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SOME WEIGHTED INTEGRAL INEQUALITIES FOR DIFFERENTIABLE PREINVEX AND PREQUASIINVEX FUNCTIONS

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Abstract. In this paper, a weighted identity for functions defined on an open invex subset of set of real numbers is established, and by using the this identity and the Hölder integral inequality we present weighted integral inequalities of Hermite-Hadamard type for functions whose powers of derivatives in absolute value are preinvex and prequasiinvex functions. Our results, on the one hand, give a weighted generalization of recent results for preinvex and prequasiinvex functions and, on the other hand, extend several results connected with the Hermite-Hadamard type inequalities for preinvex functions already exist in literature.

1. Introduction

Any paper on Hermite-Hadamard type inequalities seems to be incomplete without mentioning the famous Hermite-Hadamard inequality, which states as follows:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The inequalities in (1.1) hold in reversed direction if f is concave. Inequalities (1.1) are well-known in mathematical analysis due to its rich geometrical significance and applications (see [34]).

A number of papers have been written during the past few years which generalize, improve and extend the inequalities (1.1). For several results on Hermite-Hadamard type inequalities, we refer the interested reader to [1], [7]-[10], [14]-[17], [22], [33]-[35] and [38]-[43].

To estimate the difference between the middle and the leftmost terms in (1.1) has been an important question in mathematical analysis see for instance [14, 15, 33, 50]. The most representative work to give the answer of the above raised question are articles of Kırmacı [14] and Pearce and Pečarić [33]. The main results from these papers are the following.

Theorem 1. [14] *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$, and $f^{'} \in L([a, b])$. If $\left|f^{'}\right|$ is convex function on $[a, b]$, the*

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following inequality holds:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}\left[\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right]. \tag{1.2}$$

Theorem 2. [33] *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$, and $f^{'} \in L([a,b])$ and $q \geq 1$. If $\left|f^{'}\right|^q$ is a convex function on $[a,b]$, the following inequality holds:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left[\frac{\left|f^{'}(a)\right|^q+\left|f^{'}(b)\right|^q}{2}\right]^{1/q}. \tag{1.3}$$

Now, we recall that the notion of quasi-convex functions generalizes the notion of convex functions. More exactly, a function $f:[a,b] \to \mathbb{R}$ is said quasi-convex on $[a,b]$ if

$$f(tx+(1-t)y) \leq \max\{f(x), f(y)\}$$

for all $x, y \in [a,b]$ and $t \in [0,1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex, (see [13]). For more results on Hermite-Hadamard type inequalities for quasi-convex functions we refer the interested reader to [2, 9, 10, 13, 36] and the references therein.

In [10], Hwang established the following results for convex and quasi-convex functions those results provide a weighted generalization of the results given in Theorem 1 and Theorem 2.

Theorem 3. [10] *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I^{\circ}$ with $a < b$ and let $g:[a,b] \to [0,\infty)$ be continuous positive mapping and symmetric to $\frac{a+b}{2}$. If $\left|f^{'}\right|$ is convex function on $[a,b]$, the following inequality holds*

$$\left|\int_a^b f(x)g(x)dx - f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx\right| \leq \frac{b-a}{2}\left[\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right]\int_0^1 M(g;a,b,t)\,dt, \tag{1.4}$$

where $M(g;a,b,t) = \int_a^{L(a,b,t)} g(x)\,dt$ and $L(a,b,t) = \frac{1+t}{2}a + \frac{1-t}{2}b$.

Theorem 4. [10] *Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I^{\circ}$ with $a < b$ and let $g:[a,b] \to [0,\infty)$ be continuous positive mapping and symmetric to $\frac{a+b}{2}$ and $q \geq 1$. If $\left|f^{'}\right|^q$ is convex function on $[a,b]$, the following inequality holds*

$$\left|\int_a^b f(x)g(x)dx - f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx\right| \leq \frac{b-a}{2}\left[\frac{\left|f^{'}(a)\right|^q+\left|f^{'}(b)\right|^q}{2}\right]^{\frac{1}{q}}\int_0^1 M(g;a,b,t)\,dt, \tag{1.5}$$

where $M(g;a,b,t)$ and $L(a,b,t)$ are defined as in Theorem 3.

Theorem 5. [10] *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I^{\circ}$ with $a < b$ and let $g : [a, b] \to [0, \infty)$ be continuous positive mapping and symmetric to $\frac{a+b}{2}$. If $\left|f'\right|$ is quasi-convex function on $[a, b]$, the following inequality holds*

$$\begin{aligned}
(1.6)\quad & \left|\int_a^b f(x)g(x)dx - f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx\right| \\
&\le \frac{b-a}{2}\left[\max\left\{\left|f'(a)\right|, \left|f'\left(\frac{a+b}{2}\right)\right|\right\}\right. \\
&\quad \left. + \max\left\{\left|f'(b)\right|, \left|f'\left(\frac{a+b}{2}\right)\right|\right\}\right]\int_0^1 M(g; a, b, t)\, dt,
\end{aligned}$$

where $M(g; a, b, t)$ and $L(a, b, t)$ are defined as in Theorem 3.

Theorem 6. [10] *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I^{\circ}$ with $a < b$ and let $g : [a, b] \to [0, \infty)$ be continuous positive mapping and symmetric to $\frac{a+b}{2}$ and $q \ge 1$. If $\left|f'\right|^q$ is quasi-convex function on $[a, b]$, the following inequality holds*

$$\begin{aligned}
(1.7)\quad & \left|\int_a^b f(x)g(x)dx - f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx\right| \\
&\le \frac{b-a}{2}\left[\left(\max\left\{\left|f'(a)\right|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\max\left\{\left|f'(b)\right|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right]\int_0^1 M(g; a, b, t)\, dt,
\end{aligned}$$

where $M(g; a, b, t)$ and $L(a, b, t)$ are defined as in Theorem 3.

Let us recall the notions of preinvexity and prequasiinvexity which are significant generalizations of the notions of convexity and qusi-convexity respectively, and some related results.

Definition 1. [44] *Let K be a non-empty subset in $\mathbb{R}^n$ and $\eta : K \times K \to \mathbb{R}^n$. Let $x \in K$, then the set K is said to be invex at x with respect to $\eta(\cdot, \cdot)$, if*

$$x + t\eta(y, x) \in K, \forall x, y \in K, t \in [0, 1].$$

K is said to be an invex set with respect to η if K is invex at each $x \in K$. The invex set K is also called an η-connected set.

Definition 2. [44] *A function $f : K \to \mathbb{R}$ on an invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect to η, if*

$$f(u + t\eta(v, u)) \le (1-t)\, f(u) + t f(v), \forall u, v \in K, t \in [0, 1].$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

It is to be noted that every convex function is preinvex with respect to the map $\eta(x, y) = x - y$ but the converse is not true see for instance [1].

Definition 3. [3] *A function $f : K \to \mathbb{R}$ on an invex set $K \subseteq \mathbb{R}^n$ is said to be prequasiinvex with respect to η, if*

$$f(u + t\eta(v,u)) \leq \max\{f(u), f(v)\}, \forall u, v \in K, t \in [0,1].$$

Also every quasi-convex function is a prequasiinvex with respect to the map $\eta(v,u) = v - u$ but the converse does not hold, see for example [3].

In a recent paper, Noor [27] has obtained the following Hermite-Hadamard type inequalities for the preinvex functions.

Theorem 7. [27]*Let $f : [a, a+\eta(b,a)] \to (0,\infty)$ be a preinvex function on the interval of the real numbers K° (the interior of K) and a, $b \in K^\circ$ with $\eta(b,a) > 0$. Then the following inequalities holds*

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.8}$$

For several recent results on inequalities for preinvex and prequasiinvex functions which are connected to (1.8), we refer the interested reader to [4, 21, 26] and the references therein.

In the present paper, we give new weighted integral inequalities of Hermite-Hadamard type for functions whose derivatives in absolute value are preinvex and prequasiinvex. Our results generalize those results presented in a very recent paper of Hwang [10] and also provide weighted version of those results for preinvex and prequasiinvex functions which estimate the deference between the middle and the leftmost terms in (1.8).

2. Main Results

The following Lemma is essential in establishing our main results of this section:

Lemma 1. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a+\eta(b,a)])$, where a, $b \in K$ with $\eta(b,a) > 0$. If $h : [a, a+\eta(b,a)] \to [0,\infty)$ be a differentiable mapping, then the following equality holds*

$$\begin{aligned}
&\frac{h(a)}{2}[f(a)+f(b)] - h(a+\eta(b,a)) f\left(\frac{a+b}{2}\right) \\
&+ \frac{\eta(b,a)}{4}\int_0^1 \left[f\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right) + f\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right] \\
&\times \left[h'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right) + h'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right] dt \\
&= \frac{\eta(b,a)}{4}\left\{\int_0^1 \left[h\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right) - h\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right.\right. \\
&\left.\left. h(a+\eta(b,a))\right] \times \left[-f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right) + f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right] dt\right\}.
\end{aligned} \tag{2.1}$$

Proof. It suffices to note that

$$
\begin{aligned}
(2.2)\quad I_1 &= -\int_0^1 \left[h\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right) - h\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right.\\
&\qquad \left. + h(a+\eta(b,a))\right] f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right) dt \\
&= \frac{2}{\eta(b,a)}\left[h\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right) - h\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right.\\
&\qquad \left. + h(a+\eta(b,a))\right] \times f\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\Big|_0^1 \\
&\quad + \int_0^1 \left[h'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right) + h'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right] \\
&\qquad \times f\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right) \\
&= \frac{2}{\eta(b,a)}\left[h(a)f(a) - h(a+\eta(b,a))f\left(a+\frac{1}{2}\eta(b,a)\right)\right] \\
&\quad + \int_0^1 \left[h'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right) + h'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right] \\
&\qquad \times f\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right).
\end{aligned}
$$

and

$$
\begin{aligned}
(2.3)\quad I_2 &= \int_0^1 \left[h\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right) - h\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right.\\
&\qquad \left. + h(a+\eta(b,a))\right] f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right) dt \\
&= \frac{2}{\eta(b,a)}\left[h\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right) - h\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right.\\
&\qquad \left. + h(a+\eta(b,a))\right] f\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\Big|_0^1 \\
&\quad + \int_0^1 \left[h'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right) + h'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right] \\
&\qquad \times f\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right) \\
&= \frac{2}{\eta(b,a)}\left[h(a)f(a+\eta(b,a)) - h(a+\eta(b,a))f\left(a+\frac{1}{2}\eta(b,a)\right)\right] \\
&\quad + \int_0^1 \left[h'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right) + h'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right] \\
&\qquad \times f\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right).
\end{aligned}
$$

Thus from (2.2) and (2.3), we have

$$\frac{\eta(b,a)}{4}[I_1+I_2]=\frac{h(a)}{2}[f(a)+f(b)]-h(a+\eta(b,a))f\left(\frac{a+b}{2}\right)$$
$$+\frac{\eta(b,a)}{4}\left\{\int_0^1\left[h\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)-h\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right]\right.$$
$$\left.\times\left[-f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)+f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right]dt\right\}.$$

which is the required result. □

Remark 1. *If we take $\eta(b,a)=b-a$, then Lemma 1 reduces to Lemma 2.1 from* [10].

Now using Lemma 1, we shall propose some new upper bounds for the difference between the leftmost and the middle terms of weighted version of the Hermite-Hadamard type inequality (1.8) using preinvex and prequasiinvex mappings.

In what follows we use the notations $L^{'}(a,b,t)=a+\left(\frac{1-t}{2}\right)\eta(b,a)$ and $U^{'}(a,b,t)=a+\left(\frac{1+t}{2}\right)\eta(b,a)$ for our convenience.

Theorem 8. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$. Suppose $f:K\to\mathbb{R}$ is a differentiable mapping on K and $w:[a,a+\eta(b,a)]\to[0,\infty)$ be continuous and symmetric to $a+\frac{1}{2}\eta(b,a)$, where $a,b\in K$ with $\eta(b,a)>0$. If $\left|f^{'}\right|$ is preinvex on K, we have the following inequality*

$$\left|\int_a^{a+\eta(b,a)}f(x)w(x)\,dx-f\left(a+\frac{1}{2}\eta(b,a)\right)\int_a^{a+\eta(b,a)}w(x)\,dx\right|$$
$$\leq\frac{\eta(b,a)}{2}\left[\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right]\int_0^1M^{'}(w;a,b,t)\,dt,\tag{2.4}$$

where $M^{'}(w;a,b,t)=\int_a^{L^{'}(a,b,t)}w(x)\,dx$ for all $t\in[0,1]$.

Proof. Let $h(t)=\int_a^t w(x)\,dx$ for all $t\in[a,a+\eta(b,a)]$ in Lemma 1, we have

$$-f\left(a+\frac{1}{2}\eta(b,a)\right)\int_a^{a+\eta(b,a)}w(x)\,dx\tag{2.5}$$
$$+\frac{\eta(b,a)}{4}\int_0^1\left[f\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)+f\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right]$$
$$\times\left[w\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)+w\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right]dt$$
$$=\frac{\eta(b,a)}{4}\left\{\int_0^1\left[\int_a^{L^{'}(a,b,t)}w(x)\,dx+\int_{U^{'}(a,b,t)}^{a+\eta(b,a)}w(x)\,dx\right]\right.$$
$$\left.\times\left[-f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)+f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right]dt\right\}.$$

Since $w(x)$ is symmetric to $a+\frac{1}{2}\eta(b,a)$, we have

$$w\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)=w\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\tag{2.6}$$

and

$$\int_{a}^{L^{'}(a,b,t)} w(x)\,dx = \int_{U^{'}(a,b,t)}^{a+\eta(b,a)} w(x)\,dx \tag{2.7}$$

for all $t \in [0,1]$. Hence by using (2.6), we have

$$\begin{aligned}
&\frac{\eta(b,a)}{4}\int_{0}^{1}\left[f\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)+f\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right] \\
&\quad\times\left[w\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)+w\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right]dt \\
&\quad=\frac{\eta(b,a)}{2}\int_{0}^{1} f\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)w\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)dt \\
&\quad+\frac{\eta(b,a)}{2}\int_{0}^{1} f\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)w\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)dt \\
&=\int_{a}^{a+\frac{1}{2}\eta(b,a)} f(x)\,w(x)\,dx+\int_{a+\frac{1}{2}\eta(b,a)}^{a+\eta(b,a)} f(x)\,w(x)\,dx=\int_{a}^{a+\eta(b,a)} f(x)\,w(x)\,dx.
\end{aligned} \tag{2.8}$$

Using (2.7) and (2.8) in (2.5), we get

$$\begin{aligned}
&\left|\int_{a}^{a+\eta(b,a)} f(x)\,w(x)\,dx - f\left(a+\frac{1}{2}\eta(b,a)\right)\int_{a}^{a+\eta(b,a)} w(x)\,dx\right| \\
&\qquad\leq \frac{\eta(b,a)}{2}\int_{0}^{1} M^{'}(w;a,b,t) \\
&\qquad\times\left[\left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|+\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|\right]dt.
\end{aligned} \tag{2.9}$$

Now by using the preinvexity of $\left|f^{'}\right|$ on K, we obtain

$$\begin{aligned}
&\left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|+\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right| \\
&\qquad\leq \left|f^{'}(a)\right|+\left|f^{'}(b)\right|
\end{aligned} \tag{2.10}$$

for all $t \in [0,1]$. From (2.9) and (2.10) we get the the required inequality (2.4). This completes the proof of the theorem. □

Remark 2. *In Theorem 8, if we take* $w(x) = \frac{1}{\eta(b,a)}$ *for all* $x \in [a, a+\eta(b,a)]$, *then (2.4) becomes the inequality from* [46, Corollary 3.2].

Remark 3. *If* $\eta(b,a) = b-a$ *in Theorem 8, then (2.4) reduces to the inequality (1.4) from* [10].

Remark 4. *If* $\eta(b,a) = b-a$ *and* $w(x) = \frac{1}{b-a}$ *for all* $x \in [a,b]$ *in Theorem 8, we get the inequality (1.2).*

Theorem 9. *Let* $K \subseteq \mathbb{R}$ *be an open invex subset with respect to* $\eta : K \times K \to \mathbb{R}$. *Suppose* $f : K \to \mathbb{R}$ *is a differentiable mapping on* K *and* $w : [a, a+\eta(b,a)] \to$

$[0,\infty)$ be continuous and symmetric to $a+\frac{1}{2}\eta(b,a)$, where $a,\, b\in K$ with $\eta(b,a)>0$. If $\left|f^{'}\right|^{q}$ is preinvex on K for $q>1$, we have the following inequality

$$\left|\int_{a}^{a+\eta(b,a)} f(x)\,w(x)\,dx - f\left(a+\frac{1}{2}\eta(b,a)\right)\int_{a}^{a+\eta(b,a)} w(x)\,dx\right| \le \eta(b,a)\left[\frac{\left|f^{'}(a)\right|^{q}+\left|f^{'}(b)\right|^{q}}{2}\right]^{\frac{1}{q}}\left(\int_{0}^{1}\left[M^{'}(w;a,b,t)\right]^{p}dt\right)^{\frac{1}{p}}, \tag{2.11}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $M^{'}(w;a,b,t)$ is defined as in Theorem 8.

Proof. Continuing from inequality (2.9) in the proof of Theorem 8 and using the Hölder's integral inequality, we have

$$\left|\int_{a}^{a+\eta(b,a)} f(x)\,w(x)\,dx - f\left(a+\frac{1}{2}\eta(b,a)\right)\int_{a}^{a+\eta(b,a)} w(x)\,dx\right| \tag{2.12}$$
$$\le \frac{\eta(b,a)}{2}\left(\int_{0}^{1}\left[M^{'}(w;a,b,t)\right]^{p}dt\right)^{\frac{1}{p}}\left[\left(\int_{0}^{1}\left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^{q}dt\right)^{\frac{1}{q}} + \left(\int_{0}^{1}\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^{q}dt\right)^{\frac{1}{q}}\right].$$

By the power-mean inequality $t^{r}+s^{r}<2^{1-r}(t+s)^{r}$ for $t>0,\ s>0$ and $r<1$ and by the the preinvexity of $\left|f^{'}\right|^{q}$ on K for $q>1$, we have for every $a,b\in K$ with $\eta(b,a)>0$ the following inequality

(2.13)
$$\left(\int_{0}^{1}\left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^{q}dt\right)^{\frac{1}{q}}+\left(\int_{0}^{1}\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^{q}dt\right)^{\frac{1}{q}}$$
$$\le 2^{1-\frac{1}{q}}\left[\int_{0}^{1}\left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^{q}dt+\int_{0}^{1}\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^{q}dt\right]^{\frac{1}{q}}$$
$$\le 2^{1-\frac{1}{q}}\left[\int_{0}^{1}\left\{\left(\frac{1+t}{2}\right)\left|f^{'}(a)\right|^{q}+\left(\frac{1-t}{2}\right)\left|f^{'}(b)\right|^{q}\right.\right.$$
$$\left.\left.+\left(\frac{1-t}{2}\right)\left|f^{'}(a)\right|^{q}+\left(\frac{1+t}{2}\right)\left|f^{'}(b)\right|^{q}\right\}dt\right]^{\frac{1}{q}} = 2^{1-\frac{1}{q}}\left[\left|f^{'}(a)\right|^{q}+\left|f^{'}(b)\right|^{q}\right]^{\frac{1}{q}}.$$

Using the last inequality (2.13) in (2.12), we get the desired inequality. This completes the proof of the theorem as well. □

Remark 5. *In Theorem 9, if we take* $w(x) = \frac{1}{\eta(b,a)}$ *for all* $x \in [a, a+\eta(b,a)]$ *with* $\eta(b,a) > 0$*, then (2.11) becomes the following inequality*

$$\left| \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx - f\left(a + \frac{1}{2}\eta(b,a)\right) \right| \leq \frac{\eta(b,a)}{2(1+p)^{\frac{1}{p}}} \left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2} \right]^{\frac{1}{q}}, \tag{2.14}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 6. *If we take* $\eta(b,a) = b - a$ *in Theorem 9, then (2.11) becomes the following inequality*

$$\left| \int_a^b f(x)\,w(x)\,dx - f\left(\frac{a+b}{2}\right) \int_a^b w(x)\,dx \right| \leq (b-a) \left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2} \right]^{\frac{1}{q}} \left(\int_0^1 \left[M(w;a,b,t)\right]^p dt \right)^{\frac{1}{p}}, \tag{2.15}$$

where $M(w;a,b,t) = \int_a^{L(a,b,t)} w(x)\,dx$, $L(a,b,t) = \left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)b$ *for all* $t \in [0,1]$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

A similar result may be stated as follows.

Theorem 10. *Let* $K \subseteq \mathbb{R}$ *be an open invex subset with respect to* $\eta : K \times K \to \mathbb{R}$. *Suppose* $f : K \to \mathbb{R}$ *is a differentiable mapping on* K *and* $w : [a, a+\eta(b,a)] \to [0,\infty)$ *be continuous and symmetric to* $a + \frac{1}{2}\eta(b,a)$*, where* $a, b \in K$ *with* $\eta(b,a) > 0$. *If* $\left|f'\right|^q$ *is preinvex on* K *for* $q \geq 1$*, we have the following inequality*

$$\left| \int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx - f\left(a + \frac{1}{2}\eta(b,a)\right) \int_a^{a+\eta(b,a)} w(x)\,dx \right| \leq \eta(b,a) \left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2} \right]^{\frac{1}{q}} \int_0^1 M'(w;a,b,t)\,dt, \tag{2.16}$$

where $M'(w;a,b,t)$ *is defined as in Theorem 8.*

Proof. Continuing from inequality (2.9) in the proof of Theorem 8 and using the well known Hölder's integral inequality, we have

$$\left|\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx - f\left(a+\frac{1}{2}\eta(b,a)\right)\int_a^{a+\eta(b,a)} w(x)\,dx\right| \tag{2.17}$$
$$\leq \frac{\eta(b,a)}{2}\left(\int_0^1 M'(w;a,b,t)\,dt\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left[\int_0^1 M'(w;a,b,t)\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\int_0^1 M'(w;a,b,t)\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right]^{\frac{1}{q}}\right\}.$$

By the power-mean inequality $t^r + s^r < 2^{1-r}(t+s)^r$ for $t > 0$, $s > 0$, $r < 1$, and by the the preinvexity of $\left|f'\right|^q$ on K for $q > 1$, we have for every $a, b \in K$ with $\eta(b,a) > 0$ the following inequality

$$\left[\int_0^1 M'(w;a,b,t)\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right]^{\frac{1}{q}} \tag{2.18}$$
$$\left.+\left[\int_0^1 M'(w;a,b,t)\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right]^{\frac{1}{q}}\right\}$$
$$\leq 2^{1-\frac{1}{q}}\left(\int_0^1 M'(w;a,b,t)\,dt\right)^{\frac{1}{q}}\left[\left|f'(a)\right|^q + \left|f'(b)\right|^q\right]^{\frac{1}{q}}.$$

Utilizing inequality (2.18) in (2.17), we get the inequality (2.16). This completes the proof of the theorem. □

Corollary 1. *Suppose all the assumptions of Theorem 10 are satisfied and if* $w(x) = \frac{1}{\eta(b,a)}$ *for all* $x \in [a, a+\eta(b,a)]$ *with* $\eta(b,a) > 0$*, then we have the following inequality*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx - f\left(a+\frac{1}{2}\eta(b,a)\right)\right| \tag{2.19}$$
$$\leq \frac{\eta(b,a)}{4}\left[\frac{\left|f'(a)\right|^q + \left|f'(b)\right|^q}{2}\right]^{\frac{1}{q}}.$$

Remark 7. *If we take* $\eta(b,a) = b-a$ *in Theorem 10, then the inequality reduces to the inequality (1.5).*

Remark 8. *For* $q = 1$*, (2.19) recaptures the inequality proved in* [46, Corollary 3.2]. *If* $q = \frac{p}{p-1}$ $(p > 1)$*, we have* $2^p > p+1$ *for* $p > 1$ *and accordingly*

$$\frac{1}{4} < \frac{1}{2(p+1)^{\frac{1}{p}}}.$$

This reveals that the inequality (2.19) is better than the one given by (2.14). Moreover, for $\eta(b,a) = b-a$ *the inequality (2.19) takes the form of the inequality (1.3).*

Now we give our results for prequasiinvex functions.

Theorem 11. *Let K be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K and $w : [a, a+\eta(b,a)] \to [0,\infty)$ be continuous and symmetric to $a+\frac{1}{2}\eta(b,a)$, where $a, b \in K$ with $\eta(b,a) > 0$. If $\left|f'\right|$ is prequasiinvex on K, we have the following inequality*

$$\begin{aligned} (2.20)\qquad & \left| \int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx - f\left(a+\frac{1}{2}\eta(b,a)\right)\int_a^{a+\eta(b,a)} w(x)\,dx \right| \\ & \leq \frac{\eta(b,a)}{2}\left\{ \max\left(\left|f'(a)\right|, \left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right) \right. \\ & \left. + \max\left(\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|, \left|f'(a+\eta(b,a))\right|\right)\right\} \int_0^1 M'(w;a,b,t), \end{aligned}$$

where $M'(w;a,b,t)$ is defined as in Theorem 8.

Proof. We continue the inequality (2.9) in the proof of Theorem 8. Since $\left|f'\right|$ is prequasiinvex on K hence for every $t \in [0,1]$, we obtain,

$$(2.21)\qquad \left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right| \leq \max\left(\left|f'(a)\right|, \left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right)$$

and

$$(2.22)\qquad \left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right| \leq \max\left(\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|, \left|f'(a+\eta(b,a))\right|\right).$$

A combination of (2.9), (2.21) and (2.22) gives the required inequality (2.20). □

Corollary 2. *Suppose all the conditions of Theorem 11 are satisfied. Moreover,*

(1) *If $\left|f'\right|$ is non-decreasing on K, then the following inequality holds*

$$\begin{aligned} (2.23)\qquad & \left| \int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx - f\left(a+\frac{1}{2}\eta(b,a)\right)\int_a^{a+\eta(b,a)} w(x)\,dx \right| \\ & \leq \frac{\eta(b,a)}{2}\left[\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right| + \left|f'(a+\eta(b,a))\right|\right] \int_0^1 M'(w;a,b,t). \end{aligned}$$

and

(2) *If $\left|f'\right|$ is non-increasing on K, then the following inequality holds*

$$\begin{aligned} (2.24)\qquad & \left| \int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx - f\left(a+\frac{1}{2}\eta(b,a)\right)\int_a^{a+\eta(b,a)} w(x)\,dx \right| \\ & \leq \frac{\eta(b,a)}{2}\left[\left|f'(a)\right| + \left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right] \int_0^1 M'(w;a,b,t). \end{aligned}$$

Remark 9. *If in Theorem 11, we take* $w(x) = \frac{1}{\eta(b,a)}$ *for all* $x \in [a, a+\eta(b,a)]$ *with* $\eta(b,a) > 0$*, then we have the following inequality*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - f\left(a+\frac{1}{2}\eta(b,a)\right)\right| \le \frac{\eta(b,a)}{8}\left\{\max\left(\left|f'(a)\right|, \left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right) + \max\left(\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|, \left|f'(a+\eta(b,a))\right|\right)\right\}. \tag{2.25}$$

he inequality (2.25) represents a new refinement of the bound for the difference between the middle and the rightmost terms in (1.8) for prequasiinvex functions and hence for preinvex functions. Moreover,

(1) *If* $\left|f'\right|$ *is non-decreasing* K*, then the following inequality holds*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - f\left(a+\frac{1}{2}\eta(b,a)\right)\right| \le \frac{\eta(b,a)}{8}\left[\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right| + \left|f'(a+\eta(b,a))\right|\right] \tag{2.26}$$

and

(2) *If* $\left|f'\right|$ *is non-increasing* K*, then the following inequality holds*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - f\left(a+\frac{1}{2}\eta(b,a)\right)\right| \le \frac{\eta(b,a)}{8}\left[\left|f'(a)\right| + \left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right]. \tag{2.27}$$

Remark 10. *If* $\eta(b,a) = b-a$ *in Theorem 11, then (2.20) reduces to the inequality (1.6) established in Theorem 5 and the inequalities (2.26) and (2.27) recapture the related inequalities given in the corollary of Theorem 5 from* [10].

Remark 11. *If* $\eta(b,a) = b-a$ *in Remark 9, then (2.25), we get the following new results*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{8}\left\{\max\left(\left|f'(a)\right|, \left|f'\left(\frac{a+b}{2}\right)\right|\right) + \max\left(\left|f'\left(\frac{a+b}{2}\right)\right|, \left|f'(b)\right|\right)\right\}. \tag{2.28}$$

Moreover,

(1) *If* $\left|f'\right|$ *is non-decreasing* $[a,b]$*, then the following inequality holds*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{\eta(b,a)}{8}\left[\left|f'\left(\frac{a+b}{2}\right)\right| + \left|f'(b)\right|\right] \tag{2.29}$$

and

(2) *If* $\left|f^{'}\right|$ *is non-increasing* $[a,b]$, *then the following inequality holds*

$$\left|\frac{1}{b-a}\int_{a}^{b}f\left(x\right)dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{\eta\left(b,a\right)}{8}\left[\left|f^{'}\left(a\right)\right|+\left|f^{'}\left(\frac{a+b}{2}\right)\right|\right]. \tag{2.30}$$

Theorem 12. *Let* K *be an open invex subset with respect to* $\eta:K\times K\rightarrow\mathbb{R}$. *Suppose* $f:K\rightarrow\mathbb{R}$ *is a differentiable mapping on* K *and* $w:[a,a+\eta\left(b,a\right)]\rightarrow[0,\infty)$ *be continuous and symmetric to* $a+\frac{1}{2}\eta\left(b,a\right)$, *where* $a,b\in K$ *with* $\eta\left(b,a\right)>0$. *If* $\left|f^{'}\right|^{q}$ *is prequasiinvex on* K *for* $q>1$, *we have the following inequality:*

$$\left|\int_{a}^{a+\eta(b,a)}f\left(x\right)w\left(x\right)dx-f\left(a+\frac{1}{2}\eta\left(b,a\right)\right)\int_{a}^{a+\eta(b,a)}w\left(x\right)dx\right| \tag{2.31}$$
$$\leq\frac{\eta\left(b,a\right)}{2}\left(\int_{0}^{1}\left[M^{'}\left(w;a,b,t\right)\right]^{p}dt\right)^{\frac{1}{p}}\left\{\left[\max\left(\left|f^{'}\left(a\right)\right|^{q},\left|f^{'}\left(a+\frac{1}{2}\eta\left(b,a\right)\right)\right|^{q}\right)\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\max\left(\left|f^{'}\left(a+\frac{1}{2}\eta\left(b,a\right)\right)\right|^{q},\left|f^{'}\left(a+\eta\left(b,a\right)\right)\right|^{q}\right)\right]^{\frac{1}{q}}\right\},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. We continue inequality (2.12) in the proof of Theorem 9. By the prequasiinvexity of $\left|f^{'}\right|^{q}$ on K for $q>1$, we have for every $t\in[0,1]$

$$\left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta\left(b,a\right)\right)\right|^{q}\leq\max\left\{\left|f^{'}\left(a\right)\right|^{q},\left|f^{'}\left(a+\frac{1}{2}\eta\left(b,a\right)\right)\right|^{q}\right\} \tag{2.32}$$

and

$$\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta\left(b,a\right)\right)\right|^{q} \tag{2.33}$$
$$\leq\max\left\{\left|f^{'}\left(a+\frac{1}{2}\eta\left(b,a\right)\right)\right|^{q},\left|f^{'}\left(a+\eta\left(b,a\right)\right)\right|^{q}\right\}.$$

A combination of (2.12), (2.32) and (2.33) gives us the required inequality (2.31). This completes the proof of the Theorem. □

Corollary 3. *Suppose all the conditions of Theorem 12 are satisfied. Moreover*

(1) *If* $\left|f^{'}\right|^{q}$ *is non-decreasing on* K, *then the following inequality holds*

$$\left|\int_{a}^{a+\eta(b,a)}f\left(x\right)w\left(x\right)dx-f\left(a+\frac{1}{2}\eta\left(b,a\right)\right)\int_{a}^{a+\eta(b,a)}w\left(x\right)dx\right| \tag{2.34}$$
$$\leq\frac{\eta\left(b,a\right)}{2}\left[\left|f^{'}\left(a+\frac{1}{2}\eta\left(b,a\right)\right)\right|+\left|f^{'}\left(a+\eta\left(b,a\right)\right)\right|\right]\left(\int_{0}^{1}\left[M^{'}\left(w;a,b,t\right)\right]^{p}dt\right)^{\frac{1}{p}}$$

and

(2) *If* $\left|f^{'}\right|^{q}$ *is non-increasing on* K*, then the following inequality holds*

$$\left|\int_a^{a+\eta(b,a)} f(x)\,w(x)\,dx - f\left(a+\frac{1}{2}\eta(b,a)\right)\int_a^{a+\eta(b,a)} w(x)\,dx\right| \tag{2.35}$$
$$\leq \frac{\eta(b,a)}{2}\left[\left|f^{'}(a)\right|+\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right]\left(\int_0^1\left[M^{'}(w;a,b,t)\right]^p dt\right)^{\frac{1}{p}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Remark 12. *If in Theorem 12, we take* $w(x)=\frac{1}{\eta(b,a)}$ *for all* $x\in[a,a+\eta(b,a)]$ *with* $\eta(b,a)>0$*, then we have the following inequality:*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - f\left(a+\frac{1}{2}\eta(b,a)\right)\right| \tag{2.36}$$
$$\leq \frac{\eta(b,a)}{4(p+1)^{\frac{1}{p}}}\left[\max\left(\left|f^{'}(a)\right|,\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right)\right.$$
$$\left.+\max\left(\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|,\left|f^{'}(a+\eta(b,a))\right|\right)\right].$$

The inequality (2.36) represents a new refinement of the bound for the difference between the middle and the rightmost terms in (1.8) for prequasiinvex functions and hence for preinvex functions. Moreover,

(1) *If* $\left|f^{'}\right|$ *is non-decreasing on* K*, the following inequality holds*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - f\left(a+\frac{1}{2}\eta(b,a)\right)\right| \tag{2.37}$$
$$\leq \frac{\eta(b,a)}{4(p+1)^{\frac{1}{p}}}\left[\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|+\left|f^{'}(a+\eta(b,a))\right|\right]$$

and

(2) *If* $\left|f^{'}\right|$ *is non-increasing on* K*, the following inequality holds:*

$$\left|\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx - f\left(a+\frac{1}{2}\eta(b,a)\right)\right| \tag{2.38}$$
$$\leq \frac{\eta(b,a)}{4(p+1)^{\frac{1}{p}}}\left[\left|f^{'}(a)\right|+\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Remark 13. *If we take* $\eta(b,a)=b-a$ *in Remark 12, we get the results for quasi-convex functions.*

Theorem 13. *Let* K *be an open invex subset with respect to* $\eta: K\times K\to\mathbb{R}$. *Suppose* $f: K\to\mathbb{R}$ *is a differentiable mapping on* K *and* $w:[a,a+\eta(b,a)]\to[0,\infty)$ *be continuous and symmetric to* $a+\frac{1}{2}\eta(b,a)$*, where* $a,b\in K$ *with* $\eta(b,a)>0$.

If $\left|f^{'}\right|^{q}$ *is prequasiinvex on* K *for* $q\geq 1$*, we have the following inequality*

$$\begin{aligned}(2.39)\quad &\left|\int_{a}^{a+\eta(b,a)} f\left(x\right)w\left(x\right)dx - f\left(a+\frac{1}{2}\eta\left(b,a\right)\right)\int_{a}^{a+\eta(b,a)} w\left(x\right)dx\right| \\ &\leq \frac{\eta\left(b,a\right)}{2}\left\{\left[\max\left(\left|f^{'}\left(a+\frac{1}{2}\eta\left(b,a\right)\right)\right|^{q},\left|f^{'}\left(a+\eta\left(b,a\right)\right)\right|^{q}\right)\right]^{\frac{1}{q}}\right. \\ &\quad \left.+\left[\max\left(\left|f^{'}\left(a\right)\right|^{q},\left|f^{'}\left(a+\frac{1}{2}\eta\left(b,a\right)\right)\right|^{q}\right)\right]^{\frac{1}{q}}\right\}\int_{0}^{1}M^{'}\left(w;a,b,t\right)dt,\end{aligned}$$

where $M^{'}\left(w;a,b,t\right)$ *is defined as in Theorem 8.*

Proof. We continue inequality (2.17) in the proof of Theorem 10. By the prequasiinvex of $\left|f^{'}\right|^{q}$ on K for $q\geq 1$, we have for every $t\in[0,1]$

$$(2.40)\quad \left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta\left(b,a\right)\right)\right|^{q}\leq\max\left\{\left|f^{'}\left(a\right)\right|^{q},\left|f^{'}\left(a+\frac{1}{2}\eta\left(b,a\right)\right)\right|^{q}\right\}$$

and
(2.41)

$$\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta\left(b,a\right)\right)\right|^{q}\leq\max\left\{\left|f^{'}\left(a+\frac{1}{2}\eta\left(b,a\right)\right)\right|^{q},\left|f^{'}\left(a+\eta\left(b,a\right)\right)\right|^{q}\right\}.$$

A combination of (2.17), (2.40) and (2.41) gives us the required inequality (2.39). This completes the proof of the Theorem. □

Corollary 4. *Suppose all the conditions of Theorem 13 are satisfied. Moreover*

(1) *If* $\left|f^{'}\right|^{q}$ *is non-decreasing on* K*, then the following inequality holds*

$$\begin{aligned}(2.42)\quad &\left|\int_{a}^{a+\eta(b,a)} f\left(x\right)w\left(x\right)dx - f\left(a+\frac{1}{2}\eta\left(b,a\right)\right)\int_{a}^{a+\eta(b,a)} w\left(x\right)dx\right| \\ &\leq \frac{\eta\left(b,a\right)}{2}\left[\left|f^{'}\left(a+\frac{1}{2}\eta\left(b,a\right)\right)\right|+\left|f^{'}\left(a+\eta\left(b,a\right)\right)\right|\right]\int_{0}^{1}M^{'}\left(w;a,b,t\right)dt.\end{aligned}$$

and

(2) *If* $\left|f^{'}\right|^{q}$ *is non-increasing on* K*, then the the following inequality holds*

$$\begin{aligned}(2.43)\quad &\left|\int_{a}^{a+\eta(b,a)} f\left(x\right)w\left(x\right)dx - f\left(a+\frac{1}{2}\eta\left(b,a\right)\right)\int_{a}^{a+\eta(b,a)} w\left(x\right)dx\right| \\ &\leq \frac{\eta\left(b,a\right)}{2}\left[\left|f^{'}\left(a\right)\right|+\left|f^{'}\left(a+\frac{1}{2}\eta\left(b,a\right)\right)\right|\right]\int_{0}^{1}M^{'}\left(w;a,b,t\right)dt.\end{aligned}$$

Remark 14. *If in Theorem 13, we take* $w(x) = \frac{1}{\eta(b,a)}$ *for all* $x \in [a, a+\eta(b,a)]$ *with* $\eta(b,a) > 0$*, we have the following inequality*

$$\left| \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx - f\left(a + \frac{1}{2}\eta(b,a)\right) \right| \tag{2.44}$$
$$\leq \frac{\eta(b,a)}{8} \left\{ \left[\max\left(\left| f'\left(a + \frac{1}{2}\eta(b,a)\right) \right|^q, \left| f'(a+\eta(b,a)) \right|^q \right) \right]^{\frac{1}{q}} \right.$$
$$\left. + \left[\max\left(\left| f'(a) \right|^q, \left| f'\left(a + \frac{1}{2}\eta(b,a)\right) \right|^q \right) \right]^{\frac{1}{q}} \right\}.$$

Moreover,

1. *If* $\left|f'\right|^q$ *is non-decreasing on* K*, the inequality (2.26) holds and*
2. *If* $\left|f'\right|^q$ *is non-increasing on* K*, the inequality (2.27) holds.*

Remark 15. *If* $\eta(b,a) = b - a$ *in Theorem 13, then (2.39) reduces to the inequality (1.7) established in Theorem 6 and the inequalities (2.42) and (2.43) recapture the related inequalities established in the corollary of Theorem 6 from* [10].

Remark 16. *If* $\eta(b,a) = b - a$ *in Remark 14, we get results for qusi-convex functions.*

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