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# ON HERMITE -HADAMARD TYPE INEQUALITIES FOR $\varphi-$CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

MEHMET ZEKI SARIKAYA AND HATICE YALDIZ

Abstract. In this paper, we establish integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals for $\varphi$-convex functions and some new inequalities of right-hand side of Hermite-Hadamard type are given for functions whose first derivatives absolute values $\varphi-$convex functions via Riemann-Liouville fractional integrals.

## 1. Introduction

The function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$, is said to be convex if the following inequality holds

$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. We say that $f$ is concave if $(-f)$ is convex.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g.,[11, p.137], [7]). These inequalities state that if $f:I\to\mathbb{R}$ is a convex function on the interval $I$ of real numbers and $a,b\in I$ with $a<b$, then

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if $f$ is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [1, 7, 8]) and the references cited therein.

In [8], Dragomir and Agarwal proved the following results connected with the right part of (1.1).

**Lemma 1.** *Let $f:I^\circ\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on $I^\circ$, $a,b\in I^\circ$ with $a<b$. If $f'\in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx=\frac{b-a}{2}\int_0^1(1-2t)f'(ta+(1-t)b)dt. \tag{1.2}$$

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**Theorem 1.** *Let $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^{\circ}$, $a, b \in I^{\circ}$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)}{8} \left( |f'(a)| + |f'(b)| \right). \tag{1.3}$$

Meanwhile, Sarikaya et al.[14] presented the following important integral identity including the first-order derivative of $f$ to establish many interesting Hermite-Hadamard type inequalities for convexity functions via Riemann-Liouville fractional integrals of the order $\alpha > 0$.

**Lemma 2.** *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on $(a, b)$ with $a < b$. If $f' \in L[a, b]$, then the following equality for fractional integrals holds:*

$$\begin{aligned} & \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[ J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) \right] \\ = \; & \frac{b-a}{2} \int_0^1 \left[ (1-t)^{\alpha} - t^{\alpha} \right] f'(ta + (1-t)b)\, dt. \end{aligned} \tag{1.4}$$

It is remarkable that Sarikaya et al.[14] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

**Theorem 2.** *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If $f$ is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left( \frac{a+b}{2} \right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[ J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) \right] \leq \frac{f(a) + f(b)}{2} \tag{1.5}$$

*with $\alpha > 0$.*

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult [9, 10].

**Definition 1.** *Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

*and*

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

*respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.*

For some recent results connected with fractional integral inequalities see ([2, 3, 4, 5, 6],[12],[15]).

In [16], Youness have defined the $\varphi$-convex function as follows:

**Definition 2.** *Let $\varphi : [a, b] \subset \mathbb{R} \to [a, b]$. A function $f : [a, b] \to \mathbb{R}$ is said to be $\varphi-$convex on $[a, b]$ if, for every $x, y \in [a, b]$ and $\lambda \in [0, 1]$, the following inequality holds:*

$$f(\lambda\varphi(x) + (1-\lambda)\varphi(y)) \leq \lambda f(\varphi(x)) + (1-\lambda) f(\varphi(y)).$$

Obviously, if $\varphi(x) = x$, then the classical convexity is obtained from the previous defition.

In [13], Sarikaya et. all gave the following important inequalities for $\varphi-$convex mappings :

**Theorem 3.** *Let $J$ be an interval $a, b \in J$ with $a < b$ and $\varphi : J \to \mathbb{R}$ a continuous increasing function. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a $\varphi$-convex function on $I = [\varphi(a), \varphi(b)]$, then we have*

(1.6)
$$f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) \leq \frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\, d\varphi(x) \leq \frac{f(\varphi(a)) + f(\varphi(b))}{2}.$$

In this paper, by using $\varphi-$convex mappings, we give Hermite-Hadamard's inequalities for Riemann-Liouville fractional integral and some other integral inequalities using the identity is obtained for fractional integrals.

## 2. Main Results

Hermite-Hadamard's inequalities can be represented in fractional integral forms as follows:

**Theorem 4.** *Let $J$ be an interval $a, b \in J$ with $a < b$ and $\varphi : J \to \mathbb{R}$ a continuous increasing function. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a $\varphi$-convex function on $I = [\varphi(a), \varphi(b)]$, then the following inequalities for fractional integrals hold:*

$$\begin{aligned} & f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) \\ \leq\ & \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}} \left[ J^{\alpha}_{\varphi(a)^+} f(\varphi(b)) + J^{\alpha}_{\varphi(b)^-} f(\varphi(a)) \right] \\ \leq\ & \frac{f(\varphi(a)) + f(\varphi(b))}{2} \end{aligned} \tag{2.1}$$

*with $\alpha > 0$.*

*Proof.* Since $f$ is a $\varphi$-convex function on $[a, b]$, we have for $\varphi(x), \varphi(y) \in [\varphi(a), \varphi(b)]$ with $\lambda = \frac{1}{2}$

$$f\left(\frac{\varphi(x)+\varphi(y)}{2}\right) \leq \frac{f(\varphi(x)) + f(\varphi(y))}{2} \tag{2.2}$$

i.e., with $\varphi(x) = t\varphi(a) + (1-t)\varphi(b)$, $\varphi(y) = (1-t)\varphi(a) + t\varphi(b)$,

$$2f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) \leq f(t\varphi(a) + (1-t)\varphi(b)) + f((1-t)\varphi(a) + t\varphi(b)). \tag{2.3}$$

Multiplying both sides of (2.3) by $t^{\alpha-1}$, then integrating the resulting inequality with respest to $t$ over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{2}{\alpha}f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\\
\leq\;& \int_0^1 t^{\alpha-1}f\left(t\varphi(a)+(1-t)\varphi(b)\right)dt+\int_0^1 t^{\alpha-1}f\left((1-t)\varphi(a)+t\varphi(b)\right)dt\\
=\;& \int_{\varphi(b)}^{\varphi(a)}\left(\frac{\varphi(b)-\varphi(u)}{\varphi(b)-\varphi(a)}\right)^{\alpha-1}f(\varphi(u))\frac{d\varphi(u)}{\varphi(a)-\varphi(b)}\\
&+\int_{\varphi(a)}^{\varphi(b)}\left(\frac{\varphi(v)-\varphi(a)}{\varphi(b)-\varphi(a)}\right)^{\alpha-1}f(\varphi(v))\frac{d\varphi(v)}{\varphi(b)-\varphi(a)}\\
=\;& \frac{\Gamma(\alpha)}{(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)^+}f(\varphi(b))+J^{\alpha}_{\varphi(b)^-}f(\varphi(a))\right]
\end{aligned}$$

i.e.

$$f\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)^+}f(\varphi(b))+J^{\alpha}_{\varphi(b)^-}f(\varphi(a))\right]$$

and the first inequality is proved.

For the proof of the second inequality in (2.2) we first note that if $f$ is a $\varphi$-convex function, then, for $t\in[0,1]$, it yields

$$f\left(t\varphi(a)+(1-t)\varphi(b)\right)\leq tf(\varphi(a))+(1-t)f(\varphi(b))$$

and

$$f\left((1-t)\varphi(a)+t\varphi(b)\right)\leq(1-t)f(\varphi(a))+tf(\varphi(b)).$$

By adding these inequalities we have

$$\begin{aligned}
&f\left(t\varphi(a)+(1-t)\varphi(b)\right)+f\left((1-t)\varphi(a)+t\varphi(b)\right)\\
\leq\;& tf(\varphi(a))+(1-t)f(\varphi(b))+(1-t)f(\varphi(a))+tf(\varphi(b)).
\end{aligned}\tag{2.4}$$

Then multiplying both sides of (2.4) by $t^{\alpha-1}$and integrating the resulting inequality with respest to $t$ over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 t^{\alpha-1}f\left(t\varphi(a)+(1-t)\varphi(b)\right)dt+\int_0^1 t^{\alpha-1}f\left((1-t)\varphi(a)+t\varphi(b)\right)dt\\
\leq\;& [f(\varphi(a))+f(\varphi(b))]\int_0^1 t^{\alpha-1}dt
\end{aligned}$$

i.e.

$$\frac{\Gamma(\alpha)}{(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)^+}f(\varphi(b))+J^{\alpha}_{\varphi(b)^-}f(\varphi(a))\right]\leq\frac{f(\varphi(a))+f(\varphi(b))}{\alpha}.$$

The proof is completed. □

**Remark 1.** *If in Theorem 4, we let* $\alpha=1$, *then the inequalities (2.1) become the inequalities (1.6) of Theorem 3.*

To prove our main results, we need the following lemma:

**Lemma 3.** *Let $J$ be an interval $a,b \in J$ with $0 \leq a < b$ and $\varphi : J \to \mathbb{R}$ a continuous increasing function. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differantiable function on $I^{\circ}$ (the interior $I$). If $f' \in L_1[\varphi(a), \varphi(b)]$ for $\varphi(a), \varphi(b) \in I$, then the following equality holds:*

$$\begin{aligned} & \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)^{+}} f(\varphi(b)) + J^{\alpha}_{\varphi(b)^{-}} f(\varphi(a))\right] \\ (2.5)\quad & = \frac{\varphi(b)-\varphi(a)}{2}\int_0^1 \left[(1-t)^{\alpha} - t^{\alpha}\right] f'(t\varphi(a) + (1-t)\varphi(b))\,dt. \end{aligned}$$

*where $\alpha > 0$.*

*Proof.* It suffices to note that

$$\begin{aligned} (2.6)\qquad I &= \int_0^1 \left[(1-t)^{\alpha} - t^{\alpha}\right] f'(t\varphi(a) + (1-t)\varphi(b))\,dt \\ &= \left[\int_0^1 (1-t)^{\alpha} f'(t\varphi(a) + (1-t)\varphi(b))\,dt\right] \\ &\quad + \left[-\int_0^1 t^{\alpha} f'(t\varphi(a) + (1-t)\varphi(b))\,dt\right] \\ &= I_1 + I_2. \end{aligned}$$

Integrating by parts

$$\begin{aligned} I_1 &= \int_0^1 (1-t)^{\alpha} f'(t\varphi(a) + (1-t)\varphi(b))\,dt \\ &= (1-t)^{\alpha}\frac{f(t\varphi(a) + (1-t)\varphi(b))}{\varphi(a)-\varphi(b)}\Big|_0^1 + \int_0^1 \alpha(1-t)^{\alpha}\frac{f(t\varphi(a) + (1-t)\varphi(b))}{\varphi(a)-\varphi(b)}dt \\ (2.7)\quad &= \frac{f(\varphi(b))}{\varphi(b)-\varphi(a)} - \frac{\alpha}{\varphi(b)-\varphi(a)}\int_{\varphi(b)}^{\varphi(a)}\left(\frac{\varphi(a)-\varphi(x)}{\varphi(a)-\varphi(b)}\right)^{\alpha-1}\frac{f(\varphi(x))}{\varphi(a)-\varphi(b)}d\varphi(x) \\ &= \frac{f(\varphi(b))}{\varphi(b)-\varphi(a)} - \frac{\Gamma(\alpha+1)}{(\varphi(b)-\varphi(a))^{\alpha+1}} J^{\alpha}_{\varphi(b)^{-}} f(\varphi(a)) \end{aligned}$$

and similarly we get,

$$\begin{aligned} I_2 &= -\int_0^1 t^{\alpha} f'(t\varphi(a) + (1-t)\varphi(b))\,dt \\ (2.8)\quad & -\frac{t^{\alpha} f(t\varphi(a) + (1-t)\varphi(b))}{\varphi(a)-\varphi(b)}\Big|_0^1 + \alpha\int_0^1 t^{\alpha-1}\frac{f(t\varphi(a) + (1-t)\varphi(b))}{\varphi(a)-\varphi(b)}dt \\ &= \frac{f(\varphi(a))}{\varphi(b)-\varphi(a)} - \frac{\alpha}{\varphi(b)-\varphi(a)}\int_{\varphi(b)}^{\varphi(a)}\left(\frac{\varphi(b)-\varphi(x)}{\varphi(b)-\varphi(a)}\right)^{\alpha-1}\frac{f(\varphi(x))}{\varphi(a)-\varphi(b)}d\varphi(x) \\ &= \frac{f(\varphi(a))}{\varphi(b)-\varphi(a)} - \frac{\Gamma(\alpha+1)}{(\varphi(b)-\varphi(a))^{\alpha+1}} J^{\alpha}_{\varphi(a)^{+}} f(\varphi(b)). \end{aligned}$$

Using (2.7) and (2.8) in (2.6), it follows that

$$I=\frac{f(\varphi(a))+f(\varphi(b))}{\varphi(b)-\varphi(a)}-\frac{\Gamma(\alpha+1)}{(\varphi(b)-\varphi(a))^{\alpha+1}}\left[J^{\alpha}_{\varphi(a)^{+}}f(\varphi(b))+J^{\alpha}_{\varphi(b)^{-}}f(\varphi(a))\right].$$

Thus, by multiplying the both sides by $\frac{\varphi(b)-\varphi(a)}{2}$, we have the conclusion (2.5). □

**Remark 2.** *If we take $\varphi(x)=x$ in Lemma 3, then the inedentity (2.5) reduces to the identity (1.4).*

By using this Lemma, we can obtain the following fractional integral inequality:

**Theorem 5.** *Let $J$ be an interval $a,b\in J$ with $0\leq a<b$ and $\varphi:J\to\mathbb{R}$ a continuous increasing function. Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differantiable function on $I^{\circ}$(the interior $I$) and $f'\in L_1[\varphi(a),\varphi(b)]$ for $\varphi(a),\varphi(b)\in I$. If $|f'|^q$ is the $\varphi$-convex on $[\varphi(a),\varphi(b)]$, $q\geq1$, then the following inequaliy holds:*

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\right.$$
$$\left.\times\left[J^{\alpha}_{(\varphi(a))^{+}}f(\varphi(b))+J^{\alpha}_{(\varphi(b))^{-}}f(\varphi(a)\right]\right|$$
$$\leq\frac{\varphi(b)-\varphi(a)}{2^{\frac{2-q}{q}}}\left[\frac{1}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\right]\left[|f'(\varphi(a))|^q+|f'(\varphi(b))|^q\right]^{\frac{1}{q}}. \tag{2.9}$$

where $\alpha>0$.

*Proof.* Firstly, we suppose that $q=1$. Using Lemma 3 and $\varphi$-convexity of $|f'|^q$, we find that

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{(\varphi(a))^{+}}f(\varphi(b))+J^{\alpha}_{(\varphi(b))^{-}}f(\varphi(a))\right]\right|$$
$$\leq\frac{\varphi(b)-\varphi(a)}{2}\int_0^1|(1-t)^{\alpha}-t^{\alpha}|\left|f'\left[t(\varphi(a))+(1-t)(\varphi(b))\right]\right|dt$$
$$\leq\frac{\varphi(b)-\varphi(a)}{2}\int_0^1|(1-t)^{\alpha}-t^{\alpha}|\left[t\left|f'(\varphi(a))\right|+(1-t)\left|f'(\varphi(b))\right|\right]dt$$
$$=\frac{\varphi(b)-\varphi(a)}{2}\left\{\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]\left[t\left|f'(\varphi(a))\right|+(1-t)\left|f'(\varphi(b))\right|\right]dt\right.$$
$$\left.+\int_{\frac{1}{2}}^1\left[t^{\alpha}-(1-t)^{\alpha}\right]\left[t\left|f'(\varphi(a))\right|+(1-t)\left|f'(\varphi(b))\right|\right]dt\right\}$$
$$\frac{\varphi(b)-\varphi(a)}{2}\left\{K_1+K_2\right\}. \tag{2.10}$$

Hence, conculating $K_1$ ve $K_2$, we have

$$
\begin{aligned}
K_1 &= \int_0^{\frac{1}{2}} \left[(1-t)^\alpha - t^\alpha\right]\left[t\left|f'(\varphi(a))\right| + (1-t)\left|f'(\varphi(b))\right|\right] dt \\
&= \left|f'(\varphi(a))\right| \int_0^{\frac{1}{2}} \left[(1-t)^\alpha t - t^{\alpha+1}\right] dt + \left|f'(\varphi(b))\right| \int_0^{\frac{1}{2}} \left[(1-t)^{\alpha+1} - t^\alpha(1-t)\right] dt \\
&= \left|f'(\varphi(a))\right| \left[\frac{1}{(\alpha+1)(\alpha+2)} - \frac{1}{2^{\alpha+1}(\alpha+1)}\right] + \left|f'(\varphi(b))\right| \left[\frac{1}{\alpha+2} - \frac{1}{2^{\alpha+1}(\alpha+1)}\right]
\end{aligned}
\tag{2.11}
$$

and

$$
\begin{aligned}
K_2 &= \int_{\frac{1}{2}}^{1} \left[t^\alpha - (1-t)^\alpha\right]\left[t\left|f'(\varphi(a))\right| + (1-t)\left|f'(\varphi(b))\right|\right] dt \\
&= \left|f'(\varphi(a))\right| \left[\frac{1}{\alpha+2} - \frac{1}{2^{\alpha+1}(\alpha+1)}\right] + \left|f'(\varphi(b))\right| \left[\frac{1}{(\alpha+2)(\alpha+1)} - \frac{1}{2^{\alpha+1}(\alpha+1)}\right].
\end{aligned}
\tag{2.12}
$$

Using (2.11) and (2.12) in (2.10), it follows that

$$
\begin{aligned}
&\left|\frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^\alpha}\left[J^\alpha_{(\varphi(a))+} f(\varphi(b)) + J^\alpha_{(\varphi(b))-} f(\varphi(a))\right]\right| \\
\leq\ & \frac{\varphi(b)-\varphi(a)}{2}\frac{1}{\alpha+1}\left[1 - \frac{1}{2^\alpha}\right]\left[\left|f'(\varphi(a))\right| + \left|f'(\varphi(b))\right|\right].
\end{aligned}
$$

Secondly, we suppose that $q > 1$. Using Lemma 3 and power mean inequality, we obtain

$$
\begin{aligned}
&\int_0^1 \left|(1-t)^\alpha - t^\alpha\right| \left|f'\left[t(\varphi(a)) + (1-t)(\varphi(b))\right]\right| dt \\
\leq\ & \left(\int_0^1 \left|(1-t)^\alpha - t^\alpha\right| dt\right)^{1-\frac{1}{q}} \left(\int_0^1 \left|(1-t)^\alpha - t^\alpha\right| \left|f'\left[t(\varphi(a)) + (1-t)(\varphi(b))\right]\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}
\tag{2.13}
$$

Hence, using $\varphi$-convexity of $|f'|^q$ and (2.13) we obtain

$$
\begin{aligned}
&\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{(\varphi(a))+}f(\varphi(b))+J^{\alpha}_{(\varphi(b))-}f(\varphi(a))\right]\right| \\
\leq\;& \frac{\varphi(b)-\varphi(a)}{2}\left(\int_0^1 |(1-t)^{\alpha}-t^{\alpha}|\,dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 |(1-t)^{\alpha}-t^{\alpha}|\,|f'[t(\varphi(a))+(1-t)(\varphi(b))]|^q\,dt\right)^{\frac{1}{q}} \\
\leq\;& \frac{\varphi(b)-\varphi(a)}{2}\left(\int_0^{\frac{1}{2}} [(1-t)^{\alpha}-t^{\alpha}]\,dt+\int_{\frac{1}{2}}^1 [t^{\alpha}-(1-t)^{\alpha}]\,dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 |(1-t)^{\alpha}-t^{\alpha}|\left[t\,|f'(\varphi(a))|^q+(1-t)\,|f'(\varphi(b))|^q\right]dt\right)^{\frac{1}{q}} \\
\leq\;& \frac{\varphi(b)-\varphi(a)}{2}2^{\frac{q-1}{q}}\left[\frac{1}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\right]\left[|f'(\varphi(a))|^q+|f'(\varphi(b))|^q\right]^{\frac{1}{q}}
\end{aligned}
$$

which completes the proof. $\square$

**Theorem 6.** *Let $J$ be an interval $a,b\in J$ with $0\leq a<b$ and $\varphi:J\to\mathbb{R}$ a continuous increasing function. Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differantiable function on $I^{\circ}$(the interior $I$) and $f'\in L_1[\varphi(a),\varphi(b)]$ for $\varphi(a),\varphi(b)\in I$. If $|f'|^q$ is the $\varphi$-convex on $[\varphi(a),\varphi(b)]$ , $q>1$, then the following inequaliy holds:*

$$
\begin{aligned}
&\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)+}f(\varphi(b))+J^{\alpha}_{\varphi(b)-}f(\varphi(a))\right]\right| \\
&\leq \frac{\varphi(b)-\varphi(a)}{2}\left[\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right]^{\frac{1}{p}}\left(\frac{|f'(\varphi(a))|^q+|f'(\varphi(b))|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}
\tag{2.14}
$$

*where $\alpha>0$ and $\frac{1}{p}+\frac{1}{q}=1$.*

*Proof.* Using Lemma 3, $\varphi$-convexity of $|f|^q$ and well-known Hölder's inequality, we obtain

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)^{+}}f(\varphi(b))+J^{\alpha}_{\varphi(b)^{-}}f(\varphi(a))\right]\right|$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\int_0^1 |(1-t)^{\alpha}-t^{\alpha}|\,|f'[t(\varphi(a))+(1-t)(\varphi(b))]|\,dt$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\left(\int_0^1 |(1-t)^{\alpha}-t^{\alpha}|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'[t(\varphi(a))+(1-t)(\varphi(b))]|^q\,dt\right)^{\frac{1}{q}}$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\left(\int_0^{\frac{1}{2}} [(1-t)^{\alpha}-t^{\alpha}]^p\,dt+\int_{\frac{1}{2}}^1 [t^{\alpha}-(1-t)^{\alpha}]^p\,dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \left[t\,|f'(\varphi(a))|^q+(1-t)\,|f'(\varphi(b))|^q\right]dt\right)^{\frac{1}{q}}$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\left(\int_0^{\frac{1}{2}} [(1-t)^{\alpha p}-t^{\alpha p}]\,dt+\int_{\frac{1}{2}}^1 [t^{\alpha p}-(1-t)^{\alpha p}]\,dt\right)^{\frac{1}{p}}$$

$$\times\left(\frac{|f'(\varphi(a))|^q+|f'(\varphi(b))|^q}{2}\right)^{\frac{1}{q}}$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\left[\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right]^{\frac{1}{p}}\left(\frac{|f'(\varphi(a))|^q+|f'(\varphi(b))|^q}{2}\right)^{\frac{1}{q}}.$$

Here, we use $(A-B)^p \leq A^p - B^p$ for any $A > B \geq 0$ and $p \geq 1$. □

**Theorem 7.** *Let $J$ be an interval $a, b \in J$ with $0 \leq a < b$ and $\varphi : J \to \mathbb{R}$ a continuous increasing function. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differantiable function on $I^{\circ}$(the interior $I$) and $f' \in L_1[\varphi(a), \varphi(b)]$ for $\varphi(a), \varphi(b) \in I$. If $|f'|^q$ $\varphi$- convex on $[\varphi(a), \varphi(b)]$ for same fixed $q \geq 1$,then the following inequality for fractional integrals holds:*

$$\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)^{+}}f(\varphi(b))+J^{\alpha}_{\varphi(b)^{-}}f(\varphi(a))\right]$$

$$\leq \left[\frac{1}{q\alpha+1}\left(1-\frac{1}{2^{q\alpha+1}}\right)\right]^{\frac{1}{q}}\left(\frac{|f'(\varphi(a))|^q+|f'(\varphi(b))|^q}{2}\right)^{\frac{1}{q}}$$

*where $\alpha > 0$.*

*Proof.* Using Lemma 3, $\varphi$-convexity of $|f'|^q$, and well-known Hölder's inequality, we have

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)^+}f(\varphi(b))+J^{\alpha}_{\varphi(b)^-}f(\varphi(b))\right]\right|$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\int_0^1 |(1-t)^{\alpha}-t^{\alpha}|\,|f'[t(\varphi(a))+(1-t)(\varphi(b))]|\,dt$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\left(\int_0^1 1^p dt\right)^{\frac{1}{p}}\left(\int_0^1 |(1-t)^{\alpha}-t^{\alpha}|^q\,|f'[t(\varphi(a))+(1-t)(\varphi(b))]^q|\,dt\right)^{\frac{1}{q}}$$

$$= \frac{\varphi(b)-\varphi(a)}{2}\left(\int_0^{\frac{1}{2}}[(1-t)^{\alpha}-t^{\alpha}]^q\,|f'[t(\varphi(a))+(1-t)(\varphi(b)]|^q\,dt\right.$$

$$\left.+\int_0^{\frac{1}{2}}[t^{\alpha}-(1-t)^{\alpha}]^q\,|f'[t(\varphi(a))+(1-t)(\varphi(b))]|^q\,dt\right)^{\frac{1}{q}}$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\left(|f'(\varphi(a))|^q\int_0^{\frac{1}{2}}\left[(1-t)^{q\alpha}t-t^{q\alpha+1}\right]dt\right.$$

$$+|f'(\varphi(b))|^q\int_0^{\frac{1}{2}}\left[(1-t)^{q\alpha+1}-t^{q\alpha}(1-t)\right]dt+|f'(\varphi(a))|^q\int_{\frac{1}{2}}^1\left[t^{q\alpha+1}-(1-t)^{q\alpha}t\right]dt$$

$$\left.+|f'(\varphi(b))|^q\int_{\frac{1}{2}}^1\left[t^{q\alpha}(1-t)-(1-t)^{q\alpha+1}\right]dt\right)^{\frac{1}{q}}$$

$$= \frac{\varphi(b)-\varphi(a)}{2}\left(\frac{1}{\alpha+1}\left[1-\frac{1}{2^{\alpha}}\right]\right)^{\frac{1}{q}}\left[|f'(\varphi(a))|+|f'(\varphi(b))|\right]^{\frac{1}{q}}.$$

Here, we use $(A-B)^p \leq A^p - B^p$, for any $A > B \geq 0$ and $q \geq 1$. □

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Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY

*E-mail address*: sarikayamz@gmail.com

*E-mail address*: yaldizhatice@gmail.com