

Bounding the Čebyšev Functional for a Function that is Convex in Absolute Value and Applications

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ABSTRACT. Some sharp bounds for the Čebyšev functional of a function that is convex in absolute value and applications for functions of selfadjoint operators in Hilbert spaces via the spectral representation theorem are given.

1. Introduction

For two Lebesgue integrable functions $f, g : [a, b] \rightarrow \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the Čebyšev functional defined by

$$C(f, g) := \frac{1}{b-a} \int_a^b f(t)g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt.$$

In 1934, G. Grüss [14] showed that

$$(1.1) \quad |C(f, g)| \leq \frac{1}{4} (M - m)(N - n),$$

provided m, M, n, N are real numbers with the property that

$$(1.2) \quad -\infty < m \leq f \leq M < \infty, \quad -\infty < n \leq g \leq N < \infty \quad \text{a.e. on } [a, b].$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

Another lesser known inequality for $C(f, g)$ was derived in 1882 by Čebyšev [3] under the assumption that f', g' exist and are continuous on $[a, b]$, and is given by

$$(1.3) \quad |C(f, g)| \leq \frac{1}{12} \|f'\|_\infty \|g'\|_\infty (b-a)^2,$$

where $\|f'\|_\infty := \sup_{t \in [a, b]} |f'(t)| < \infty$.

The constant $\frac{1}{12}$ cannot be improved in general in (1.3).

Čebyšev's inequality (1.3) also holds if $f, g : [a, b] \rightarrow \mathbb{R}$ are assumed to be absolutely continuous and $f', g' \in L_\infty[a, b]$.

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In 1970, A.M. Ostrowski [22] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results:

$$(1.4) \quad |C(f, g)| \leq \frac{1}{8} (b-a) (M-m) \|g'\|_\infty,$$

provided f is Lebesgue integrable on $[a, b]$ and satisfying (1.2) while $g : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous and $g' \in L_\infty[a, b]$. Here the constant $\frac{1}{8}$ is also sharp.

In 1973, A. Lupaş [16] (see also [19] p. 210) obtained the following result as well:

$$(1.5) \quad |C(f, g)| \leq \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b-a),$$

provided f, g are absolutely continuous and $f', g' \in L_2[a, b]$.

Here the constant $\frac{1}{\pi^2}$ is the best possible as well.

In [1], P. Cerone and S.S. Dragomir proved the following inequalities:

$$(1.6) \quad |C(f, g)| \leq \begin{cases} \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt, \\ \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_q \cdot \frac{1}{b-a} \left(\int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right|^p dt \right)^{\frac{1}{p}} \\ \text{where } p > 1, \quad 1/p + 1/q = 1. \end{cases}$$

For $\gamma = 0$, we get from the first inequality in (1.6)

$$(1.7) \quad |C(f, g)| \leq \|g\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \leq g \leq M$ for a.e. $x \in [a, b]$, then $\|g - \frac{m+M}{2}\|_\infty \leq \frac{1}{2}(M-m)$ and by the first inequality in (1.6) we can deduce the following result obtained by Cheng and Sun [4]

$$(1.8) \quad |C(f, g)| \leq \frac{1}{2} (M-m) \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt.$$

The constant $\frac{1}{2}$ is best in (1.8) as shown by Cerone and Dragomir in [2].

The following result holds [10].

THEOREM 1. *Let $f : [a, b] \rightarrow \mathbb{C}$ be of bounded variation on $[a, b]$ and $g : [a, b] \rightarrow \mathbb{C}$ a Lebesgue integrable function on $[a, b]$. Then*

$$(1.9) \quad |C(f, g)| \leq \frac{1}{2} \bigvee_a^b(f) \cdot \frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt$$

where $\bigvee_a^b(f)$ denotes the total variation of f on the interval $[a, b]$.

The constant $\frac{1}{2}$ is best possible in (1.9).

We denote the *variance* of the function $f : [a, b] \rightarrow \mathbb{C}$ by $D(f)$ and defined as

$$(1.10) \quad D(f) = [C(f, \bar{f})]^{1/2} = \left[\frac{1}{b-a} \int_a^b |f(t)|^2 dt - \left| \frac{1}{b-a} \int_a^b f(t) dt \right|^2 \right]^{1/2},$$

where \bar{f} denotes the complex conjugate function of f .

We have [10]:

COROLLARY 1. *If the function $f : [a, b] \rightarrow \mathbb{C}$ is of bounded variation on $[a, b]$, then*

$$(1.11) \quad D(f) \leq \frac{1}{2} \bigvee_a^b(f).$$

The constant $\frac{1}{2}$ is best possible in (1.11).

Now we can state the following result when both functions are of bounded variation [10]:

COROLLARY 2. *If $f, g : [a, b] \rightarrow \mathbb{C}$ are of bounded variation on $[a, b]$, then*

$$(1.12) \quad |C(f, g)| \leq \frac{1}{4} \bigvee_a^b(f) \bigvee_a^b(g).$$

The constant $\frac{1}{4}$ is best possible in (1.12).

REMARK 1. *We can consider the following quantity associated with a complex valued function $f : [a, b] \rightarrow \mathbb{C}$,*

$$E(f) := |C(f, f)|^{1/2} = \left| \frac{1}{b-a} \int_a^b f^2(t) dt - \left(\frac{1}{b-a} \int_a^b f(t) dt \right)^2 \right|^{1/2}.$$

Utilising the above results we can state that

$$(1.13) \quad \begin{aligned} E^2(f) &\leq \frac{1}{2} \bigvee_a^b(f) \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt \\ &\leq \frac{1}{2} \bigvee_a^b(f) D(f) \leq \frac{1}{4} \left[\bigvee_a^b(f) \right]^2. \end{aligned}$$

If we consider

$$\begin{aligned} G(f) &:= |C(f, |f|)|^{1/2} \\ &= \left| \frac{1}{b-a} \int_a^b f(t) |f(t)| dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b |f(t)| dt \right|^{1/2}, \end{aligned}$$

then we also have

$$(1.14) \quad \begin{aligned} G^2(f) &\leq \frac{1}{2} \bigvee_a^b(f) \cdot \frac{1}{b-a} \int_a^b \left| |f(t)| - \frac{1}{b-a} \int_a^b |f(s)| ds \right| dt \\ &\leq \frac{1}{2} \bigvee_a^b(f) D(|f|) \leq \frac{1}{4} \bigvee_a^b(f) \bigvee_a^b(|f|) \leq \frac{1}{4} \left[\bigvee_a^b(f) \right]^2 \end{aligned}$$

and

$$(1.15) \quad \begin{aligned} G^2(f) &\leq \frac{1}{2} \bigvee_a^b(|f|) \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt \\ &\leq \frac{1}{2} \bigvee_a^b(|f|) D(f) \leq \frac{1}{4} \bigvee_a^b(f) \bigvee_a^b(|f|) \leq \frac{1}{4} \left[\bigvee_a^b(f) \right]^2. \end{aligned}$$

Motivated by the results presented above, we establish in this paper some new bounds for the magnitude of $C(f, g)$ in the case when one of the complex valued function, say f , is convex in absolute value while the other is Lebesgue integrable on $[a, b]$. Applications for functions of selfadjoint operators in Hilbert spaces via the spectral representation theorem are also given.

2. New Results for Čebyšev Functional

Recall that a function $g : [a, b] \rightarrow \mathbb{R}$ is *convex* (*strictly convex*) on the interval $[a, b]$, if

$$g((1-t)x + ty) \leq (<) (1-t)g(x) + tg(y)$$

for any $x, y \in [a, b]$ ($x \neq y$) and $t \in [0, 1]$ ($(0, 1)$).

We observe that the *constant function* $k(t) = k, t \in [a, b]$ and the *identity function* $e(t) = t, t \in [a, b]$ can then be interpreted as convex functions. However, they are not strictly convex functions on $[a, b]$.

We have the following result:

THEOREM 2. *Let $f : [a, b] \rightarrow \mathbb{C}$ be a measurable function such that $|f|$ is convex on $[a, b]$ and $g : [a, b] \rightarrow \mathbb{C}$ is a Lebesgue integrable function on $[a, b]$. Then*

$$(2.1) \quad |C(f, g)| \leq \max\{|f(a)|, |f(b)|\} \frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt.$$

The inequality (2.1) is sharp.

PROOF. We use Sonin's identity

$$(2.2) \quad C(f, g) = \frac{1}{b-a} \int_a^b (f(t) - \lambda) \left[g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right] dt,$$

for $\lambda = 0$ to get

$$(2.3) \quad C(f, g) = \frac{1}{b-a} \int_a^b f(t) \left[g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right] dt.$$

Taking the modulus and utilizing the convexity of $|f|$ on $[a, b]$ we have

$$(2.4) \quad \begin{aligned} |C(f, g)| &\leq \frac{1}{b-a} \int_a^b |f(t)| \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt \\ &\leq \frac{1}{b-a} \int_a^b \left[\frac{(b-t)|f(a)| + (t-a)|f(b)|}{b-a} \right] \\ &\quad \times \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt \end{aligned}$$

If we denote the right side of (2.4) by I , then we have

$$\begin{aligned} I &\leq \sup_{t \in [a, b]} \left[\frac{(b-t)|f(a)| + (t-a)|f(b)|}{b-a} \right] \\ &\quad \times \frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt \\ &= \max\{|f(a)|, |f(b)|\} \frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt \end{aligned}$$

and by (2.4) we get (2.1).

Assume that the inequality (2.1) holds with a constant $K > 0$, namely

$$(2.5) \quad |C(f, g)| \leq K \max\{|f(a)|, |f(b)|\} \frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt.$$

Consider the functions $f, g : [a, b] \rightarrow \mathbb{R}$ defined by

$$f(t) := \begin{cases} -1, & t \in [a, \frac{a+b}{2}] \\ 1, & t \in (\frac{a+b}{2}, b] \end{cases}$$

and $g : [a, b] \rightarrow \mathbb{R}$, $g(t) = t - \frac{a+b}{2}$.

We have $|f| = 1$, which satisfy the convexity condition with equality and

$$\begin{aligned} C(f, g) &= \frac{1}{b-a} \int_a^b \left| t - \frac{a+b}{2} \right| dt = \frac{b-a}{4}, \\ \max\{|f(a)|, |f(b)|\} &= 1 \end{aligned}$$

and

$$\frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt = \frac{1}{b-a} \int_a^b \left| t - \frac{a+b}{2} \right| dt = \frac{b-a}{4}$$

and by (2.5) we have

$$\frac{b-a}{4} \leq K \frac{b-a}{4},$$

which shows that $K \geq 1$. □

With the notations from the introduction we have:

COROLLARY 3. *Let $f : [a, b] \rightarrow \mathbb{C}$ be a measurable function such that $|f|$ is convex on $[a, b]$. Then*

$$(2.6) \quad \begin{aligned} &D^2(f), E^2(f) \\ &\leq \max\{|f(a)|, |f(b)|\} \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right| dt, \end{aligned}$$

and

$$(2.7) \quad G^2(f) \leq \max\{|f(a)|, |f(b)|\} \frac{1}{b-a} \int_a^b \left| |f(t)| - \frac{1}{b-a} \int_a^b |f(s)| ds \right| dt.$$

We recall the *p*-logarithmic mean defined by

$$L_p(m, n) := \begin{cases} \left[\frac{m^{p+1} - n^{p+1}}{(p+1)(M-m)} \right]^{1/p}, & m \neq n \\ m, & m = n \end{cases}$$

where $p \neq -1, 0$ and $m, n > 0$.

The case of *p*-norm of the deviation

$$\left| f - \frac{1}{b-a} \int_a^b f(s) ds \right|$$

is as follows:

THEOREM 3. *Let $f : [a, b] \rightarrow \mathbb{C}$ be a measurable function such that $|f|$ is convex on $[a, b]$ and, for $p > 1$, $g : [a, b] \rightarrow \mathbb{C}$ is in the Lebesgue space $L_p[a, b]$. Then*

$$(2.8) \quad |C(f, g)| \leq L_q(|f(a)|, |f(b)|) \left[\frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^p dt \right]^{1/p},$$

where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

The inequality (2.8) is sharp in the sense that there is no constant $K \in (0, 1)$ such that

$$(2.9) \quad |C(f, g)| \leq KL_q(|f(a)|, |f(b)|) \times \left[\frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^p dt \right]^{1/p},$$

for any $p > 1$ and f, g as above.

PROOF. Making use of Hölder's inequality, we have

$$(2.10) \quad (b-a)I = \int_a^b \left[\frac{(b-t)|f(a)| + (t-a)|f(b)|}{b-a} \right] \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt \\ \leq \left(\int_a^b \left[\frac{(b-t)|f(a)| + (t-a)|f(b)|}{b-a} \right]^q dt \right)^{1/q} \\ \times \left(\int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^p dt \right)^{1/p}.$$

Observe that, by changing the variable $u = \frac{t-a}{b-a}$ we have

$$\int_a^b \left[\frac{(b-t)|f(a)| + (t-a)|f(b)|}{b-a} \right]^q dt \\ = (b-a) \int_0^1 [u|f(b)| + (1-u)|f(a)|]^q du.$$

Changing the variable again

$$v = u|f(b)| + (1-u)|f(a)|$$

we have for $|f(a)| \neq |f(b)|$

$$\begin{aligned} (b-a) \int_0^1 [u|f(b)| + (1-u)|f(a)|]^q du &= \frac{b-a}{|f(b)| - |f(a)|} \int_0^1 v^q dv \\ &= (b-a) L_q^q(|f(a)|, |f(b)|). \end{aligned}$$

For $|f(a)| = |f(b)|$ we also have

$$\begin{aligned} (b-a) \int_0^1 [u|f(b)| + (1-u)|f(a)|]^q du &= (b-a) |f(a)|^q \\ &= (b-a) L_q^q(|f(a)|, |f(b)|). \end{aligned}$$

Therefore

$$\begin{aligned} (b-a) I &\leq ((b-a) L_q^q(|f(a)|, |f(b)|))^{1/q} \left(\int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^p dt \right)^{1/p} \\ &= (b-a)^{1/q} L_q(|f(a)|, |f(b)|) \left(\int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^p dt \right)^{1/p}, \end{aligned}$$

which implies

$$I \leq L_q(|f(a)|, |f(b)|) \left(\frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^p dt \right)^{1/p}.$$

Making use of (2.10) we get the desired result (2.8).

Assume that (2.9) holds with a constant $K > 0$.

Consider the functions $f, g : [a, b] \rightarrow \mathbb{R}$ defined by

$$f(t) := \begin{cases} -1, & t \in [a, \frac{a+b}{2}], \\ 1, & t \in (\frac{a+b}{2}, b] \end{cases}$$

and $g : [a, b] \rightarrow \mathbb{R}$, $g(t) = t - \frac{a+b}{2}$.

We have $|f| = 1$, which satisfy the convexity condition with equality and

$$C(f, g) = \frac{1}{b-a} \int_a^b \left| t - \frac{a+b}{2} \right| dt = \frac{b-a}{4}.$$

We also have $L_q(|f(a)|, |f(b)|) = 1$ and

$$\begin{aligned} &\left(\frac{1}{b-a} \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|^p dt \right)^{1/p} \\ &= \left(\frac{1}{b-a} \int_a^b \left| t - \frac{a+b}{2} \right|^p dt \right)^{1/p} = \left(\frac{2}{b-a} \int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2} \right)^p dt \right)^{1/p} \\ &= \left(\frac{2}{b-a} \frac{\left(\frac{b-a}{2} \right)^{p+1}}{p+1} \right)^{1/p} = \frac{b-a}{2(p+1)^{1/p}}. \end{aligned}$$

If we replace these values in (2.9) we get

$$(2.11) \quad \frac{b-a}{4} \leq \frac{K(b-a)}{2(p+1)^{1/p}}$$

for any $p > 1$.

Now, if we let $p \rightarrow 1+$ in (2.11) we get $K \geq 1$, which proves the desired sharpness. \square

The case $p = q = 2$ is of interest.

COROLLARY 4. *Let $f : [a, b] \rightarrow \mathbb{C}$ be a measurable function such that $|f|$ is convex on $[a, b]$ and $g : [a, b] \rightarrow \mathbb{C}$ is in the Lebesgue space $L_2[a, b]$. Then*

$$(2.12) \quad |C(f, g)| \leq \left(\frac{|f(a)|^2 + |f(a)f(b)| + |f(b)|^2}{3} \right)^{1/2} D(g).$$

The following particular cases are of interest as well:

COROLLARY 5. *Let $f : [a, b] \rightarrow \mathbb{C}$ be a measurable function such that $|f|$ is convex on $[a, b]$. Then*

$$(2.13) \quad D^2(f), E^2(f) \leq L_q(|f(a)|, |f(b)|) \times \left[\frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right|^p dt \right]^{1/p},$$

and

$$(2.14) \quad G^2(f) \leq L_q(|f(a)|, |f(b)|) \times \left[\frac{1}{b-a} \int_a^b \left| |f(t)| - \frac{1}{b-a} \int_a^b |f(s)| ds \right|^p dt \right]^{1/p},$$

where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

In particular, we have

$$(2.15) \quad D^2(f), E^2(f) \leq \left(\frac{|f(a)|^2 + |f(a)f(b)| + |f(b)|^2}{3} \right)^{1/2} D(f),$$

and

$$(2.16) \quad G^2(f) \leq \left(\frac{|f(a)|^2 + |f(a)f(b)| + |f(b)|^2}{3} \right)^{1/2} D(|f|).$$

The first inequality in (2.15) is equivalent to

$$(2.17) \quad D(f) \leq \left(\frac{|f(a)|^2 + |f(a)f(b)| + |f(b)|^2}{3} \right)^{1/2}.$$

The following result also holds:

THEOREM 4. *Let $f : [a, b] \rightarrow \mathbb{C}$ be a measurable function such that $|f|$ is convex on $[a, b]$ and $g : [a, b] \rightarrow \mathbb{C}$ is essentially bounded on $[a, b]$. Then*

$$(2.18) \quad |C(f, g)| \leq \frac{1}{2} [|f(a)| + |f(b)|] \sup_{t \in [a, b]} \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|.$$

The constant $\frac{1}{2}$ is best possible in (2.18).

PROOF. We have

$$\begin{aligned}
 (2.19) \quad I &\leq \frac{1}{b-a} \int_a^b \left[\frac{(b-t)|f(a)| + (t-a)|f(b)|}{b-a} \right] \\
 &\times \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| dt \\
 &\leq \operatorname{ess\,sup}_{t \in [a,b]} \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| \\
 &\times \frac{1}{b-a} \int_a^b \left[\frac{(b-t)|f(a)| + (t-a)|f(b)|}{b-a} \right] dt \\
 &= \frac{|f(a)| + |f(b)|}{2} \operatorname{ess\,sup}_{t \in [a,b]} \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|
 \end{aligned}$$

and by (2.19) we get the desired result (2.18).

Assume that the inequality (2.18) holds with a constant $D > 0$

$$(2.20) \quad |C(f, g)| \leq D [|f(a)| + |f(b)|] \sup_{t \in [a,b]} \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right|.$$

Consider the functions $f, g : [a, b] \rightarrow \mathbb{R}$ defined by

$$f(t) = g(t) := \begin{cases} -1, & t \in [a, \frac{a+b}{2}], \\ 1, & t \in (\frac{a+b}{2}, b]. \end{cases}$$

We have $|f| = 1$, which satisfy the convexity condition with equality and

$$C(f, g) := \frac{1}{b-a} \int_a^b dt = 1, \quad |f(a)| = |f(b)| = 1$$

while

$$\sup_{t \in [a,b]} \left| g(t) - \frac{1}{b-a} \int_a^b g(s) ds \right| = 1.$$

From (2.20) we have $1 \leq 2D$, i.e. $D \geq \frac{1}{2}$. □

COROLLARY 6. *Let $f : [a, b] \rightarrow \mathbb{C}$ be a measurable function such that $|f|$ is convex on $[a, b]$. Then*

$$(2.21) \quad D^2(f), E^2(f) \leq \frac{1}{2} [|f(a)| + |f(b)|] \sup_{t \in [a,b]} \left| f(t) - \frac{1}{b-a} \int_a^b f(s) ds \right|$$

and

$$(2.22) \quad G^2(f) \leq \frac{1}{2} [|f(a)| + |f(b)|] \sup_{t \in [a,b]} \left| |f(t)| - \frac{1}{b-a} \int_a^b |f(s)| ds \right|.$$

3. Application for Riemann-Stieltjes Integral

The following representation is of interest in itself. The result was firstly obtained in [5] (see also [6]). For the sake a completeness we give here a short proof as well.

LEMMA 1. *If $v : [a, b] \rightarrow \mathbb{C}$ is continuous (of bounded variation) on $[a, b]$ and $h : [a, b] \rightarrow \mathbb{C}$ is of bounded variation (continuous) on $[a, b]$, then we have the identity*

$$(3.1) \quad \frac{v(b) \int_a^b (t-a) dh(t) + v(a) \int_a^b (b-t) dh(t)}{b-a} - \int_a^b v(t) dh(t) \\ = \int_a^b h(t) dv(t) - \frac{v(b) - v(a)}{b-a} \int_a^b h(t) dt.$$

PROOF. Integrating by parts in the Riemann-Stieltjes integral we have

$$(3.2) \quad \frac{v(b) \int_a^b (t-a) dh(t) + v(a) \int_a^b (b-t) dh(t)}{b-a} - \int_a^b v(t) dh(t) \\ = \int_a^b \left[\frac{v(b)(t-a) + v(a)(b-t)}{b-a} - v(t) \right] dh(t) \\ = \left[\frac{(t-a)v(b) + (b-t)v(a)}{b-a} - v(t) \right] h(t) \Big|_a^b \\ - \int_a^b h(t) d \left[\frac{(t-a)v(b) + (b-t)v(a)}{b-a} - v(t) \right] \\ = [v(b) - v(a)] h(b) - [v(a) - v(a)] h(a) \\ - \int_a^b h(t) \left[\frac{v(b) - v(a)}{b-a} dt - dv(t) \right] \\ = \int_a^b h(t) dv(t) - \frac{v(b) - v(a)}{b-a} \int_a^b h(t) dt$$

and the identity is proven. \square

We can provide now the following application for Riemann-Stieltjes integral:

PROPOSITION 1. *If $v : I \rightarrow \mathbb{C}$ is differentiable on the interior of the interval I denoted \dot{I} and $[a, b] \subset \dot{I}$, $|v'|$ is convex on $[a, b]$ and $h : [a, b] \rightarrow \mathbb{C}$ is integrable on*

$[a, b]$, then we have the inequalities

$$(3.3) \quad \left| \frac{v(b) \int_a^b (t-a) dh(t) + v(a) \int_a^b (b-t) dh(t)}{b-a} - \int_a^b v(t) dh(t) \right|$$

$$\leq \begin{cases} \max\{|v'(a)|, |v'(b)|\} \int_a^b \left| h(t) - \frac{1}{b-a} \int_a^b h(s) ds \right| dt, \\ (b-a) L_q(|v'(a)|, |v'(b)|) \left[\frac{1}{b-a} \int_a^b \left| h(t) - \frac{1}{b-a} \int_a^b h(s) ds \right|^p dt \right]^{1/p} \\ \text{where } q > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{2} (b-a) [|v'(a)| + |v'(b)|] \sup_{t \in [a,b]} \left| h(t) - \frac{1}{b-a} \int_a^b h(s) ds \right|. \end{cases}$$

PROOF. From (3.1) we have

$$(3.4) \quad \frac{v(b) \int_a^b (t-a) dh(t) + v(a) \int_a^b (b-t) dh(t)}{b-a} - \int_a^b v(t) dh(t)$$

$$= \int_a^b h(t) v'(t) dt - \frac{v(b) - v(a)}{b-a} \int_a^b h(t) dt = (b-a) C(v', h).$$

Since $|v'|$ is convex on $[a, b]$, then by applying Theorem 2-Theorem 4 for $f = v'$ and $g = h$ we deduce the desired result (3.3). \square

REMARK 2. If $p = q = 2$, then by (3.3) we get

$$(3.5) \quad \left| \frac{v(b) \int_a^b (t-a) dh(t) + v(a) \int_a^b (b-t) dh(t)}{b-a} - \int_a^b v(t) dh(t) \right|$$

$$\leq (b-a) \left(\frac{|v'(a)|^2 + |v'(a)v'(b)| + |v'(b)|^2}{3} \right)^{1/2}$$

$$\times \left[\frac{1}{b-a} \int_a^b \left| h(t) - \frac{1}{b-a} \int_a^b h(s) ds \right|^2 dt \right]^{1/2},$$

provided that $|v'|$ is convex on $[a, b]$ and $h : [a, b] \rightarrow \mathbb{C}$ is integrable on $[a, b]$.

4. Applications for Selfadjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$. Let $A \in \mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda \in \mathbb{R}$ as follows

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \leq \lambda, \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

Then for every $\lambda \in \mathbb{R}$ the operator

$$(4.1) \quad E_\lambda := \varphi_\lambda(A)$$

is a projection which reduces A .

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [15, p. 256]:

THEOREM 5 (Spectral Representation Theorem). *Let A be a bounded selfadjoint operator on the Hilbert space H and let $m = \min \{\lambda \mid \lambda \in Sp(A)\} =: \min Sp(A)$ and $M = \max \{\lambda \mid \lambda \in Sp(A)\} =: \max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda \in \mathbb{R}}$, called the spectral family of A , with the following properties*

- a) $E_\lambda \leq E_{\lambda'}$ for $\lambda \leq \lambda'$;
- b) $E_{m-0} = 0, E_M = I$ and $E_{\lambda+0} = E_\lambda$ for all $\lambda \in \mathbb{R}$;
- c) We have the representation

$$(4.2) \quad A = \int_{m-0}^M \lambda dE_\lambda.$$

More generally, for every continuous complex-valued function φ defined on \mathbb{R} and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$(4.3) \quad \left\| \varphi(A) - \sum_{k=1}^n \varphi(\lambda'_k) [E_{\lambda_k} - E_{\lambda_{k-1}}] \right\| \leq \varepsilon$$

whenever

$$(4.4) \quad \begin{cases} \lambda_0 < m = \lambda_1 < \dots < \lambda_{n-1} < \lambda_n = M, \\ \lambda_k - \lambda_{k-1} \leq \delta \text{ for } 1 \leq k \leq n, \\ \lambda'_k \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \leq k \leq n \end{cases}$$

this means that

$$(4.5) \quad \varphi(A) = \int_{m-0}^M \varphi(\lambda) dE_\lambda,$$

where the integral is of Riemann-Stieltjes type.

COROLLARY 7. *With the assumptions of Theorem 5 for A, E_λ and φ we have the representations*

$$(4.6) \quad \varphi(A)x = \int_{m-0}^M \varphi(\lambda) dE_\lambda x \quad \text{for all } x \in H$$

and

$$(4.7) \quad \langle \varphi(A)x, y \rangle = \int_{m-0}^M \varphi(\lambda) d\langle E_\lambda x, y \rangle \quad \text{for all } x, y \in H.$$

In particular,

$$(4.8) \quad \langle \varphi(A)x, x \rangle = \int_{m-0}^M \varphi(\lambda) d\langle E_\lambda x, x \rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$(4.9) \quad \|\varphi(A)x\|^2 = \int_{m-0}^M |\varphi(\lambda)|^2 d\|E_\lambda x\|^2 \quad \text{for all } x \in H.$$

The next result shows that it is legitimate to talk about "the" spectral family of the bounded selfadjoint operator A since it is uniquely determined by the requirements a), b) and c) in Theorem 5, see for instance [15, p. 258]:

THEOREM 6. *Let A be a bounded selfadjoint operator on the Hilbert space H and let $m = \min Sp(A)$ and $M = \max Sp(A)$. If $\{F_\lambda\}_{\lambda \in \mathbb{R}}$ is a family of projections satisfying the requirements a), b) and c) in Theorem 5, then $F_\lambda = E_\lambda$ for all $\lambda \in \mathbb{R}$ where E_λ is defined by (4.1).*

By the above two theorems, the spectral family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ uniquely determines and in turn is uniquely determined by the bounded selfadjoint operator A .

We can state now the following generalized trapezoid inequality for functions of selfadjoint operators:

THEOREM 7. *Let A be a bounded selfadjoint operator on the Hilbert space H and let $m = \min \{\lambda \mid \lambda \in Sp(A)\} =: \min Sp(A)$ and $M = \max \{\lambda \mid \lambda \in Sp(A)\} =: \max Sp(A)$. Consider also the spectral family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ of A .*

If $f : I \rightarrow \mathbb{C}$ is differentiable on the interior of the interval I , denoted \mathring{I} and $[m, M] \subset \mathring{I}$, $|f'|$ is convex on $[m, M]$, then we have the inequalities

$$\begin{aligned}
 (4.10) \quad & \left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A)x, y \rangle \right| \\
 & \leq \frac{1}{2} [|f'(m)| + |f'(M)|] (M - m) \\
 & \quad \times \sup_{t \in [m, M]} \left[\frac{t - m}{M - m} \bigvee_{m-0}^t (\langle E_{(\cdot)} x, y \rangle) + \frac{M - t}{M - m} \bigvee_t^M (\langle E_{(\cdot)} x, y \rangle) \right] \\
 & \leq \frac{1}{2} [|f'(m)| + |f'(M)|] (M - m) \bigvee_{m-0}^M (\langle E_{(\cdot)} x, y \rangle) \\
 & \leq \frac{1}{2} [|f'(m)| + |f'(M)|] (M - m) \|x\| \|y\|
 \end{aligned}$$

for any $x, y \in H$.

PROOF. Let $x, y \in H$ and consider $h : \mathbb{R} \rightarrow \mathbb{C}$, $h(t) := \langle E_t x, y \rangle$. If we use the third inequality in (3.3) for the interval $[m - \varepsilon, M]$ with small $\varepsilon > 0$, we have

$$\begin{aligned}
 (4.11) \quad & \left| \frac{f(M) \int_{m-\varepsilon}^M (t - m + \varepsilon) d \langle E_t x, y \rangle + f(m - \varepsilon) \int_{m-\varepsilon}^M (M - t) d \langle E_t x, y \rangle}{M - m + \varepsilon} \right. \\
 & \quad \left. - \int_{m-\varepsilon}^M f(t) d \langle E_t x, y \rangle \right| \\
 & \leq \frac{1}{2} [|f'(M)| + |f'(m - \varepsilon)|] (M - m + \varepsilon) \\
 & \quad \times \sup_{t \in [m-\varepsilon, M]} \left| \langle E_t x, y \rangle - \frac{1}{M - m + \varepsilon} \int_{m-\varepsilon}^M \langle E_s x, y \rangle ds \right|.
 \end{aligned}$$

Taking the limit over $\varepsilon \rightarrow 0+$ and using the Spectral representation theorem, we have

$$(4.12) \quad \left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A)x, y \rangle \right| \\ \leq \frac{1}{2} [|f'(m)| + |f'(M)|] (M - m) \\ \times \sup_{t \in [m, M]} \left| \langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^M \langle E_s x, y \rangle ds \right|$$

for any $x, y \in H$.

It is well known that if $p : [a, b] \rightarrow \mathbb{C}$ is a bounded function, $v : [a, b] \rightarrow \mathbb{C}$ is of bounded variation and the Riemann-Stieltjes integral $\int_a^b p(t) dv(t)$ exists, then the following inequality holds

$$(4.13) \quad \left| \int_a^b p(t) dv(t) \right| \leq \sup_{t \in [a, b]} |p(t)| \bigvee_a^b(v),$$

where $\bigvee_a^b(v)$ denotes the total variation of v on $[a, b]$.

Now, a simple integration by parts in the Riemann-Stieltjes integral reveals the following equality of interest

$$(4.14) \quad \langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^M \langle E_s x, y \rangle ds \\ = \frac{1}{M - m} \left[\int_{m-0}^t (s - m) d \langle E_s x, y \rangle + \int_t^M (s - M) d \langle E_s x, y \rangle \right]$$

that holds for any $t \in [m, M]$ and for any $x, y \in H$.

Since the function $v(s) := \langle E_s x, y \rangle$ is of bounded variation on $[m, M]$ for any $x, y \in H$, then on applying the inequality (4.13), we get

$$(4.15) \quad \left| \langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^M \langle E_s x, y \rangle ds \right| \\ \leq \frac{1}{M - m} \left[\left| \int_{m-0}^t (s - m) d \langle E_s x, y \rangle \right| + \left| \int_t^M (s - M) d \langle E_s x, y \rangle \right| \right] \\ \leq \frac{t - m}{M - m} \bigvee_{m-0}^t (\langle E_{(\cdot)} x, y \rangle) + \frac{M - t}{M - m} \bigvee_t^M (\langle E_{(\cdot)} x, y \rangle) \\ \leq \max \left\{ \frac{t - m}{M - m}, \frac{M - t}{M - m} \right\} \bigvee_{m-0}^M (\langle E_{(\cdot)} x, y \rangle) \\ = \left[\frac{1}{2} + \left| t - \frac{m+M}{2} \right| \right] \bigvee_{m-0}^M (\langle E_{(\cdot)} x, y \rangle)$$

that holds for any $t \in [m, M]$ and for any $x, y \in H$.

This implies that

$$\begin{aligned}
 (4.16) \quad & \sup_{t \in [m, M]} \left| \langle E_t x, y \rangle - \frac{1}{M-m} \int_{m-0}^M \langle E_s x, y \rangle ds \right| \\
 & \leq \sup_{t \in [m, M]} \left[\frac{t-m}{M-m} \bigvee_{m-0}^t (\langle E_{(\cdot)} x, y \rangle) + \frac{M-t}{M-m} \bigvee_t^M (\langle E_{(\cdot)} x, y \rangle) \right] \\
 & \leq \sup_{t \in [m, M]} \left[\frac{1}{2} + \left| t - \frac{m+M}{2} \right| \right] \bigvee_{m-0}^M (\langle E_{(\cdot)} x, y \rangle) = \bigvee_{m-0}^M (\langle E_{(\cdot)} x, y \rangle)
 \end{aligned}$$

for any $x, y \in H$.

The proof of the inequality

$$\bigvee_{m-0}^M (\langle E_{(\cdot)} x, y \rangle) \leq \|x\| \|y\|$$

for any $x, y \in H$, can be found in [12, p. 9]. \square

We also have:

THEOREM 8. *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m = \min \{\lambda \mid \lambda \in Sp(A)\} =: \min Sp(A)$ and $M = \max \{\lambda \mid \lambda \in Sp(A)\} =: \max Sp(A)$. Consider also the spectral family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ of A .*

If $f : I \rightarrow \mathbb{C}$ is differentiable on \hat{I} , $[m, M] \subset \hat{I}$ and $|f'|$ is convex on $[m, M]$, then we have the inequalities

$$\begin{aligned}
 (4.17) \quad & \left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M-m} \right] x, y \right\rangle - \langle f(A)x, y \rangle \right| \\
 & \leq \max \{|f'(m)|, |f'(M)|\} \int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M-m} \int_{m-0}^M \langle E_s x, y \rangle ds \right| dt \\
 & \leq \frac{1}{2} \max \{|f'(m)|, |f'(M)|\} (M-m) \|x\| \|y\|
 \end{aligned}$$

for any $x, y \in H$.

PROOF. Let $x, y \in H$ and consider $h : \mathbb{R} \rightarrow \mathbb{C}$, $h(t) := \langle E_t x, y \rangle$. If we use the first inequality in (3.3) for the interval $[m-\varepsilon, M]$ with small $\varepsilon > 0$, we have

$$\begin{aligned}
 (4.18) \quad & \left| \frac{f(M) \int_{m-\varepsilon}^M (t-m+\varepsilon) d \langle E_t x, y \rangle + f(m-\varepsilon) \int_{m-\varepsilon}^M (M-t) d \langle E_t x, y \rangle}{M-m+\varepsilon} \right. \\
 & \left. - \int_{m-\varepsilon}^M f(t) d \langle E_t x, y \rangle \right| \\
 & \leq \max \{|f'(M)|, |f'(m-\varepsilon)|\} (M-m+\varepsilon) \\
 & \times \frac{1}{M-m+\varepsilon} \int_{m-\varepsilon}^M \left| \langle E_t x, y \rangle - \frac{1}{M-m+\varepsilon} \int_{m-\varepsilon}^M \langle E_s x, y \rangle ds \right| dt.
 \end{aligned}$$

Taking the limit over $\varepsilon \rightarrow 0+$ and using the Spectral representation theorem, we have

$$(4.19) \quad \left| \left\langle \left[\frac{f(m)(M1_H - A) + f(M)(A - m1_H)}{M - m} \right] x, y \right\rangle - \langle f(A)x, y \rangle \right| \\ \leq \max\{|f'(m)|, |f'(M)|\} \int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^M \langle E_s x, y \rangle ds \right| dt$$

for any $x, y \in H$.

By the Schwarz inequality in H we have that

$$(4.20) \quad \int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^M \langle E_s x, y \rangle ds \right| dt \\ = \int_{m-0}^M \left| \left\langle \left[E_t x - \frac{1}{M - m} \int_{m-0}^M E_s x ds \right], y \right\rangle \right| dt \\ \leq \|y\| \int_{m-0}^M \left\| E_t x - \frac{1}{M - m} \int_{m-0}^M E_s x ds \right\| dt$$

for any $x, y \in H$.

On utilizing the Cauchy-Buniakovski-Schwarz integral inequality we may state that

$$(4.21) \quad \int_{m-0}^M \left\| E_t x - \frac{1}{M - m} \int_{m-0}^M E_s x ds \right\| dt \\ \leq (M - m)^{1/2} \left(\int_{m-0}^M \left\| E_t x - \frac{1}{M - m} \int_{m-0}^M E_s x ds \right\|^2 dt \right)^{1/2}$$

for any $x \in H$.

Observe that the following equalities of interest hold and they can be easily proved by direct calculations

$$(4.22) \quad \frac{1}{M - m} \int_{m-0}^M \left\| E_t x - \frac{1}{M - m} \int_{m-0}^M E_s x ds \right\|^2 dt \\ = \frac{1}{M - m} \int_{m-0}^M \|E_t x\|^2 dt - \left\| \frac{1}{M - m} \int_{m-0}^M E_s x ds \right\|^2$$

and

$$(4.23) \quad \frac{1}{M - m} \int_{m-0}^M \|E_t x\|^2 dt - \left\| \frac{1}{M - m} \int_{m-0}^M E_s x ds \right\|^2 \\ = \frac{1}{M - m} \int_{m-0}^M \left\langle E_t x - \frac{1}{M - m} \int_{m-0}^M E_s x ds, E_t x - \frac{1}{2} x \right\rangle dt$$

for any $x \in H$.

By (4.21), (4.22) and (4.23) we get

$$\begin{aligned}
 (4.24) \quad & \int_{m-0}^M \left\| E_t x - \frac{1}{M-m} \int_{m-0}^M E_s x ds \right\| dt \\
 & \leq (M-m)^{1/2} \left(\int_{m-0}^M \left\langle E_t x - \frac{1}{M-m} \int_{m-0}^M E_s x ds, E_t x - \frac{1}{2} x \right\rangle dt \right)^{1/2}
 \end{aligned}$$

for any $x \in H$.

On making use of the Schwarz inequality in H we also have

$$\begin{aligned}
 (4.25) \quad & \int_{m-0}^M \left\langle E_t x - \frac{1}{M-m} \int_{m-0}^M E_s x ds, E_t x - \frac{1}{2} x \right\rangle dt \\
 & \leq \int_{m-0}^M \left\| E_t x - \frac{1}{M-m} \int_{m-0}^M E_s x ds \right\| \left\| E_t x - \frac{1}{2} x \right\| dt \\
 & = \frac{1}{2} \|x\| \int_{m-0}^M \left\| E_t x - \frac{1}{M-m} \int_{m-0}^M E_s x ds \right\| dt,
 \end{aligned}$$

where we used the fact that E_t are projectors, and in this case we have

$$\left\| E_t x - \frac{1}{2} x \right\|^2 = \|E_t x\|^2 - \langle E_t x, x \rangle + \frac{1}{4} \|x\|^2 = \frac{1}{4} \|x\|^2$$

for any $t \in [m, M]$ for any $x \in H$.

From (4.24) and (4.25) we get

$$\begin{aligned}
 (4.26) \quad & \int_{m-0}^M \left\| E_t x - \frac{1}{M-m} \int_{m-0}^M E_s x ds \right\| dt \\
 & \leq (M-m)^{1/2} \left(\frac{1}{2} \|x\| \int_{m-0}^M \left\| E_t x - \frac{1}{M-m} \int_{m-0}^M E_s x ds \right\| dt \right)^{1/2},
 \end{aligned}$$

which is clearly equivalent with the following inequality of interest in itself

$$(4.27) \quad \int_{m-0}^M \left\| E_t x - \frac{1}{M-m} \int_{m-0}^M E_s x ds \right\| dt \leq \frac{1}{2} \|x\| (M-m)$$

for any $x \in H$.

From (4.20) we then get

$$\frac{1}{M-m} \int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M-m} \int_{m-0}^M \langle E_s x, y \rangle ds \right| dt \leq \frac{1}{2} \|x\| \|y\|$$

for any $x, y \in H$. □

Finally, we also have:

THEOREM 9. *Let A be a bonded selfadjoint operator on the Hilbert space H and let $m = \min \{ \lambda \mid \lambda \in Sp(A) \} =: \min Sp(A)$ and $M = \max \{ \lambda \mid \lambda \in Sp(A) \} =: \max Sp(A)$. Consider also the spectral family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ of A .*

If $f : I \rightarrow \mathbb{C}$ is differentiable on \hat{I} , $[m, M] \subset \hat{I}$ and $|f'|$ is convex on $[m, M]$, then we have the inequalities

$$\begin{aligned}
(4.28) \quad & \left| \left\langle \left[\frac{f(m)(M\mathbf{1}_H - A) + f(M)(A - m\mathbf{1}_H)}{M - m} \right] x, y \right\rangle - \langle f(A)x, y \rangle \right| \\
& \leq \left(\frac{|f'(M)|^2 + |f'(M)f'(m)| + |f'(m)|^2}{3} \right)^{1/2} (M - m) \\
& \quad \times \left(\frac{1}{M - m} \int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^M \langle E_s x, y \rangle ds \right|^2 dt \right)^{1/2} \\
& \leq \frac{1}{2} \left(\frac{|f'(M)|^2 + |f'(M)f'(m)| + |f'(m)|^2}{3} \right)^{1/2} (M - m) \|x\| \|y\|
\end{aligned}$$

for any $x, y \in H$.

PROOF. Utilising the inequality (3.5) we can prove in a similar manner as above the first inequality in (4.28).

By the Schwarz inequality in H we have that

$$\begin{aligned}
(4.29) \quad & \frac{1}{M - m} \int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^M \langle E_s x, y \rangle ds \right|^2 dt \\
& = \frac{1}{M - m} \int_{m-0}^M \left| \left\langle \left[E_t x - \frac{1}{M - m} \int_{m-0}^M E_s x ds \right], y \right\rangle \right|^2 dt \\
& \leq \|y\|^2 \frac{1}{M - m} \int_{m-0}^M \left\| E_t x - \frac{1}{M - m} \int_{m-0}^M E_s x ds \right\|^2 dt
\end{aligned}$$

for any $x, y \in H$.

As in the proof of Theorem 8 we also have

$$\begin{aligned}
(4.30) \quad & \frac{1}{M - m} \int_{m-0}^M \left\| E_t x - \frac{1}{M - m} \int_{m-0}^M E_s x ds \right\|^2 dt \\
& \leq \frac{1}{2} \|x\| \frac{1}{M - m} \int_{m-0}^M \left\| E_t x - \frac{1}{M - m} \int_{m-0}^M E_s x ds \right\| dt \leq \frac{1}{4} \|x\|^2.
\end{aligned}$$

By (4.29) and (4.30) we then get

$$\frac{1}{M - m} \int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^M \langle E_s x, y \rangle ds \right|^2 dt \leq \frac{1}{4} \|x\|^2 \|y\|^2,$$

namely

$$\left[\frac{1}{M - m} \int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^M \langle E_s x, y \rangle ds \right|^2 dt \right]^{1/2} \leq \frac{1}{2} \|x\| \|y\|,$$

for any $x, y \in H$.

This proves the last part of (4.28). \square

EXAMPLE 1. a) Let A be a bonded selfadjoint operator on the Hilbert space H and let $m = \min \{\lambda | \lambda \in Sp(A)\} =: \min Sp(A) \geq 0$ and $M = \max \{\lambda | \lambda \in Sp(A)\} =: \max Sp(A)$. Consider also the spectral family $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ of A . Then by Theorem 7-9 we have for $f(t) = t^p$, $p \geq 2$ that

$$\begin{aligned}
 (4.31) \quad & \left| \left\langle \left[\frac{m^p (M1_H - A) + M^p (A - m1_H)}{M - m} \right] x, y \right\rangle - \langle A^p x, y \rangle \right| \\
 & \leq \frac{1}{2} p (m^{p-1} + M^{p-1}) (M - m) \\
 & \quad \times \sup_{t \in [m, M]} \left[\frac{t - m}{M - m} \bigvee_{m-0}^t (\langle E_{(\cdot)} x, y \rangle) + \frac{M - t}{M - m} \bigvee_t^M (\langle E_{(\cdot)} x, y \rangle) \right] \\
 & \leq \frac{1}{2} p (m^{p-1} + M^{p-1}) (M - m) \bigvee_{m-0}^M (\langle E_{(\cdot)} x, y \rangle) \\
 & \leq \frac{1}{2} p (m^{p-1} + M^{p-1}) (M - m) \|x\| \|y\|,
 \end{aligned}$$

$$\begin{aligned}
 (4.32) \quad & \left| \left\langle \left[\frac{m^p (M1_H - A) + M^p (A - m1_H)}{M - m} \right] x, y \right\rangle - \langle A^p x, y \rangle \right| \\
 & \leq p M^{p-1} \int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^M \langle E_s x, y \rangle ds \right| dt \\
 & \leq \frac{1}{2} p M^{p-1} (M - m) \|x\| \|y\|
 \end{aligned}$$

and

$$\begin{aligned}
 (4.33) \quad & \left| \left\langle \left[\frac{m^p (M1_H - A) + M^p (A - m1_H)}{M - m} \right] x, y \right\rangle - \langle A^p x, y \rangle \right| \\
 & \leq p \left(\frac{M^{2(p-1)} + (Mm)^{p-1} + m^{2(p-1)}}{3} \right)^{1/2} (M - m) \\
 & \quad \times \left(\frac{1}{M - m} \int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^M \langle E_s x, y \rangle ds \right|^2 dt \right)^{1/2} \\
 & \leq \frac{1}{2} \left(\frac{M^{2(p-1)} + (Mm)^{p-1} + m^{2(p-1)}}{3} \right)^{1/2} (M - m) \|x\| \|y\|
 \end{aligned}$$

for any $x, y \in H$.

b) With the assumptions of a) and if $m > 0$, then by Theorem 7-9 we have for $f(t) = \ln t$, that

$$\begin{aligned}
 (4.34) \quad & \left| \left\langle \left[\frac{\ln m (M1_H - A) + \ln M (A - m1_H)}{M - m} \right] x, y \right\rangle - \langle \ln Ax, y \rangle \right| \\
 & \leq \frac{m + M}{2mM} (M - m) \\
 & \times \sup_{t \in [m, M]} \left[\frac{t - m}{M - m} \bigvee_{m-0}^t (\langle E_{(\cdot)} x, y \rangle) + \frac{M - t}{M - m} \bigvee_t^M (\langle E_{(\cdot)} x, y \rangle) \right] \\
 & \leq \frac{m + M}{2mM} (M - m) \bigvee_{m-0}^M (\langle E_{(\cdot)} x, y \rangle) \leq \frac{m + M}{2mM} (M - m) \|x\| \|y\|,
 \end{aligned}$$

$$\begin{aligned}
 (4.35) \quad & \left| \left\langle \left[\frac{\ln m (M1_H - A) + \ln M (A - m1_H)}{M - m} \right] x, y \right\rangle - \langle \ln Ax, y \rangle \right| \\
 & \leq \frac{1}{m} \int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^M \langle E_s x, y \rangle ds \right| dt \\
 & \leq \frac{1}{2m} \|x\| \|y\| (M - m)
 \end{aligned}$$

and

$$\begin{aligned}
 (4.36) \quad & \left| \left\langle \left[\frac{\ln m (M1_H - A) + \ln M (A - m1_H)}{M - m} \right] x, y \right\rangle - \langle \ln Ax, y \rangle \right| \\
 & \leq \left(\frac{M^2 + mM + m^2}{3m^2M^2} \right)^{1/2} (M - m) \\
 & \times \left(\frac{1}{M - m} \int_{m-0}^M \left| \langle E_t x, y \rangle - \frac{1}{M - m} \int_{m-0}^M \langle E_s x, y \rangle ds \right|^2 dt \right)^{1/2} \\
 & \leq \frac{1}{2mM} \left(\frac{M^2 + mM + m^2}{3} \right)^{1/2} (M - m) \|x\| \|y\|
 \end{aligned}$$

for any $x, y \in H$.

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