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HERMITE HADAMARD INEQUALITY FOR s-CONVEX FUNCTIONS IN THE SECOND SENSE FOR FUZZY INTEGRALS

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ABSTRACT. In this paper we prove a Hermite–Hadamard type inequality for s-convex functions in the second sense for fuzzy integrals. Some examples are given to illustrate the results.

1. INTRODUCTION

The theory of fuzzy measures and fuzzy integral was initiated by Sugeno [13] as a tool for modeling non-deterministic problems. The theory of fuzzy integrals attracted the attention of many mathematicians and therefor the properties and applications of the Sugeno integral have been studied by many authors. Ralescu and Adams [8] gave several equivalent definitions of fuzzy integrals, Román-Flores et al. [9, 10] studied the level-continuity of fuzzy integrals and H-continuity of fuzzy measures and Wang and Klir [15] has given a general overview on fuzzy measurement and fuzzy integration theory. Recently, Román-Flores et al. [11, 12] and Flores-Franulič et al. [3] presented some fuzzy integral inequalities. In this paper we prove a Hermite-Hadamard type inequality for the Sugeno integral for functions which are s-convex in the second sense. In order to proceed to our results we first give some basic notation and properties about Sugeno integral. For details on Sugeno integral we refers the reader to [13, 15].

Suppose that Σ is σ-algebra of subsets of $\mathbb{R}$ and that $\mu : \Sigma \to [0, \infty)$, is non-negative, extended real valued set function, then μ is said to be fuzzy measure if and only if:

(1) $\mu(\emptyset) = 0$,
(2) E, $F \in \Sigma$ and $E \subset F$ imply that $\mu(E) \leq \mu(F)$ (monotonicity),
(3) $\{E_n\} \subset \Sigma$, $E_1 \subset E_2 \subset ...$, imply $\lim\limits_{n\to\infty} \mu(E_n) = \mu(\bigcup\limits_{n=1}^{\infty} E_n)$ (continuity from below) and
(4) $\{E_n\} \subset \Sigma$, $E_1 \supset E_2 \supset ...$, $\mu(E_1) < \infty$, imply $\lim\limits_{n\to\infty} \mu(E_n) = \mu(\bigcap\limits_{n=1}^{\infty} E_n)$ (continuity from above).

If f is a non-negative real-valued function defined on $\mathbb{R}$, we will denote by $L_\alpha f = \{x \in \mathbb{R} : f(x) \geq \alpha\} = \{f \geq \alpha\}$ the α-level of f, for $\alpha > 0$ and $L_0 f = \{x \in \mathbb{R} : f(x) \geq 0\} = supp f$ is the support of f. Observe that if $\alpha \leq \beta$, then $\{f \leq \alpha\} \subset \{f \leq \beta\}$. If μ is fuzzy measure on $(\mathbb{R}, \Sigma)$, by $\mathcal{F}^\mu(\mathbb{R})$, we denote all μ-measurable functions from $\mathbb{R}$ to $[0, \infty)$.

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Suppose that μ is a fuzzy measure on $(\mathbb{R},\Sigma)$. If $f\in\mathcal{F}^{\mu}(\mathbb{R})$ and $A\subset\Sigma$ then the Sugeno integral (or fuzzy integral) of f on A with respect to the fuzzy measure μ is defined as

$$\fint_A f d\mu=\bigvee_{\alpha\geq 0}\left[\alpha\wedge\mu\left(A\cap\{f\geq\alpha\}\right)\right],$$

where $\vee$ and $\wedge$ denote the operations sup and inf on $[0,\infty)$, respectively. The following properties of the Sugeno integral are well known and can be found in [15].

Proposition 1. *μ is a fuzzy measure on $(\mathbb{R},\Sigma)$, $A\subset\Sigma$ and f, $g\in\mathcal{F}^{\mu}(\mathbb{R})$ then*

(1) $\fint_A f d\mu\leq\mu(A)$.
(2) $\fint_A k d\mu=k\wedge\mu(A)$.
(3) *If* $f\leq g$ *on* A *then* $\fint_A f d\mu\leq\fint_A g d\mu$.
(4) $\mu\left(A\cap\{f\geq\alpha\}\right)\geq\alpha\Rightarrow\fint_A f d\mu\geq\alpha$.
(5) $\mu\left(A\cap\{f\geq\alpha\}\right)\leq\alpha\Rightarrow\fint_A f d\mu\leq\alpha$.
(6) $\fint_A f d\mu<\alpha\Longleftrightarrow$ *there exists* $\gamma<\alpha$ *such that* $\mu\left(A\cap\{f\geq\gamma\}\right)<\alpha$.
(7) $\fint_A f d\mu>\alpha\Longleftrightarrow$ *there exists* $\gamma>\alpha$ *such that* $\mu\left(A\cap\{f\geq\gamma\}\right)>\alpha$.

Remark 1. *Consider the distribution function F associated to f on A, that is, $F(\alpha)=\mu\left(A\cap\{f\geq\gamma\}\right)$. Then from (4) and (5) of Proposition 1, we have that*

$$F(\alpha)=\alpha\Rightarrow\fint_A f d\mu=\alpha.$$

Therefore it follows that any fuzzy integral can be calculated by solving the equation $F(\alpha)=\alpha$.

2. s-Hermite-Hadamard Inequality for Fuzzy Integrals

In [1], J. Caballero, K. Sadarangani has proven with the help of certain examples that the classical Hermite–Hadamard inequalities (see [4, 5] for the history of these inequalities):

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2},$$

where $f:[a,b]\to\mathbb{R}$ is a convex function, do not hold true for fuzzy integrals in general . In [1], the authors proved some Hermite-Hadamard type inequalities for fuzzy integrals and some examples were also given to illustrate their results .

In this section we aim to prove some Hermite-Hadamard type inequalities for functions which are s-convex in the second sense for fuzzy integrals.

It is known that a function $f:[0,\infty)\to\mathbb{R}$ is said to be s-convex in the second sense, or that f belongs to the class K_s^2, if

$$f(\alpha x+(1-\alpha)y)\leq\alpha^s f(x)+(1-\alpha)^s f(y),$$

holds for all x, $y\in[0,\infty)$, $\alpha\in[0,1]$, for some fixed $s\in(0,1]$.

The inequalities:

$$2^{s-1}f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{s+1},$$

where $f:[0,\infty)\to[0,\infty)$ is an s-convex function in the second sense $s\in(0,1]$ are known as s-Hermite-Hadamard inequalities. For more about the properties on s-convex functions and s-Hermite-Hadamard inequalities we refer the interested readers to [2]. Unfortunately, as we will see, the s-Hermite-Hadamard inequalities do not valid as well for fuzzy integrals in general.

We quote below a very important example of s-convex functions from [2] to be used in the sequel:

Let $s \in (0,1)$ and $a, b, c \in \mathbb{R}$. We define a function $f : [0,\infty) \to \mathbb{R}$, by

$$f(t) = \begin{cases} a & t = 0, \\ bt^s + c & t > 0. \end{cases}$$

If $b \geq 0$ and $0 \leq c \leq a$, then $f \in K_s^2$. Hence, for $a = c = 0$, $b = 1$, we have $f : [0,1] \to [0,1]$, $f(t) = t^s$, $0 < s < 1$, $f \in K_s^2$ (see e.g.[2]).

Example 1. *Consider $X = [0,1]$ and let μ be the usual Lebesgue measure on X. If we take the function $f(x) = \sqrt{x}$, $x \in [0,1]$, then $f \in K_s^2$. Now we calculate the the Sugeno integral $⨍_0^1 \sqrt{x} d\mu$ by using Remark 1. Consider the distribution function F associated to f on $[0,1]$, that is*

$$F(\alpha) = \mu\left([0,1] \cap \{f \geq \alpha\}\right) = \mu\left([0,1] \cap \left\{\sqrt{x} \geq \alpha\right\}\right) = \mu\left([0,1] \cap \left\{x \geq \alpha^2\right\}\right) = 1 - \alpha^2$$

and we solve the equation $1 - \alpha^2 = \alpha$, we obtain that $\alpha = \frac{-1+\sqrt{5}}{2}$. By Remark 1 we have

$$⨍_0^1 \sqrt{x} d\mu = \frac{-1+\sqrt{5}}{2} \approx 0.618033$$

on the other hand $\sqrt{2} f\left(\frac{1}{2}\right) = \sqrt{2} \times \frac{1}{\sqrt{2}} = 1$. This shows that the left part of the s-Hermite-Hadamard inequality is not valid in the fuzzy context.

Example 2. *Consider $X = [0,1]$ and let μ be the usual Lebesgue measure on X. Then if we take the function $f(x) = \frac{\sqrt{x}}{3}$, then $f \in K_s^2$. Now again we calculate the the Sugeno integral $⨍_0^1 \frac{\sqrt{x}}{3} d\mu$ by using Remark 1. Consider the distribution function F associated to f on $[0,1]$, that is*

$$F(\alpha) = \mu\left([0,1] \cap \{f \geq \alpha\}\right) = \mu\left([0,1] \cap \left\{\frac{\sqrt{x}}{3} \geq \alpha\right\}\right) = \mu\left([0,1] \cap \left\{x \geq 9\alpha^2\right\}\right)$$
$$= 1 - 9\alpha^2$$

and we solve the equation $1 - 9\alpha^2 = \alpha$, we obtain that $\alpha = \frac{-1+\sqrt{37}}{18}$. By Remark 1 we have

$$⨍_0^1 \frac{\sqrt{x}}{2} d\mu = \frac{-1+\sqrt{37}}{18} \approx 0.2823756,$$

but $\frac{2[f(1)+f(0)]}{3} = \frac{2}{9} \approx 0.22222222$. Thus the right side of the s-Hermite-Hadamard inequality is not satisfied for fuzzy integrals.

Now we present s-Hermite-Hadamard inequalities for for the Sugeno integral.

Lemma 1. *Let $0 < p \leq 1$ and $x \geq 0$, $y \geq 0$. Then*

$$(x+y)^p \leq x^p + y^p.$$

Proof. When $p = 1$, then the result is obviously true. So assume that $0 < p < 1$ and consider the function

$$f(t) = 1 + t^p - (1+t)^p, \; t \geq 0.$$

Then

$$f^{'}(t) = pt^{p-1} - p(1+t)^{p-1},\ t \geq 0.$$

Since $p-1<0$, hence $f^{'}(t) \geq 0$, $t \geq 0$. Thus

$$f(t) \geq 0,\ t \geq 0.$$

That is

$$(1+t)^p \leq t^p + 1,\ t \geq 0. \tag{2.1}$$

If $y=0$ then $(x+y)^p \leq x^p + y^p$ is true with equality sign, so let $y>0$ and take $t = \frac{x}{y}$ in (2.1), we have

$$(1+y)^p \leq \left(\frac{x}{y}\right)^p + 1, y>0$$

so that

$$(x+y)^p \leq x^p + y^p.$$

This completes the proof. □

Theorem 1. *$g : [0,1] \to (0,\infty]$ be an s-convex function in the second sense $s \in (0,1)$ and μ the Lebesgue measure on $\mathbb{R}$. Then*

$$⨍_0^1 g d\mu \leq \min\{1, \alpha_1\},$$

where α_1 is a positive real solution of the equation $1 - \left(\frac{\alpha - g(0)}{g(0)+g(1)}\right)^{\frac{1}{s}} = \alpha$.

Proof. Since g is an s-convex function in the second sense $s \in (0,1)$, therefore for $x \in [0,1]$ we have

$$g(x) = g\left((1-x)\,0 + 1 \cdot x\right) \leq (1-x)^s\, g(0) + x^s g(1) = h(x)$$

and hence by (3) of Proposition 1, we have that

$$⨍_0^1 g d\mu \leq ⨍_0^1 \left((1-x)\, g(0) + x g(1)\right) d\mu = ⨍_0^1 h(x) d\mu$$

In order to calculate the integral in the right-hand part of the last inequality, we consider the distribution function F given by

$$F(\alpha) = \mu\left([0,1] \cap \{h \geq \alpha\}\right) = \mu\left([0,1] \cap \{(1-x)^s\, g(0) + x^s g(1) \geq \alpha\}\right). \tag{2.2}$$

We observe that $1-x \geq 0$, therefore we have that

$$1 - x = |1-x| \leq 1 + x.$$

and hence by Lemma1, we have

$$(1-x)^s \leq (1+x)^s \leq 1 + x^s.$$

Thus from (2.2) we get that

$$\begin{aligned} F(\alpha) &= \mu\left([0,1] \cap \{h \geq \alpha\}\right) \\ &= \mu\left([0,1] \cap \left\{x \geq \left(\frac{\alpha - g(0)}{g(0)+g(1)}\right)^{\frac{1}{s}}\right\}\right) \\ &= 1 - \left(\frac{\alpha - g(0)}{g(0)+g(1)}\right)^{\frac{1}{s}}. \end{aligned} \tag{2.3}$$

Therefore we have

$$1-\left(\frac{\alpha-g(0)}{g(0)+g(1)}\right)^{\frac{1}{s}}=\alpha \tag{2.4}$$

From (1) of Proposition 1 we get that

$$\fint_0^1 h(x)d\mu \leq \mu\left([0,1]\right)=1 \tag{2.5}$$

By Remark 1 and (2.4), we have

$$\fint_0^1 gd\mu \leq \min\left\{1,\alpha_1\right\},$$

where α_1 is a positive real solution of (2.4). □

Corollary 1. *$g:[0,1]\to(0,\infty]$ be an s-convex function in the second sense $s\in(0,1]$ and μ the Lebesgue measure on $\mathbb{R}$. If $g(0)=g(1)$, then*

$$\fint_0^1 gd\mu \leq \min\left\{1,\alpha_1\right\},$$

where α_1 is a positive solution of the equation $1-\left(\frac{\alpha-g(0)}{2g(0)}\right)^{\frac{1}{s}}=\alpha$.

Proof. It is direct consequence of the above theorem. □

Now we give example to illustrate our result:

Example 3. *Consider the function $f(x)=x^{\frac{1}{3}}$ on $[0,1]$, then this function is an s-convex function with $s=\frac{1}{3}$. Moreover $f(0)=0$ and $f(1)=1$, thus we have from the equation $1-\left(\frac{\alpha-f(0)}{f(0)+f(1)}\right)^{3}=\alpha$, which gives by solving by numerical methods, the positive real solution $\alpha_1=\sqrt[3]{\frac{1}{108}\sqrt{31}\sqrt{108}+\frac{1}{2}}-\frac{1}{3\sqrt[3]{\frac{1}{108}\sqrt{31}\sqrt{108}+\frac{1}{2}}}\approx 0.682\,33$. Therefore from Theorem 1 we have*

$$\fint_0^1 fd\mu \leq \sqrt[3]{\frac{1}{108}\sqrt{31}\sqrt{108}+\frac{1}{2}}-\frac{1}{3\sqrt[3]{\frac{1}{108}\sqrt{31}\sqrt{108}+\frac{1}{2}}}.$$

Example 4. *Consider the function $f(x)=x^{\frac{3}{4}}$ on $[0,1]$, then this function is an s-convex function with $s=\frac{3}{4}$. Moreover $f(0)=0$ and $f(1)=1$, thus we have from the equation $1-\left(\frac{\alpha-f(0)}{f(0)+f(1)}\right)^{\frac{4}{3}}=\alpha$, which gives by solving by numerical methods, the positive real solution $\alpha_1\approx 0.549\,7$. Therefore from Theorem 1 we have*

$$\fint_0^1 fd\mu \leq 0.549\,7.$$

Now we prove general case of Theorem 1 as follow:

Theorem 2. (1) *$g:[a,b]\to(0,\infty]$ be an s-convex function in the second sense $s\in(0,1)$, $0\leq a\leq b$ and μ the Lebesgue measure on $\mathbb{R}$. If $x-a\geq 0$, then*

$$\fint_a^b gd\mu \leq \min\left\{b-a,\alpha_1\right\},$$

where α_1 *is a positive solution of the equation* $(b-a)\left[1-\left(\frac{\alpha-g(a)}{g(a)+g(b)}\right)^{\frac{1}{s}}\right]=\alpha$.

(2) *If* $x-a<0$, $\frac{1}{s}$ *is an odd integer and* $g(b)>g(a)$, *then*

$$\fint_a^b g d\mu \le \min\{b-a,\alpha_1\},$$

where α_1 *is a positive solution of the equation* $(b-a)\left[1-\left(\frac{\alpha-g(a)}{g(b)-g(a)}\right)^{\frac{1}{s}}\right]=\alpha$.

(3) *If* $x-a<0$, $\frac{1}{s}$ *is an odd integer and* $g(b)<g(a)$, *then*

$$\fint_a^b g d\mu \le \min\{b-a,\alpha_1\},$$

where α_1 *is a positive solution of the equation* $(a-b)\left(\frac{\alpha-g(a)}{g(b)-g(a)}\right)^{\frac{1}{s}}=\alpha$

Proof. Since g is an s-convex function in the second sense $s\in(0,1)$, therefore for $x\in[0,1]$ we have

$$g(x)=g\left(\left(1-\frac{x-a}{b-a}\right)a+\frac{x-a}{b-a}\cdot b\right)\le\left(1-\frac{x-a}{b-a}\right)^s g(a)+\left(\frac{x-a}{b-a}\right)^s g(b)=h(x)$$

and hence by (3) of Proposition 1, we have that

$$\fint_a^b g d\mu \le \fint_a^b \left[\left(1-\frac{x-a}{b-a}\right)^s g(a)+\left(\frac{x-a}{b-a}\right)^s g(b)\right]d\mu=\fint_a^b h(x)d\mu$$

Let us consider the distribution function F given by

$$\begin{aligned}F(\alpha)&=\mu\left([a,b]\cap\{h\ge\alpha\}\right)\\&=\mu\left([a,b]\cap\left\{\left(1-\frac{x-a}{b-a}\right)^s g(a)+\left(\frac{x-a}{b-a}\right)^s g(b)\ge\alpha\right\}\right).\end{aligned}\tag{2.6}$$

Arguing similarly as in Theorem 1, we have that

$$1-\frac{x-a}{b-a}=\left|1-\frac{x-a}{b-a}\right|\le 1+\left|\frac{x-a}{b-a}\right|.$$

and hence by Lemma1, we have

$$\left(1-\frac{x-a}{b-a}\right)^s\le\left(1+\left|\frac{x-a}{b-a}\right|\right)^s\le 1+\left|\frac{x-a}{b-a}\right|^s.$$

Now we consider the following cases:

Case I

If $x-a \geq 0$, then from (2.6) we get that

$$
\begin{aligned}
F(\alpha) &= \mu\left([0,1] \cap \{h \geq \alpha\}\right) \\
&= \mu\left([a,b] \cap \left\{\left(1+\left|\frac{x-a}{b-a}\right|^{s}\right) g(a) + \left(\frac{x-a}{b-a}\right)^{s} g(b) \geq \alpha\right\}\right) \\
&= \mu\left([0,1] \cap \left\{x \geq a + (b-a)\left(\frac{\alpha - g(a)}{g(a)+g(b)}\right)^{\frac{1}{s}}\right\}\right) \\
&= (b-a)\left[1 - \left(\frac{\alpha - g(a)}{g(a)+g(b)}\right)^{\frac{1}{s}}\right].
\end{aligned} \tag{2.7}
$$

Therefore from (2.7), we have

$$
(b-a)\left[1 - \left(\frac{\alpha - g(a)}{g(a)+g(b)}\right)^{\frac{1}{s}}\right] = \alpha. \tag{2.8}
$$

From (1) of Proposition 1 we get that

$$
⨍_{a}^{b} h(x) d\mu \leq \mu\left([a,b]\right) = b-a. \tag{2.9}
$$

Thus by Remark 1

$$
⨍_{a}^{b} g d\mu \leq \min\left\{b-a, \alpha_1\right\},
$$

where α_1 is a positive real solution of (2.8).

Case II

If $x-a<0$, $\frac{1}{s}$ is an odd integer and $g(b)>g(a)$, then from (2.6) we get that

$$
\begin{aligned}
F(\alpha) &= \mu\left([0,1] \cap \{h \geq \alpha\}\right) \\
&= \mu\left([a,b] \cap \left\{\left(1+\left|\frac{x-a}{b-a}\right|^{s}\right) g(a) + \left(\frac{x-a}{b-a}\right)^{s} g(b) \geq \alpha\right\}\right) \\
&= \mu\left([0,1] \cap \left\{x \geq a + (b-a)\left(\frac{\alpha - g(a)}{g(b)-g(a)}\right)^{\frac{1}{s}}\right\}\right) \\
&= (b-a)\left[1 - \left(\frac{\alpha - g(a)}{g(b)-g(a)}\right)^{\frac{1}{s}}\right].
\end{aligned} \tag{2.10}
$$

and hence

$$
⨍_{a}^{b} g d\mu \leq \min\left\{b-a, \alpha_1\right\},
$$

where α_1 is a positive solution of the equation $(b-a)\left[1-\left(\frac{\alpha-g(a)}{g(b)-g(a)}\right)^{\frac{1}{s}}\right]=\alpha$.

Case III

If $x-a<0$, $\frac{1}{s}$ is an odd integer and $g(b)<g(a)$, then from (2.6) we get that

$$\begin{aligned}
F(\alpha) &= \mu\left([0,1]\cap\{h\geq\alpha\}\right) \\
&= \mu\left([a,b]\cap\left\{\left(1+\left|\frac{x-a}{b-a}\right|^{s}\right)g(a)+\left(\frac{x-a}{b-a}\right)^{s}g(b)\geq\alpha\right\}\right) \\
&= \mu\left([0,1]\cap\left\{x\leq a+(b-a)\left(\frac{g(a)-\alpha}{g(a)-g(b)}\right)^{\frac{1}{s}}\right\}\right) \\
&= (b-a)\left(\frac{g(a)-\alpha}{g(a)-g(b)}\right)^{\frac{1}{s}}.
\end{aligned} \tag{2.11}$$

Therefore

$$⨍_{a}^{b} g d\mu \leq \min\{b-a,\alpha_1\},$$

where α_1 is a positive solution of the equation $(b-a)\left(\frac{g(a)-\alpha}{g(a)-g(b)}\right)^{\frac{1}{s}}=\alpha$. This completes the proof of the theorem. □

Example 5. *Let $f(x)=2x^{\frac{1}{3}}$ be a function defined on $[0,2]$, then f is an s-convex function in the second sense with $s=\frac{1}{3}$. Here we have $a=0$, $b=2$, therefore for $x\in[0,1]$, $x-a\geq 0$. Moreover $f(a)=f(0)=0$ and $f(b)=f(2)=2^{\frac{4}{3}}$. Now by (a) of Theorem 2, we solve the equations $(b-a)\left[1-\left(\frac{\alpha-g(a)}{g(a)+g(b)}\right)^{\frac{1}{s}}\right]=\alpha$, that is we solve the equation $2\left[1-\frac{\alpha^3}{16}\right]=\alpha$, we get that $\alpha=\sqrt[3]{\frac{1}{27}\sqrt{27}\sqrt{2240}+8}-\frac{8}{3\sqrt[3]{\frac{1}{27}\sqrt{27}\sqrt{2240}+8}}\approx 1.5418$. Hence*

$$\begin{aligned}
⨍_{0}^{2} f d\mu &\leq \min\left\{2,\sqrt[3]{\frac{1}{27}\sqrt{27}\sqrt{2240}+8}-\frac{8}{3\sqrt[3]{\frac{1}{27}\sqrt{27}\sqrt{2240}+8}}\right\} \\
&= \sqrt[3]{\frac{1}{27}\sqrt{27}\sqrt{2240}+8}-\frac{8}{3\sqrt[3]{\frac{1}{27}\sqrt{27}\sqrt{2240}+8}}.
\end{aligned}$$

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