}\right). \qquad (38)
\end{aligned}$$

4.2. **Trapezoid type inequalities.** We now apply Theorem 15 to get some inequalities involving the special means.

Let $b > a > 0$ and $r \in \mathbb{R}$, $r \neq 0,1$. If $f(x) = x^{r+1}$ $(x \in [a,b])$, then $f'(x) = (r+1)x^r$, then $f'(x)\ell(x) - f(x) = (r+1)x^{r+1} - x^{r+1} = rx^{r+1}$. Letting $f(x) = x^{r+1}$ in (24) and multiplying the results by $\frac{1}{b-a}$, we get

$$\begin{aligned}
&\left|\frac{b-x}{b-a}b^r + \frac{x-a}{b-a}a^r - \frac{1}{b-a}\int_a^b t^r\,dt\right| \\
\leq\ & \frac{1}{b-a}\left[\frac{x-a}{a} - \log\left(\frac{x}{a}\right)\right]\|\ell^{r+1}\|_{[a,x],\infty} + \frac{1}{b-a}\left[\log\left(\frac{b}{x}\right) - \frac{b-x}{b}\right]\|\ell^{r+1}\|_{[x,b],\infty} \\
\leq\ & \frac{2}{b-a}\left[\log\left(\frac{G}{x}\right) + \frac{x-H}{H}\right]\|\ell^{r+1}\|_{[a,b],\infty}.
\end{aligned}$$

for $x \in [a,b]$. We observe that

$$\begin{aligned}
\frac{b-x}{b-a}b^r + \frac{x-a}{b-a}a^r - \frac{1}{b-a}\int_a^b t^r\,dt &= \frac{b^{r+1}-a^{r+1}}{b-a} - x\frac{b^r-a^r}{b-a} - (L_r(a,b))^r \\
&= (r+1)(L_r(a,b))^r - rx(L_{r-1}(a,b))^{r-1} - (L_r(a,b))^r \\
&= r\left[L_r(a,b)^r - x(L_{r-1}(a,b))^{r-1}\right].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\left|L_r(a,b)^r - x(L_{r-1}(a,b))^{r-1}\right| \qquad (39) \\
\leq\ & \frac{1}{b-a}\left[\frac{x-a}{a} - \log\left(\frac{x}{a}\right)\right]\|\ell^{r+1}\|_{[a,x],\infty} \\
&+\frac{1}{b-a}\left[\log\left(\frac{b}{x}\right) - \frac{b-x}{b}\right]\|\ell^{r+1}\|_{[x,b],\infty} \\
\leq\ & \frac{2}{b-a}\left[\log\left(\frac{G}{x}\right) + \frac{x-H}{H}\right]\|\ell^{r+1}\|_{[a,b],\infty}
\end{aligned}$$

for $x \in [a,b]$. In particular, for $r > -1$, $r \neq 0$, we have

$$\begin{aligned}
\left|L_r(a,b)^r - x(L_{r-1}(a,b))^{r-1}\right| &\leq \frac{x^{r+1}}{b-a}\left[\frac{x-a}{a} - \log\left(\frac{x}{a}\right)\right] + \frac{b^{r+1}}{b-a}\left[\log\left(\frac{b}{x}\right) - \frac{b-x}{b}\right] \\
&\leq \frac{2b^{r+1}}{b-a}\left[\log\left(\frac{G}{x}\right) + \frac{x-H}{H}\right]\|\ell^{r+1}\|_{[a,b],\infty}
\end{aligned}$$

for $x \in [a,b]$; and for $r < -1$, we have the following for any $x \in [a,b]$:

$$\begin{aligned}
\left|L_r(a,b)^r - x(L_{r-1}(a,b))^{r-1}\right| &\leq \frac{a^{r+1}}{b-a}\left[\frac{x-a}{a} - \log\left(\frac{x}{a}\right)\right] + \frac{x^{r+1}}{b-a}\left[\log\left(\frac{b}{x}\right) - \frac{b-x}{b}\right] \\
&\leq \frac{2a^{r+1}}{b-a}\left[\log\left(\frac{G}{x}\right) + \frac{x-H}{H}\right]\|\ell^{r+1}\|_{[a,b],\infty}.
\end{aligned}$$

Let $b > a > 0$. If $f(x) = 1$ $(x \in [a, b])$, then $f'(x) = 0$ and $f'(x)\ell(x) - f(x) = -1$. Letting $f(x) = 1$ in (24) and multiplying the results by $\frac{1}{b-a}$, we get

$$\begin{aligned}
&\left|\frac{b-x}{b(b-a)} + \frac{x-a}{a(b-a)} - \frac{1}{b-a}\int_a^b \frac{f(t)}{t}\,dt\right| \\
\leq\ & \frac{1}{b-a}\left[\frac{x-a}{a} - \log\left(\frac{x}{a}\right)\right] + \frac{1}{b-a}\left[\log\left(\frac{b}{x}\right) - \frac{b-x}{b}\right] \\
=\ & \frac{2}{b-a}\left[\log\left(\frac{G}{x}\right) + \frac{x-H}{H}\right]
\end{aligned}$$

for $x \in [a, b]$. Observe that

$$\frac{b-x}{b(b-a)} + \frac{x-a}{a(b-a)} = \frac{x}{ab} = \frac{x}{G^2}.$$

Thus,

$$\left|\frac{x}{G^2} - L^{-1}\right| \leq \frac{2}{b-a}\left[\log\left(\frac{G}{x}\right) + \frac{x-H}{H}\right], \quad x \in [a, b]. \tag{40}$$

Let $b > a > 0$. If $f(x) = -x\log(x)$ $(x \in [a, b])$, then $f'(x) = -\log(x) - 1$, and $f'(x)\ell(x) - f(x) = -x$. Letting $f(x) = -x\log(x)$ in (24) and multiplying the results by $\frac{1}{b-a}$, we get

$$\begin{aligned}
&\left|-\frac{b-x}{b-a}\log(b) - \frac{x-a}{b-a}\log(a) + \log(I(a,b))\right| \\
=\ & \left|-\frac{b-x}{b-a}\log(b) - \frac{x-a}{b-a}\log(a) + \frac{1}{b-a}\int_a^b \log(t)\,dt\right| \\
\leq\ & \frac{x}{b-a}\left[\frac{x-a}{a} - \log\left(\frac{x}{a}\right)\right] + \frac{b}{b-a}\left[\log\left(\frac{b}{x}\right) - \frac{b-x}{b}\right] \\
\leq\ & \frac{2b}{b-a}\left[\log\left(\frac{G}{x}\right) + \frac{x-H}{H}\right]
\end{aligned} \tag{41}$$

for $x \in [a, b]$. Note the use of (37).

Let $x = (a+b)/2 = A$ in Remark 19 and multiply the results by $1/(b-a)$, we now have:

$$\begin{aligned}
&\left|\frac{h(a)+h(b)}{2} - \frac{1}{b-a}\int_a^b h(t)\,dt\right| \\
\leq\ & \left[\frac{1}{2a} - \frac{1}{b-a}\log\left(\frac{A}{a}\right)\right]\|\ell^2 h'\|_{[a,A],\infty} + \left[\frac{1}{b-a}\log\left(\frac{b}{A}\right) - \frac{1}{2b}\right]\|\ell^2 h'\|_{[A,b],\infty} \\
\leq\ & \frac{2}{b-a}\left[\frac{A-H}{H} - \log\left(\frac{A}{G}\right)\right]\|\ell^2 h'\|_{[a,b],\infty}.
\end{aligned} \tag{42}$$

In what follows, we present some special cases of (42):

(1) Let $b > a > 0$. If $h(x) = x^r$ $(x \in [a, b])$, where $r \neq 0, -1$, then $h'(x) = rx^{r-1}$, and $\ell^2(x)h'(x) = rx^{r+1}$. Thus, we have

$$\begin{aligned}
&\left|A(a^r,b^r)-\left(L_r(a,b)\right)^r\right| = \left|\frac{a^r+b^r}{2}-\frac{1}{b-a}\int_a^b t^r\,dt\right| \\
\leq\; & \left[\frac{1}{2a}-\frac{1}{b-a}\log\left(\frac{A}{a}\right)\right]\|r\ell^{r+1}\|_{[a,A],\infty}+\left[\frac{1}{b-a}\log\left(\frac{b}{A}\right)-\frac{1}{2b}\right]\|r\ell^{r+1}\|_{[A,b],\infty} \\
\leq\; & \frac{2}{b-a}\left[\frac{A-H}{H}-\log\left(\frac{A}{G}\right)\right]\|r\ell^{r+1}\|_{[a,b],\infty}.
\end{aligned}$$

In particular, for $r>-1$, $r\neq 0$, we have

$$\begin{aligned}
&\left|A(a^r,b^r)-\left(L_r(a,b)\right)^r\right| \\
\leq\; & |r|A^{r+1}\left[\frac{1}{2a}-\frac{1}{b-a}\log\left(\frac{A}{a}\right)\right]+|r|b^{r+1}\left[\frac{1}{b-a}\log\left(\frac{b}{A}\right)-\frac{1}{2b}\right] \\
\leq\; & \frac{2|r|b^{r+1}}{b-a}\left[\frac{A-H}{H}-\log\left(\frac{A}{G}\right)\right],
\end{aligned}$$

and for $r<-1$, we have

$$\begin{aligned}
&\left|A(a^r,b^r)-\left(L_r(a,b)\right)^r\right| \\
\leq\; & ra^{r+1}\left[\frac{1}{b-a}\log\left(\frac{A}{a}\right)-\frac{1}{2a}\right]+rA^{r+1}\left[\frac{1}{2b}-\frac{1}{b-a}\log\left(\frac{b}{A}\right)\right] \\
\leq\; & \frac{2ra^{r+1}}{b-a}\left[\log\left(\frac{A}{G}\right)-\frac{A-H}{H}\right].
\end{aligned}$$

(2) Let $b>a>0$. If we let $h(x)=1/x$ ($x\in[a,b]$), then $h'(x)=-1/x^2$, and $\ell^2(x)h'(x)=-1$. Therefore, we have

$$\begin{aligned}
\left|\frac{\frac{1}{a}+\frac{1}{b}}{2}-\frac{1}{b-a}\int_a^b\frac{1}{t}\,dt\right| &\leq \left[\frac{1}{2a}-\frac{1}{b-a}\log\left(\frac{A}{a}\right)\right]+\left[\frac{1}{b-a}\log\left(\frac{b}{A}\right)-\frac{1}{2b}\right] \\
&= \frac{2}{b-a}\left[\frac{A-H}{H}-\log\left(\frac{A}{G}\right)\right].
\end{aligned}$$

In terms of the special means, we have

$$0\leq\frac{1}{H}-\frac{1}{L}\leq\frac{2}{b-a}\left[\frac{A-H}{H}-\log\left(\frac{A}{G}\right)\right],$$

since $H\leq L$ for $b>a>0$.

(3) Let $b>a>0$. If we let $h(x)=-\log(x)$ ($x\in[a,b]$), then $h'(x)=-1/x$ and $\ell^2(x)h'(x)=-x$. Therefore,

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b\log(t)\,dt+\frac{-\log(a)-\log(b)}{2}\right| \\
\leq\; & \left[\frac{1}{2a}-\frac{1}{b-a}\log\left(\frac{A}{a}\right)\right]A+\left[\frac{1}{b-a}\log\left(\frac{b}{A}\right)-\frac{1}{2b}\right]b \\
\leq\; & \frac{2b}{b-a}\left[\frac{A-H}{H}-\log\left(\frac{A}{G}\right)\right].
\end{aligned}$$

We note that

$$\frac{-\log(a)-\log(b)}{2}=\log\frac{1}{G}.$$

By the above identity and (37), we have the following inequalities

$$\begin{aligned}\log\left(\frac{I}{G}\right) &\leq \left[\frac{1}{2a}-\frac{1}{b-a}\log\left(\frac{A}{a}\right)\right]A+\left[\frac{1}{b-a}\log\left(\frac{b}{A}\right)-\frac{1}{2b}\right]b \\ &\leq \frac{2b}{b-a}\left[\frac{A-H}{H}-\log\left(\frac{A}{G}\right)\right].\end{aligned}$$

5. Applications to inequalities for f-divergence measures

One of the important issues in many applications of probability theory is finding an appropriate measure of *distance* (or *difference* or *discrimination*) between two probability distributions. A number of divergence measures for this purpose have been proposed and extensively studied by Jeffreys [18], Kullback and Leibler [23], Rényi [32], Havrda and Charvat [17], Kapur [21], Sharma and Mittal [35], Burbea and Rao [5], Rao [31], Lin [24], Csiszár [10], Ali and Silvey [3], Vajda [41], Shioya and Da-te [36] and others (see for example Mei [25] and the references therein).

These measures have been applied in a variety of fields such as: anthropology [31], genetics [25], finance, economics, and political science [34], [39], [40], biology [28], the analysis of contingency tables [16], approximation of probability distributions [9], [22], signal processing [19], [20] and pattern recognition [4], [8]. A number of these measures of distance are specific cases of Csiszár f-divergence and so further exploration of this concept will have a flow on effect to other measures of distance and to areas in which they are applied.

The difference between two probability measures p, q on a set $A = \{\alpha_i | 1 \leq i \leq n\}$ is commonly measured in a variety of ways. Denote by p_i, q_i, the associated point probabilities for the event $\alpha_i \in A$. To avoid triviality we assume that $p_i + q_i > 0$ for each i. The variational distance, i.e. ℓ_1-distance, information divergence (Kullback-Leibler divergence) (cf. Kullback and Leibler [23]) and the triangular discrimination (cf. Topsoe [38]), between the distributions p and q are defined respectively by

$$V(p,q) := \sum_{i=1}^{n} |p_i - q_i|, \tag{43}$$

$$D(p,q) := \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right) \tag{44}$$

$$D_{\Delta}(p,q) := \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{p_i + q_i}. \tag{45}$$

For other divergence measures, we refer the readers to the paper by Kapur [21] or the book by Taneja [37].

If $f : [0,\infty) \to \mathbb{R}$ is convex, the Csiszár f-divergence between p and q is defined by

$$I_f(p,q) := \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right). \tag{46}$$

The distances $D(p,q)$ and $D_{\Delta}(p,q)$ are particular instances of Csiszár $f-$divergence. For the basic properties of Csiszár $f-$divergence, we refer the readers to Csiszár [11], [12] and Vajda [41].

Proposition 25. *Let $R > 1 > r > 0$ and assume that $\frac{p_i}{q_i} \in [r,R]$, for all $i \in \{1,\ldots,n\}$. Let $f : [r,R] \to \mathbb{R}$ a convex function on $[r,R]$. We have the following inequalities involving the Csiszár f-divergence and Kullback-Leibler divergence* *between the distributions p and q:*

$$\left|I_f(p,q) - \frac{1}{R-r}\int_r^R \frac{f(t)}{t}\,dt\right| \leq \frac{2\|f'\ell - f\|_{[r,R],\infty}}{R-r}\left[D(p,q) - \log(G) - 1 + A\right]; \tag{47}$$

where G is the geometric mean of r and R, and A is the arithmetic mean of r and R.

Proof. Since f is convex on $[r,R]$, then f is absolutely continuous, thus we may apply Theorem 8. We let $x = p_i/q_i$ in (10), and multiply the results by $p_i/(R-r)$, to obtain

$$\left|q_i f\left(\frac{p_i}{q_i}\right) - \frac{p_i}{R-r}\int_r^R \frac{f(t)}{t}\,dt\right| \leq \frac{2}{R-r}\left[p_i\left(\log\left(\frac{p_i}{q_i}\right) - \log(G) - 1\right) + q_i A\right]\|f'\ell - f\|_{[r,R],\infty}.$$

Taking the sum from 1 to n, we have

$$\begin{aligned}\left|I_f(p,q) - \frac{1}{R-r}\int_r^R \frac{f(t)}{t}\,dt\right| &\leq \sum_{i=1}^n \left|q_i f\left(\frac{p_i}{q_i}\right) - \frac{p_i}{R-r}\int_r^R \frac{f(t)}{t}\,dt\right| \\ &\leq \frac{2}{R-r}\left[\sum_{i=1}^n p_i \log\left(\frac{p_i}{q_i}\right) - \log(G) - 1 + A\right]\|f'\ell - f\|_{[r,R],\infty};\end{aligned}$$

and by using (44), the proof is completed. □

We consider a particular case of Proposition 25 in the following.

(1) Let $R > 1 > r > 0$ and assume that $\frac{p_i}{q_i} \in [r,R]$, for all $i \in \{1,\dots,n\}$. Let $f : [r,R] \to \mathbb{R}$ a convex function on $[r,R]$. Let $f(x) = (x-1)^2/(x+1)$ in Proposition 25. Then,

$$I_f(p,q) = \sum_{i=1}^n q_i \frac{\left(\frac{p_i}{q_i} - 1\right)^2}{\frac{p_i}{q_i} + 1} = \sum_{i=1}^n \frac{(p_i - q_i)^2}{p_i + q_i} = D_\Delta(p,q),$$

and

$$\int_r^R \frac{(t-1)^2}{t(t+1)}\,dt = \int_r^R \left[1 + \frac{1}{t} - \frac{4}{t+1}\right] dt = R - r + \log\left(\frac{R(r+1)^4}{r(R+1)^4}\right).$$

We have

$$f'(x) = \frac{(x-1)(x+3)}{(x+1)^2}$$

and therefore

$$f'(x)\ell(x) - f(x) = \frac{(x-1)(3x+1)}{(x+1)^2}.$$

Denote by Φ and Ψ, the functions:

$$\Phi(x) = \frac{(1-x)(3x+1)}{(x+1)^2}, \quad \text{and} \quad \Psi(x) = \frac{(x-1)(3x+1)}{(x+1)^2}.$$

In conclusion, for p and q as above, we have

$$\begin{aligned} & \left| D_\Delta(p,q) - 1 - \frac{1}{R-r}\log\left(\frac{R(r+1)^4}{r(R+1)^4}\right)\right| \\ \leq\ & \frac{2}{R-r}\left[D(p,q) - \log(G) - 1 + A\right]\max\left\{\Phi(r), \Psi(R)\right\}. \end{aligned} \tag{48}$$

We note that Φ is decreasing on $[0,1]$ from 1 to 0, and Ψ is increasing on $[0,\infty)$ from 0 to 3. Let γ be a point on $[0,1]$ such that $\Phi(\gamma) = \Psi(\gamma - 1)$. We have the following simplification of the upper bound of (48) above:

$$\begin{aligned} & \frac{2\max\{\Phi(r),\Psi(R)\}}{R-r}\left[D(p,q) - \log(G) - 1 + A\right] \\ =\ & \begin{cases} \dfrac{2\Psi(R)}{R-r}[D(p,q) - \log(G) - 1 + A], & R \geq 1+\sqrt{2} \\ \dfrac{2\Phi(r)}{R-r}[D(p,q) - \log(G) - 1 + A], & 0 < r < \gamma, \text{ and } R < \gamma + 1 \\ \dfrac{2\Psi(R)}{R-r}[D(p,q) - \log(G) - 1 + A], & \gamma < r < 1, \text{ and } 1 + \gamma < R \\ \dfrac{2\max\{\Phi(r),\Psi(R)\}}{R-r}[D(p,q) - \log(G) - 1 + A], & \text{otherwise.} \end{cases} \end{aligned}$$

(2) Let $R > 1 > r > 0$ and assume that $\frac{p_i}{q_i} \in [r,R]$, for all $i \in \{1,\dots,n\}$. Let $f : [r,R] \to \mathbb{R}$ a convex function on $[r,R]$. Let $f(x) = -\log(x)$ in Proposition 25. Then,

$$I_f(p,q) = -\sum_{i=1}^n q_i \log\left(\frac{p_i}{q_i}\right) = D(q,p)$$

$$\frac{1}{R-r}\int_r^R \frac{\log(t)}{t}\,dt = \frac{1}{2(R-r)}(\log(x))^2\Big|_r^R = \frac{(\log(R))^2 - (\log(r))^2}{2(R-r)},$$

and

$$f'(x)\ell(x) - f(x) = -1 + \log(x).$$

Therefore, for two distributions p and q as above, we have the inequalities

$$\begin{aligned} & \left| D(q,p) - \frac{(\log(R))^2 - (\log(r))^2}{2(R-r)}\right| \\ \leq\ & \frac{2}{b-a}\left[D(p,q) - \log(G) - 1 + A\right]\max\{1 - \log(r), |1 - \log(R)|\} \\ =\ & \begin{cases} \frac{2}{b-a}\left[D(p,q) - \log(G) - 1 + A\right](1 - \log(r)), & 1 < R \leq e^2 \\ \frac{2}{b-a}\left[D(p,q) - \log(G) - 1 + A\right]\max\{1 - \log(r), \log(R) - 1\}, & R > e^2. \end{cases} \end{aligned} \tag{49}$$

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[1]Department of Mathematics and Statistics, La Trobe University, Bundoora 3086, Australia, E-mail: p.cerone@latrobe.edu.au
[2]School of Engineering and Science, Victoria University, PO Box 14428, Melbourne 8001, Victoria, Australia, E-mail: sever.dragomir@vu.edu.au
[3]School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, South Africa
[4]Department of Mathematics, University of Johannesburg, PO Box 524, Auckland Park 2006, South Africa, E-mail: ekikianty@uj.ac.za
* Corresponding Author