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HERMITE HADAMARD TYPE INEQUALITIES FOR m-CONVEX AND (α, m)-CONVEX FUNCTIONS FOR FUZZY INTEGRALS

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ABSTRACT. In this paper Hermite–Hadamard type inequalities for m-convex and (α, m)-convex functions for fuzzy integrals are given. Some examples are also given to illustrate the results.

1. Introduction

The theory of fuzzy measures and fuzzy integral was introduced by Sugeno [22] as a tool for modeling non-deterministic problems. The fuzzy integral (also known as Sugeno integral) has very fascinating properties from a mathematical point of view and those properties have been studied by many authors, including Pap [23], Ralescu and Adams [15], Román-Flores et al. [18, 19] and Wang and Klir [28]. Among others, Ralescu and Adams [15] studied several equivalent definitions of fuzzy integral, while Pap [23] and Wang and Klir [28] provided an overview of fuzzy measure theory. Fuzzy measures and Sugeno integral have also been successfully applied to various fields, e.g., to decision-making [12] and to artificial intelligence [29]. Integral inequalities are useful tools in several theoretical and applied fields. For more information on classical inequalities, we refer the interested readers to the distinguished monographs [8, 9]. The study of inequalities for the Sugeno integral was initiated by Román-Flores et al. [5, 18, 19, 20, 21] and then further developed by Ouyang et al. in [10, 13, 14]. Recnetly, J. Caballero, K. Sadarangani [3] have shown that the classical Hermite-Hadamard inequalities [6, 7]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

where $f : [a, b] \to \mathbb{R}$ is a convex function, do not hold true for fuzzy integrals in general . The authors in [3] established some Hermite-Hadamard type inequalities for Sugeno integrals and illustrated their results by providing certain examples.

In this paper we will prove some Hermite-Hadamard type inequalities for m-convex and (α, m)-convex functions for fuzzy integral (Sugeno Integral).

In order to proceed to our results we first give some basic notation and properties about Sugeno integral. For more details on Sugeno integral we refer the interested readers to [22, 28].

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Suppose that Σ is σ-algebra of subsets of $\mathbb{R}$ and that $\mu : \Sigma \to [0, \infty)$ is non-negative extended real valued set function, then μ is said to be fuzzy measure if and only if:

(1) $\mu(\emptyset) = 0$,
(2) E, $F \in \Sigma$ and $E \subset F$ imply that $\mu(E) \leq \mu(F)$ (monotonicity),
(3) $\{E_n\} \subset \Sigma$, $E_1 \subset E_2 \subset ...$, imply $\lim\limits_{n\to\infty} \mu(E_n) = \mu\left(\bigcup\limits_{n=1}^{\infty} E_n\right)$ (continuity from below)
(4) $\{E_n\} \subset \Sigma$, $E_1 \supset E_2 \supset ...$, $\mu(E_1) < \infty$, imply $\lim\limits_{n\to\infty} \mu(E_n) = \mu\left(\bigcap\limits_{n=1}^{\infty} E_n\right)$ (continuity from above).

If f is a non-negative real-valued function defined on $\mathbb{R}$, we will denote by $L_\alpha f = \{x \in \mathbb{R} : f(x) \geq \alpha\} = \{f \geq \alpha\}$ the α-level of f, for $\alpha > 0$ and $L_0 f = \{x \in \mathbb{R} : f(x) \geq 0\} = supp f$, the support of f. It may be noted that if $\alpha \leq \beta$, then $\{f \leq \alpha\} \subset \{f \leq \beta\}$. If μ is fuzzy measure on $(\mathbb{R}, \Sigma)$, by $\mathcal{F}^\mu(\mathbb{R})$, we mean all μ-measurable functions from $\mathbb{R}$ to $[0, \infty)$.

Suppose that μ is a fuzzy measure on $(\mathbb{R}, \Sigma)$. If $f \in \mathcal{F}^\mu(\mathbb{R})$ and $A \subset \Sigma$ then the Sugeno integral (or fuzzy integral) of f on A with respect to the fuzzy measure μ is defined as:

$$\int_A f d\mu = \bigvee_{\alpha \geq 0} \left[\alpha \wedge \mu\left(A \cap \{f \geq \alpha\}\right)\right],$$

where $\vee$ and $\wedge$ denote the supremum and infimum on $[0, \infty)$ respectively. The following properties of the Sugeno integral are well known and can be found in [28].

Proposition 1. *If μ is a fuzzy measure on $(\mathbb{R}, \Sigma)$, $A \subset \Sigma$ and f, $g \in \mathcal{F}^\mu(\mathbb{R})$, then*

(1) $\int_A f d\mu \leq \mu(A)$.
(2) $\int_A k\ d\mu = k\ \mu(A)$.
(3) *If* $f \leq g$ *on* A *then* $\int_A f d\mu \leq \int_A g d\mu$.
(4) $\mu\left(A \cap \{f \geq \alpha\}\right) \geq \alpha \Rightarrow \int_A f d\mu \geq \alpha$.
(5) $\mu\left(A \cap \{f \geq \alpha\}\right) \leq \alpha \Rightarrow \int_A f d\mu \leq \alpha$.
(6) $\int_A f d\mu < \alpha \iff$ *there exists* $\gamma < \alpha$ *such that* $\mu\left(A \cap \{f \geq \gamma\}\right) < \alpha$.
(7) $\int_A f d\mu > \alpha \iff$ *there exists* $\gamma > \alpha$ *such that* $\mu\left(A \cap \{f \geq \gamma\}\right) > \alpha$.

Remark 1. *Consider the distribution function F associated to f on A, that is, $F(\alpha) = \mu\left(A \cap \{f \geq \gamma\}\right)$. Then from* (4) *and* (5) *of Proposition 1*

$$F(\alpha) = \alpha \Rightarrow \int_A f d\mu = \alpha.$$

Therefore, it follows that any fuzzy integral can be calculated by solving the equation $F(\alpha) = \alpha$.

2. Main Results

Let $[0, b]$, where $b > 0$, be an interval of the real line $\mathbb{R}$. A function f is said to be convex on $[0, b]$ if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y),$$

holds for all $x, y \in [0, b]$ and $t \in [0, 1]$ and a function f is starshaped with respect to the origin on $[0, b]$ if

$$f(tx) \leq tf(x),$$

holds for all $x \in [0, b]$ and $t \in [0, 1]$.

In [26] G. Toader, (see also [1, 2, 4]) defined m-convexity as follows:

Definition 1. *The function $f:[0,b]\to\mathbb{R}$, $b>0$, is said to be m-convex, where $m\in[0,1]$, if one has*

$$f(tx+m(1-t)y)\leq tf(x)+m(1-t)f(y)$$

for all $x,y\in[0,b]$ and $t\in[0,1]$. We say that f is m-concave if $-f$ is m-convex.

The class of all m-convex functions on $[0,b]$ for which $f(0)\leq 0$ is denoted by $K_m(b)$. Obviously, for $m=1$, m-convexity is the standard convexity of functions on $[0,b]$, and for $m=0$ the concept of starshaped functions.

The following lemmas hold [1].

Lemma 1. *If f is in the class $K_m(b)$, then it is starshaped.*

Lemma 2. *If f is in the class $K_m(b)$ and $0<n<m\leq 1$, then f is in the class $K_n(b)$.*

From Lemmas 1 and 2 it follows that

$$K_1(b)\subset K_m(b)\subset K_0(b),$$

whenever $m\in(0,1)$. Note that in the class $K_1(b)$ there are only convex functions $f:[0,b]\to\mathbb{R}$ for which $f(0)\leq 0$, that is, $K_1(b)$ is a proper subclass of the class of convex functions on $[0,b]$. It is interesting to point out that for any $m\in(0,1)$ there are continuous and differentiable functions which are m-convex, but which are not convex in the standard sense [27].

The notion of m-convexity was further generalized in the following definition.

Definition 2. [11] *The function $f:[0,b]\to\mathbb{R}$, $b>0$, is said to be (α,m)-convex, where $(\alpha,m)\in[0,1]^2$, if*

$$f(tx+m(1-t)y)\leq t^{\alpha}f(x)+m(1-t^{\alpha})f(y),$$

for all $x,y\in[0,b]$ and $t\in[0,1]$.

We denote the class of all (α,m)-convex functions on $[0,b]$ for which $f(0)\leq 0$ by $K_m^{\alpha}(b)$.

If we take $(\alpha,m)=(1,m)$, it can be easily seen that (α,m)-convexity reduces to m-convexity and for $(\alpha,m)=(1,1)$, (α,m)-convexity reduces to the concept of usual convexity defined on $[0,b], b>0$.

For further results on inequalities related to m-convex and (α,m)-convex functions we refer the readers to [1, 2, 4].

In [4], S. S. Dragomir and G. Toader proved the following Hadamard type inequality for m-convex functions:

Theorem 1. *Let $f:[0,\infty)\to\mathbb{R}$ be an m-convex function with $m\in(0,1]$. If $0\leq a<b<\infty$ and $f\in L^1([a,b])$, then*

$$\frac{1}{b-a}\int_a^b f(x)dx\leq \min\left\{\frac{f(a)+mf\left(\frac{b}{m}\right)}{2},\frac{f(b)+mf\left(\frac{a}{m}\right)}{2}\right\} \tag{2.1}$$

We will see that this inequality does not valid for fuzzy integrals in general.

To prove our assertion we consider the function $f:[0,\infty)\to[0,\infty)$, $f(x)=ax^n$, $n\in\mathbb{N}$, $n\geq 2$, $a\geq 0$, then f is m-convex on $[0,\infty)$, $m\in(0,1]$.

Example 1. *Take $X=[0,1]$ and let μ be the usual Lebesgue measure on X. Let $f:[0,\infty)\to[0,\infty)$ be defined as $f(x)=\frac{x^2}{3}$ with $m=\frac{9}{10}$. Now to calculate the Sugeno integral $\int_0^1\frac{x^2}{3}d\mu$, consider the distribution function F associated to f on $[0,1]$, then*

$$F(\alpha)=\mu\left([0,1]\cap\{f\geq\alpha\}\right)=\mu\left([0,1]\cap\left\{\frac{x^2}{3}\geq\alpha\right\}\right)$$
$$=\mu\left([0,1]\cap\left\{x\geq\sqrt{3\alpha}\right\}\right)=1-\sqrt{3\alpha}$$

and we solve the equation $1-\sqrt{3\alpha}=\alpha$. It can be easily seen that the solution of this equation is $\frac{5}{2}-\frac{1}{2}\sqrt{21}$, therefore by Remark 1, we have

$$\int_0^1\frac{x^2}{3}d\mu=\frac{5}{2}-\frac{1}{2}\sqrt{21}\approx 0.208\,71.$$

Also

$$\frac{f(a)+mf\left(\frac{b}{m}\right)}{2}=\frac{5}{27}\approx 0.1851852 \quad \textit{and} \quad \frac{f(b)+mf\left(\frac{a}{m}\right)}{2}=\frac{1}{6}\approx 0.16666667.$$

Therefore,

$$\min\left\{\frac{f(a)+mf\left(\frac{b}{m}\right)}{2},\frac{f(b)+mf\left(\frac{a}{m}\right)}{2}\right\}=\frac{1}{6}\approx 0.16666667.$$

Which follows that (2.1) *is not satisfied in the fuzzy context.*

Now we prove Hdadamard type inequalities like (2.1) but for Sugeno integral (or fuzzy integral).

Theorem 2. *Let $g:[0,\infty)\to[0,\infty)$ be an m-convex function with $m\in(0,1]$ such that $mg(0)<g(1)$ and $g(0)<mg\left(\frac{1}{m}\right)$. Let μ be the Lebesgue measure on $[0,1]\subset[0,\infty)$, then*

$$\int_0^1 g\ d\mu\leq\min\left\{1,\frac{mg\left(\frac{1}{m}\right)}{1+mg\left(\frac{1}{m}\right)-g(0)},\frac{g(1)}{1+g(1)-mg(0)}\right\}. \tag{2.2}$$

Proof. Since g is an m-convex function. Therefore, for $x\in[0,1]$ and $m\in(0,1]$, we have

$$g(x)=g\left((1-x)\,0+1\cdot x\right)\leq(1-x)\,g(0)+mxg\left(\frac{1}{m}\right)=h(x),$$

and hence by (3) of Proposition 1,

$$\int_0^1 g\ d\mu\leq\int_0^1\left((1-x)\,g(0)+mxg\left(\frac{1}{m}\right)\right)d\mu=\int_0^1 h(x)\ d\mu.$$

Let F be the distribution function associated to h on $[0,1]$, then

$$F(\alpha) = \mu\left([0,1] \cap \{h \geq \alpha\}\right) = \mu\left([0,1] \cap \left\{(1-x)\,g(0) + mxg\left(\frac{1}{m}\right) \geq \alpha\right\}\right)$$
$$= \mu\left([0,1] \cap \left\{x \geq \frac{\alpha - g(0)}{mg\left(\frac{1}{m}\right) - g(0)}\right\}\right)$$
$$= 1 - \frac{\alpha - g(0)}{mg\left(\frac{1}{m}\right) - g(0)},$$

and as a solution of the equation $\alpha = 1 - \frac{\alpha - g(0)}{mg\left(\frac{1}{m}\right) - g(0)}$, we get

$$\alpha = \frac{mg\left(\frac{1}{m}\right)}{1 + mg\left(\frac{1}{m}\right) - g(0)}. \tag{2.3}$$

Analogously by the m-convexity of g, we also have

$$g(x) = g\left((1-x)\,0 + 1 \cdot x\right) \leq m\,(1-x)\,g(0) + xg(1) = h_1(x).$$

Arguing similarly, let F_1 be the distribution function associated to h_1 on $[0,1]$, then

$$\alpha = \frac{g\,(1)}{1 + g\,(1) - mg(0)}. \tag{2.4}$$

By (1) of Proposition 1, we have

$$\int_0^1 h(x)d\mu = \int_0^1 h_1(x)d\mu \leq \mu\left([0,1]\right) = 1. \tag{2.5}$$

Relations (2.3)-(2.5) and the definition of Sugeno integral give us the desired result. □

A similar result may be stated as follows, however, we leave the details for the interested readers.

Theorem 3. *Let $g : [0,\infty) \to [0,\infty)$ be an m-convex function with $m \in (0,1]$ such that $mg(0) > g(1)$ and $g(0) > mg\left(\frac{1}{m}\right)$. Let μ be the Lebesgue measure on $[0,1] \subset [0,\infty)$, then*

$$\int_0^1 g\ d\mu \leq \min\left\{1, \frac{g\,(0)}{1 - mg\left(\frac{1}{m}\right) + g(0)}, \frac{mg\,(0)}{1 - g\,(1) + mg(0)}\right\}. \tag{2.6}$$

Remark 2. *If $m = 1$, then the inequalities* (2.2) *and* (2.6) *become those inequalities proved in* [3, Theorems 1,2].

Now we give general cases of Theorems 2 and 3.

Theorem 4. *Let $f : [0,\infty) \to [0,\infty)$ be an m-convex function with $m \in (0,1]$ such that $mg(\frac{a}{m}) < g(b)$ and $g(a) < mg\left(\frac{b}{m}\right)$. Let μ be the Lebesgue measure on $[a,b]$ with $0 \leq a < b < \infty$, then*

$$\int_a^b g\ d\mu \leq \min\left\{(b-a), \frac{mg\left(\frac{b}{m}\right)(b-a)}{b - a + mg\left(\frac{b}{m}\right) - g(a)}, \frac{(b-a)\,g\,(b)}{b - a + g(b) - mg\left(\frac{a}{m}\right)}\right\}. \tag{2.7}$$

Proof. Since g is an m-convex function. Therefore, for $x \in [a,b]$, $0 \leq a < b < \infty$, we have

$$\begin{aligned} g(x) &= g\left(\left(1-\frac{x-a}{b-a}\right)a+\frac{x-a}{b-a}\cdot b\right) \\ &\leq \left(\frac{b-x}{b-a}\right)g(a)+m\left(\frac{x-a}{b-a}\right)g\left(\frac{b}{m}\right)=h(x) \end{aligned}$$

By (3) of Proposition 1, we have

$$\int_a^b g d\mu \leq \int_a^b \left[\left(\frac{b-x}{b-a}\right)g(a)+m\left(\frac{x-a}{b-a}\right)g\left(\frac{b}{m}\right)\right]d\mu = \int_a^b h(x)d\mu.$$

Let us consider the distribution function F given by

$$\begin{aligned} F(\alpha) &= \mu\left([a,b]\cap\{h\geq\alpha\}\right) \\ &= \mu\left([a,b]\cap\left\{\left(\frac{b-x}{b-a}\right)g(a)+m\left(\frac{x-a}{b-a}\right)g\left(\frac{b}{m}\right)\geq\alpha\right\}\right) \\ &= \mu\left([a,b]\cap\left\{x\geq\frac{\alpha(b-a)+mag\left(\frac{b}{m}\right)-bg(a)}{mg\left(\frac{b}{m}\right)-g(a)}\right\}\right) \\ &= b-\frac{\alpha(b-a)+mag\left(\frac{b}{m}\right)-bg(a)}{mg\left(\frac{b}{m}\right)-g(a)}, \end{aligned}$$

and as solution of the equation $b-\frac{\alpha(b-a)+mag\left(\frac{b}{m}\right)-bg(a)}{mg\left(\frac{b}{m}\right)-g(a)}=\alpha$, we get

$$\alpha=\frac{mg\left(\frac{b}{m}\right)(b-a)}{b-a+mg\left(\frac{b}{m}\right)-g(a)}. \tag{2.8}$$

Analogously by the m-convexity of g, we also have

$$\begin{aligned} g(x) &= g\left(\left(1-\frac{x-a}{b-a}\right)a+\frac{x-a}{b-a}\cdot b\right) \\ &\leq m\left(\frac{b-x}{b-a}\right)g\left(\frac{a}{m}\right)+\left(\frac{x-a}{b-a}\right)g(b)=h_1(x). \end{aligned}$$

Arguing similarly, let F_1 be the distribution function associated to h_1 on $[a,b]$, then

$$\alpha=\frac{(b-a)g(b)}{b-a+g(b)-mg\left(\frac{a}{m}\right)}. \tag{2.9}$$

Moreover by (1) of Proposition 1, we have

$$\int_a^b h(x)\,d\mu=\int_a^b h_1(x)\,d\mu\leq\mu\left([a,b]\right)=b-a. \tag{2.10}$$

From relations (2.8)-(2.10) and by the definition of fuzzy integral, we obtain (2.7). This completes the proof. □

Again, we state similar results like Theorem 4, however, the details are left to the interested readers.

Theorem 5. *Let $f : [0, \infty) \to [0, \infty)$ be an m-convex function with $m \in (0, 1]$ such that $mg(\frac{a}{m}) > g(b)$ and $g(a) > mg\left(\frac{b}{m}\right)$. Let μ be the Lebesgue measure on $[a, b]$ and $0 \le a < b < \infty$, then*

(2.11)

$$\int_a^b g \; d\mu \le \min\left\{(b-a), \frac{(b-a)\, g\,(a)}{b-a+g(a)-mg\left(\frac{b}{m}\right)}, \frac{m\,(b-a)\, g\left(\frac{a}{m}\right)}{b-a+mg\left(\frac{a}{m}\right)-g(b)}\right\}.$$

Remark 3. *If $m = 1$, then the inequalities* (2.7) *and* (2.11) *become those inequalities proved in* [3, Theorem 3].

Example 2. *Take $X = [0, 1]$ and let μ be the usual Lebesgue measure on X. Let $g : [0, \infty) \to [0, \infty)$ be defined as $g(x) = x^2$. Then g is an m-convex function on $[0, 1]$ with $mg(0) < g(1)$ and $g(0) < mg\left(\frac{1}{m}\right)$ for $m \in (0, 1]$. Now*

$$\frac{mg\left(\frac{1}{m}\right)}{1+mg\left(\frac{1}{m}\right)-g(0)} = \frac{1}{m+1} \quad \text{and} \quad \frac{g\,(1)}{1+g\,(1)-mg(0)} = \frac{1}{2}.$$

Therefore by Theorem 2, we have

$$\int_0^1 x^2 d\mu \le \frac{1}{2}.$$

Now we give our results for (α, m)-convex functions.

Theorem 6. *Let $g : [0, \infty) \to [0, \infty)$ be an (α, m)-convex function with $\alpha,\, m \in (0, 1]^2$ such that $mg(0) < g(1)$ and $g(0) < mg\left(\frac{1}{m}\right)$. Let μ be the Lebesgue measure on $[0, 1]$, then*

$$\int_0^1 g \; d\mu \le \min\left\{1, \alpha_1, \alpha_2\right\}, \tag{2.12}$$

where α_1 and α_2 are the respective positive real solutions of the equations

$$\alpha' = 1 - \left(\frac{\alpha' - g(0)}{mg\left(\frac{1}{m}\right) - g(0)}\right)^{\frac{1}{\alpha}} \quad \text{and} \quad \alpha' = 1 - \left(\frac{\alpha' - g(0)}{g\,(1) - mg(0)}\right)^{\frac{1}{\alpha}}.$$

Proof. Since g is an (α, m)-convex function. Therefore, for $x \in [a, b]$ and $\alpha,\, m \in (0, 1]^2$, we have

$$g(x) = g\left((1-x)\,0 + 1 \cdot x\right) \le (1 - x^\alpha)\, g(0) + mx^\alpha g\left(\frac{1}{m}\right) = h(x).$$

And hence by (3) of Proposition 1,

$$\int_0^1 g \; d\mu \le \int_0^1 \left((1-x^\alpha)\, g(0) + mx^\alpha g\left(\frac{1}{m}\right)\right) d\mu = \int_0^1 h(x) d\mu.$$

Let F be the distribution function associated to h on $[0,1]$, then

$$\begin{aligned} F(\alpha') &= \mu\left([0,1]\cap\left\{h\geq\alpha'\right\}\right) = \mu\left([0,1]\cap\left\{(1-x^\alpha)\,g(0)+mx^\alpha g\left(\frac{1}{m}\right)\geq\alpha'\right\}\right) \\ &= \mu\left([0,1]\cap\left\{x\geq\left(\frac{\alpha'-g(0)}{mg\left(\frac{1}{m}\right)-g(0)}\right)^{\frac{1}{\alpha}}\right\}\right) \\ &= 1-\left(\frac{\alpha'-g(0)}{mg\left(\frac{1}{m}\right)-g(0)}\right)^{\frac{1}{\alpha}}, \end{aligned}$$

and hence we get the equation

$$\alpha' = 1-\left(\frac{\alpha'-g(0)}{mg\left(\frac{1}{m}\right)-g(0)}\right)^{\frac{1}{\alpha}}. \tag{2.13}$$

Analogously by the (α,m)-convexity of g, we also have

$$g(x) = g\left((1-x)\,0+1\cdot x\right) \leq m\left(1-x^\alpha\right)g(0)+x^\alpha g(1) = h_1(x).$$

Arguing similarly, let F_1 be the distribution function associated to h_1 on $[0,1]$, then we have the following equation:

$$\alpha' = 1-\left(\frac{\alpha'-g(0)}{g\left(1\right)-mg(0)}\right)^{\frac{1}{\alpha}}. \tag{2.14}$$

By (1) of Proposition 1, we have

$$\int_a^b h(x)d\mu = \int_a^b h_1(x)d\mu \leq \mu\left([a,b]\right) = b-a. \tag{2.15}$$

Relations (2.13)-(2.15) and the definition of Sugeno integral give us the required result. □

A similar result can be stated as follow, however, the details are left to the interested readers:

Theorem 7. *Let $g:[0,\infty)\to[0,\infty)$ be an (α,m)-convex function with α, $m\in(0,1]^2$ such that $mg(0)>g(1)$ and $g(0)>mg\left(\frac{1}{m}\right)$. Let μ be the Lebesgue measure on $[0,1]\subset[0,\infty)$, then*

$$\int_0^1 g\ d\mu \leq \min\left\{1,\alpha_1,\alpha_2\right\}, \tag{2.16}$$

where α_1 and α_2 are the respective positive real solutions of the equations $\alpha' = \left(\frac{g(0)-\alpha'}{g(0)-mg\left(\frac{1}{m}\right)}\right)^{\frac{1}{\alpha}}$ and $\alpha' = \left(\frac{g(0)-\alpha'}{mg(0)-g(1)}\right)^{\frac{1}{\alpha}}$.

Remark 4. *If $(\alpha,m)=(1,1)$, then the inequalities* (2.12) *and* (2.16) *become those inequalities proved in* [3, Theorems 1,2].

The following results are the general cases of the last two results.

Theorem 8. *Let $g : [0,\infty) \to [0,\infty)$ be an (α, m)-convex function with $\alpha,\ m \in (0,1]^2$ such that $mg(\frac{a}{m}) < g(b)$ and $g(a) < mg\left(\frac{b}{m}\right)$. Let μ be the Lebesgue measure on $[a,b]$, $0 \le a < b < \infty$, then*

$$\int_a^b g\ d\mu \le \min\left\{(b-a), \alpha_1, \alpha_2\right\}, \tag{2.17}$$

where α_1 and α_2 are the respective positive real solutions of the equations $\alpha^{'} = (b-a)\left[1-\left(\frac{\alpha^{'}-g(a)}{mg\left(\frac{b}{m}\right)-g(a)}\right)^{\frac{1}{\alpha}}\right]$ and $\alpha^{'} = (b-a)\left[1-\left(\frac{\alpha^{'}-g\left(\frac{a}{m}\right)}{g(b)-mg\left(\frac{a}{m}\right)}\right)^{\frac{1}{\alpha}}\right]$.

Proof. Since g is an (α, m)-convex function. Therefore, for $x \in [0,1]$ and $\alpha,\ m \in (0,1]^2$, we have

$$\begin{aligned} g(x) &= g\left(\left(1-\frac{x-a}{b-a}\right)a + \frac{x-a}{b-a}\cdot b\right) \\ &\le \left(1-\left(\frac{x-a}{b-a}\right)^{\alpha}\right)g(a) + m\left(\frac{x-a}{b-a}\right)^{\alpha} g\left(\frac{b}{m}\right) = h(x), \end{aligned}$$

and hence by (3) of Proposition 1,

$$\int_a^b g\ d\mu \le \int_a^b \left[\left(1-\left(\frac{x-a}{b-a}\right)^{\alpha}\right)g(a) + m\left(\frac{x-a}{b-a}\right)^{\alpha} g\left(\frac{b}{m}\right)\right] d\mu = \int_a^b h(x)\ d\mu.$$

Let F be the distribution function associated to h on $[a,b]$, then

$$\begin{aligned} F(\alpha^{'}) &= \mu\left([a,b] \cap \left\{h \ge \alpha^{'}\right\}\right) \\ &= \mu\left([a,b] \cap \left\{\left(1-\left(\frac{x-a}{b-a}\right)^{\alpha}\right)g(a) + m\left(\frac{x-a}{b-a}\right)^{\alpha} g\left(\frac{b}{m}\right) \ge \alpha^{'}\right\}\right) \\ &= \mu\left([a,b] \cap \left\{x \ge a + (b-a)\left(\frac{\alpha^{'}-g(a)}{mg\left(\frac{b}{m}\right)-g(a)}\right)^{\frac{1}{\alpha}}\right\}\right) \\ &= (b-a)\left[1-\left(\frac{\alpha^{'}-g(a)}{mg\left(\frac{b}{m}\right)-g(a)}\right)^{\frac{1}{\alpha},}\right], \end{aligned}$$

and hence we get the equation

$$\alpha^{'} = (b-a)\left[1-\left(\frac{\alpha^{'}-g(a)}{mg\left(\frac{b}{m}\right)-g(a)}\right)^{\frac{1}{\alpha}}\right]. \tag{2.18}$$

Analogously by the (α, m)-convexity of g, we also have

$$\begin{aligned} g(x) &= g\left(\left(1-\frac{x-a}{b-a}\right)a + \frac{x-a}{b-a}\cdot b\right) \\ &\le m\left(1-\left(\frac{x-a}{b-a}\right)^{\alpha}\right)g\left(\frac{a}{m}\right) + \left(\frac{x-a}{b-a}\right)^{\alpha} g(b) = h_1(x). \end{aligned}$$

Arguing similarly, let F_1 be the distribution function associated to h_1 on $[a,b]$. Then we have the following equation:

$$\alpha^{'}=(b-a)\left[1-\left(\frac{\alpha^{'}-g\left(\frac{a}{m}\right)}{g\left(b\right)-mg\left(\frac{a}{m}\right)}\right)^{\frac{1}{\alpha}}\right]. \tag{2.19}$$

By (1) of Proposition 1, we have

$$\int_a^b h(x)\ d\mu=\int_a^b h_1(x)\ d\mu\leq\mu\left([a,b]\right)=b-a. \tag{2.20}$$

Relations (2.18)-(2.20) and the definition of Sugeno integral give us the required result. □

A similar result is stated below, however, the details are left:

Theorem 9. *Let $g:[0,\infty)\to[0,\infty)$ be an (α,m)-convex function with $\alpha,\ m\in(0,1]^2$ such that $mg(\frac{a}{m})>g(b)$ and $g(a)>mg\left(\frac{b}{m}\right)$. Let μ be the Lebesgue measure on $[a,b]$, $0\leq a<b<\infty$, then*

$$\int_a^b g\ d\mu\leq\min\left\{(b-a),\alpha_1,\alpha_2\right\}, \tag{2.21}$$

where α_1 and α_2 are the respective positive real solutions of the equations

$$\alpha^{'}=(b-a)\left[\left(\frac{g(a)-\alpha^{'}}{g(a)-mg\left(\frac{b}{m}\right)}\right)^{\frac{1}{\alpha}}\right]\quad\text{and}\quad\alpha^{'}=(b-a)\left[\left(\frac{g\left(\frac{a}{m}\right)-\alpha^{'}}{mg\left(\frac{a}{m}\right)-g\left(b\right)}\right)^{\frac{1}{\alpha}}\right].$$

Remark 5. *If $(\alpha,m)=(1,1)$, then the inequalities* (2.17) *and* (2.21) *become those inequalities proved in* [3, Theorem 3].

References

[1] M. K. Bakula, M. E. Özdemir and J. Pečarić,*Hadamard type inequalities for m-convex and (α,m)-convex functions, J. Inequal. Pure & Appl. Math.*, **9** (2008), Article 96. [ONLINE:htpp://jipam.vu.edu.au]

[2] M. K Bakula, J. Pečarić, and M. Ribičić, *Companion inequalities to Jensen's inequality for m-convex and (α,m)-convex functions convex functions, J. Inequal. Pure & Appl. Math.*, **7** (2006), Article194. [ONLINE:htpp://jipam.vu.edu.au]

[3] J. Caballero, K. Sadarangani, *Hermite-Hadamard inequality for fuzzy integrals, Appl. Math. & Comp.* **215** (2009) 2134-2138.

[4] S. S. Dragomir and G. Toader, *Some inequalities for m-convex functions, Studia Univ. Babeş-Bolyai Math.*, **38**(1) (1993), 21–28.

[5] A. Flores-Franulič, H. Román-Flores,*A Chebyshev type inequality for fuzzy integrals, Appl. Math. Comput.* **190** (2007) 1178-1184.

[6] J. Hadamard,*Étude sur les propietés des functions entiéres et en particulies d'une function considerée par Riemann, J. Math. Pure. Appl.* **58** (1893) 175-215.

[7] Ch. Hermite, Sur deux limites d'une intégrale définie, *Mathesis* **3** (1883) 82.

[8] G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities*, second ed., Cambridge University Press, Cambridge, 1952.

[9] M. Mastić , J. Pečarić , *On inequalities of Hadamard's type for lipschtzian mappings, Tamkang J. Math.* **32** (2001) 127-130.

[10] R. Mesiar, Y. Ouyang,*general Chebyshev type inequalities for Sugeno integrals, Fuzzy Setsand Systems* **160** (2009) 58–64.

[11] V. G. Miheşan, A generalization of the convexity, Seminar on Functional Equations, *Approx. and Convex.*, Cluj-Napoca (Romania) (1993).

[12] Y. Narukawa, V. Torra, *Fuzzy measures and integrals in evaluation of strategies, Information Sciences* **177** (2007) 4686–4695.
[13] Y. Ouyang, J. Fang, *Sugeno integral of monotone functions based on Lebesgue measure, Computers and Mathematics with Applications* **56** (2008) 367–374.
[14] Y. Ouyang, J. Fang, L. Wang, *Fuzzy Chebyshev type inequality, International Journal of Approximate Reasoning* **48** (2008) 829–835.
[15] D. Ralescu, G. Adams, *The fuzzy integral, J. Math. Anal. Appl.* **75** (1980) 562-570.
[16] H. Román-Flores, A. Flores-Franulič, R. Bassanezi, M. Rojas-Medar,*On the level-continuity of fuzzy integrals, Fuzzy Set. Syst.* **80** (1996) 339-344.
[17] H. Román-Flores, Y. Chalco-Cano,*H-Continuity of fuzzy measures and set defuzzification, Fuzzy Set. Syst.* **157** (2006) 230–242.
[18] H. Román-Flores, A- Flores- Franulič, Y. Chalco-Cano, *The fuzzy integral for monotone functions, Appl. Math. Comput.* **185** (2007) 492-498.
[19] H. Román-Flores, A. Flores-Franulič, Y. Chalco-Cano,*A convolution type inequality for fuzzy integrals, Appl. Math. Comput.* **195** (2008) 94-99.
[20] H. Román-Flores, Y.Chalco-Cano, *Sugeno integral and geometric inequalities, International Journal of Uncertainty, Fuzziness and Knowledge- Based Systems* **15** (2007)1–11.
[21] H. Román-Flores, A. Flores-Franulič, Y. Chalco-Cano, *A Jensen type inequality for fuzzy integrals, Information Sciences* **177** (2007) 3192–3201.
[22] M. Sugeno, *Theory of fuzzy integrals and its applications*, Ph.D. Thesis, Tokyo Institute of Technology, 1974.
[23] E. Pap, *Null-additive Set Functions, Kluwer*, Dordrecht, 1995.
[24] J. Tabor, J. Tabor,*Characterization of convex functions, Stud. Math.* **192** (2009) 29-37.
[25] G. H. Toader, *Some generalisations of the convexity, Proc. Colloq. Approx. Optim*, Cluj-Napoca(Romania), 1984, 329-338.
[26] G. Toader,*On a generalization of the convexity, Mathematica*, **30** (53) (1988), 83-87.
[27] S. Toader, *The order of a star-convex function, Bull. Applied & Comp. Math.*, **85-B** (1998), BAM-1473, 347–350.
[28] Z. Wang, G. Klir, *Fuzzy Measure Theory*, Plenum, New York, 1992.
[29] M. Ying,*Linguistic quantifiers modeled by Sugeno integrals, Artificial Intelligence* **170** (2006) 581–606.

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