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NEW SOME OSTROWSKI TYPE INEQUALITIES FOR CO-ORDINATED CONVEX FUNCTIONS

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Abstract. In this paper, we obtain new identity for function of two variables and apply them to give new Ostrowski type integral inequality for double integrals involving functions whose derivatives are convex function on the co-ordinates on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b,\ c < d$.

1. Introduction

Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) whoose derivative $f' : (a,b) \to \mathbb{R}$ is baunded on (a,b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then we have the inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty , \tag{1.1}$$

for all $x \in [a,b]$[13]. The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as the *Ostrowski inequality.* For some results which generalize, improve and extend the inequality (1.1) see ([5], [6],[7], [14]-[17]) and the references therein.

Let us consider now a bidemensional interval $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$, amapping $f : \Delta \to \mathbb{R}$ is said to be convexs on Δ if the inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda) f(z,w),$$

holds for all $(x,y),(z,w) \in \Delta$ and $\lambda \in [0,1]$. The mapping f is said to be concave on the co-ordinates on if the above inequality holds in reversed direction, for all $(x,y),(z,w) \in \Delta$ and $\lambda \in [0,1]$.

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1. *A function $f : \Delta \to \mathbb{R}$ will be co-ordinated convex on Δ, for all $t, s \in [0,1]$ and $(x,y), (u,v) \in \Delta$, if the following inequality holds:*

$$\begin{aligned} & f(tx + (1-t)y, su + (1-s)v) \\ \leq \ & tsf(x,u) + s(1-t)f(y,u) + t(1-s)f(x,v) + (1-t)(1-s)f(y,v). \end{aligned}$$

For more information and recent developments on this topic, please refer to.

Clearly, every convex function is co-ordinated convex. Furthermore, there exist co-ordinated convex function which is not convex, (see, [3]). For several recent results concerning Hermite-Hadamard's inequality for some convex function on the

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co-ordinates on a rectangle from the plane $\mathbb{R}^2$, we refer the reader to ([1]-[4] [8]-[12],[18],[19]).

Also, in [3], Dragomir establish the following similar inequality of Hadamard's type for co-ordinated convex mapping on a rectangle from the plane $\mathbb{R}^2$.

Theorem 1. *Suppose that $f : \Delta \to \mathbb{R}$ is co-ordinated convex on Δ. Then one has the inequalities:*

$$\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
\leq\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right] \\
\leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\
\leq\ & \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right. \\
&\left.+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right] \\
\leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned} \tag{1.2}$$

The above inequalities are sharp.

In a recent paper [5], Barnett and Dragomir proved the following Ostrowski type inequality for double integrals:

Theorem 2. *Let $f : [a,b]\times[c,d] \to \mathbb{R}$ be continuous on $[a,b]\times[c,d]$, $f''_{x,y} = \frac{\partial^2 f}{\partial x\partial y}$ exists on $(a,b)\times(c,d)$ and is bounded, i.e.,*

$$\left\|f''_{x,y}\right\|_\infty = \sup_{(x,y)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right| < \infty.$$

Then, we have the inequality:

$$\begin{aligned}
&\left|\int_a^b\int_c^d f(s,t)dtds-(d-c)(b-a)f(x,y)\right. \\
&\left.-\left[(b-a)\int_c^d f(x,t)dt+(d-c)\int_a^b f(s,y)ds\right]\right| \\
\leq\ & \left[\frac{1}{4}(b-a)^2+(x-\frac{a+b}{2})^2\right]\left[\frac{1}{4}(d-c)^2+(y-\frac{d+c}{2})^2\right]\left\|f''_{x,y}\right\|_\infty
\end{aligned} \tag{1.3}$$

for all $(x,y)\in[a,b]\times[c,d]$.

The main aim of this paper is to establish some new inequalities Ostrowski type for double integrals involving functions whose derivatives are convex function on the co-ordinates on $\Delta := [a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b,\ c<d$.

2. Main Results

To establish our main results we need the following identity:

Lemma 1. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b,\ c<d$. If $f_{\lambda\alpha}=\frac{\partial^2 f}{\partial\lambda\partial\alpha}\in L_1(\Delta)$, then for any $(x,y)\in\Delta$, we have the equality:*

$$
\begin{aligned}
f(x,y) &= \frac{1}{d-c}\int_c^d f(x,s)ds+\frac{1}{b-a}\int_a^b f(t,y)dt-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \\
&\quad +\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x-t)(y-s) \qquad (2.1)\\
&\quad \times\int_0^1\int_0^1 f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)d\alpha d\lambda dsdt.
\end{aligned}
$$

Proof. For any $t,x\in[a,b]$ and $y,s\in[c,d]$, $t\neq x$, $y\neq s$, we have

$$
\begin{aligned}
\int_t^x\int_s^y f_{\sigma\tau}(\sigma,\tau)\,d\tau d\sigma &= \int_t^x\left[f_\sigma(\sigma,y)-f_\sigma(\sigma,s)\right]d\sigma \\
&= \left[f(\sigma,y)-f(\sigma,s)\right]\Big|_t^x \\
&= f(x,y)-f(x,s)-f(t,y)+f(t,s)
\end{aligned}
$$

and

$$
f(x,y)=f(x,s)+f(t,y)-f(t,s)+\int_t^x\int_s^y f_{\sigma\tau}(\sigma,\tau)\,d\tau d\sigma.
$$

For $\sigma=\lambda x+(1-\lambda)$ and $\tau=\alpha y+(1-\alpha)s$, we obtain

$$
\begin{aligned}
f(x,y) &= f(x,s)+f(t,y)-f(t,s) \\
&\quad +(x-t)(y-s)\int_0^1\int_0^1 f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)d\alpha d\lambda. \qquad (2.2)
\end{aligned}
$$

If we integrate (2.2) over t,s on Δ and divide by $(b-a)(d-c)$, we have deduced the desired equality (2.1). □

Theorem 3. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b,\ c<d$ and $\left|\frac{\partial^2 f}{\partial\lambda\partial\alpha}\right|=|f_{\lambda\alpha}|$ is a convex function on the co-ordinates on Δ.*

(i) If $f_{\lambda\alpha} \in L_\infty(\Delta)$ *then for any* $(x,y) \in \Delta$,

$$\left| f(x,y) - \frac{1}{d-c}\int_c^d f(x,s)ds - \frac{1}{b-a}\int_a^b f(t,y)dt + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \right|$$
$$\leq \frac{(b-a)(d-c)}{4}\left\| |f_{\lambda\alpha}(x,y)| + |f_{\lambda\alpha}(x,s)| + |f_{\lambda\alpha}(t,y)| + |f_{\lambda\alpha}(t,s)| \right\|_\infty$$
$$\times \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{4} + \left(\frac{y-\frac{c+d}{2}}{d-c}\right)^2\right].$$

(ii) If $f_{\lambda\alpha} \in L_p(\Delta)$ $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *then for any* $(x,y) \in \Delta$,

$$\left| f(x,y) - \frac{1}{d-c}\int_c^d f(x,s)ds - \frac{1}{b-a}\int_a^b f(t,y)dt + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \right|$$
$$\leq \frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{4(q+1)^{\frac{2}{q}}}\left[\left(\frac{b-x}{b-a}\right)^{q+1} + \left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}\left[\left(\frac{d-y}{d-c}\right)^{q+1} + \left(\frac{y-c}{d-c}\right)^{q+1}\right]^{\frac{1}{q}}$$
$$\times \left\| |f_{\lambda\alpha}(x,y)|^p + |f_{\lambda\alpha}(x,.)|^p + |f_{\lambda\alpha}(.,y)|^p + |f_{\lambda\alpha}|^p \right\|_p.$$

(iii) If $f_{\lambda\alpha} \in L_1(\Delta)$ *then for any* $(x,y) \in \Delta$,

$$\left| f(x,y) - \frac{1}{d-c}\int_c^d f(x,s)ds - \frac{1}{b-a}\int_a^b f(t,y)dt + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \right|$$
$$\leq \frac{1}{4}\left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[\frac{1}{2} + \left|\frac{y-\frac{c+d}{2}}{d-c}\right|\right]$$
$$\times \left[(b-a)(d-c)|f_{\lambda\alpha}(x,y)| + (b-a)\|f_{\lambda\alpha}(x,.)\|_1 + (d-c)\|f_{\lambda\alpha}(.,y)\|_1 + \|f_{\lambda\alpha}\|_1\right].$$

Proof. (i). Using (2.1), convexity of $|f_{\lambda\alpha}|$ and taking the modulus, it follows that

$$\begin{aligned}
&\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right| \\
\leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d |x-t|\,|y-s|\left|\int_0^1\int_0^1 f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)\,d\alpha d\lambda\right| dsdt \\
\leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\int_0^1\int_0^1 |x-t|\,|y-s|\,|f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)|\,d\alpha d\lambda dsdt \\
\leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d\int_0^1\int_0^1 |x-t|\,|y-s|\,[\lambda\alpha\,|f_{\lambda\alpha}(x,y)|+\lambda(1-\alpha)\,|f_{\lambda\alpha}(x,s)| \\
& +(1-\lambda)\alpha\,|f_{\lambda\alpha}(t,y)|+(1-\lambda)(1-\alpha)\,|f_{\lambda\alpha}(t,s)|]\,d\alpha d\lambda dsdt \\
=\ & \frac{1}{4(b-a)(d-c)} \\
& \times\int_a^b\int_c^d |x-t|\,|y-s|\,[|f_{\lambda\alpha}(x,y)|+|f_{\lambda\alpha}(x,s)|+|f_{\lambda\alpha}(t,y)|+|f_{\lambda\alpha}(t,s)|]\,dsdt
\end{aligned}$$

Since $f_{\lambda\alpha}\in L_\infty(\Delta)$, we get

$$\begin{aligned}
&\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right| \\
\leq\ & \frac{1}{4(b-a)(d-c)} ess\sup_{(t,s)\in\Delta}\{|f_{\lambda\alpha}(x,y)|+|f_{\lambda\alpha}(x,s)|+|f_{\lambda\alpha}(t,y)|+|f_{\lambda\alpha}(t,s)|\} \\
& \times\int_a^b\int_c^d |x-t|\,|y-s|\,dsdt \\
=\ & \frac{1}{4(b-a)(d-c)}\,\||f_{\lambda\alpha}(x,y)|+|f_{\lambda\alpha}(x,s)|+|f_{\lambda\alpha}(t,y)|+|f_{\lambda\alpha}(t,s)|\|_\infty \\
& \times\left(\int_a^b |x-t|\,dt\right)\left(\int_c^d |y-s|\,ds\right) \\
=\ & \frac{1}{4(b-a)(d-c)}\,\||f_{\lambda\alpha}(x,y)|+|f_{\lambda\alpha}(x,s)|+|f_{\lambda\alpha}(t,y)|+|f_{\lambda\alpha}(t,s)|\|_\infty \\
& \times\left(\frac{(x-a)^2+(b-x)^2}{2}\right)\left(\frac{(y-c)^2+(d-y)^2}{2}\right) \\
=\ & \frac{(b-a)(d-c)}{4}\,\||f_{\lambda\alpha}(x,y)|+|f_{\lambda\alpha}(x,s)|+|f_{\lambda\alpha}(t,y)|+|f_{\lambda\alpha}(t,s)|\|_\infty \\
& \times\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{4}+\left(\frac{y-\frac{c+d}{2}}{d-c}\right)^2\right].
\end{aligned}$$

(ii). As above, we have

$$\begin{aligned}
&\left| f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right| \\
\leq\; & \frac{1}{4(b-a)(d-c)} \\
&\times \int_a^b\int_c^d |x-t|\,|y-s|\left[|f_{\lambda\alpha}(x,y)|+|f_{\lambda\alpha}(x,s)|+|f_{\lambda\alpha}(t,y)|+|f_{\lambda\alpha}(t,s)|\right]dsdt.
\end{aligned}$$

Using Hölder's ineguality for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, we obtain

$$\begin{aligned}
&\left| f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right| \\
\leq\; & \frac{1}{4(b-a)(d-c)}\left(\int_a^b\int_c^d |x-t|^q\,|y-s|^q\,dsdt\right)^{\frac{1}{q}} \\
&\times\left(\int_a^b\int_c^d \left[|f_{\lambda\alpha}(x,y)|+|f_{\lambda\alpha}(x,s)|+|f_{\lambda\alpha}(t,y)|+|f_{\lambda\alpha}(t,s)|\right]^p dsdt\right)^{\frac{1}{p}} \\
=\; & \frac{1}{4(b-a)(d-c)}\left(\frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}\right)^{\frac{1}{q}}\left(\frac{(y-c)^{q+1}+(d-y)^{q+1}}{q+1}\right)^{\frac{1}{q}} \\
&\times \left\| |f_{\lambda\alpha}(x,y)|+|f_{\lambda\alpha}(x,.)|+|f_{\lambda\alpha}(.,y)|+|f_{\lambda\alpha}|\right\|_p .
\end{aligned}$$

(iii). As above, we have

$$\begin{aligned}
&\left| f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right| \\
\leq\; & \frac{1}{4(b-a)(d-c)} \\
&\times \int_a^b\int_c^d |x-t|\,|y-s|\left[|f_{\lambda\alpha}(x,y)|+|f_{\lambda\alpha}(x,s)|+|f_{\lambda\alpha}(t,y)|+|f_{\lambda\alpha}(t,s)|\right]dsdt
\end{aligned}$$

Using convexity of $|f_{\lambda\alpha}|$, we obtain

$$\begin{aligned}
&\left|f(x,y)-\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right| \\
\leq\ & \frac{1}{4(b-a)(d-c)}\sup_{t\in[a,b]}|x-t|\sup_{s\in[c,d]}|y-s| \\
&\times\int_a^b\int_c^d\left[|f_{\lambda\alpha}(x,y)|+|f_{\lambda\alpha}(x,s)|+|f_{\lambda\alpha}(t,y)|+|f_{\lambda\alpha}(t,s)|\right]dsdt \\
=\ & \frac{1}{4(b-a)(d-c)}\max\{x-a,b-x\}\max\{y-c,d-y\} \\
&\times\left[(b-a)(d-c)|f_{\lambda\alpha}(x,y)|+(b-a)\|f_{\lambda\alpha}(x,.)\|_1+(d-c)\|f_{\lambda\alpha}(.,y)\|_1+\|f_{\lambda\alpha}\|_1\right] \\
=\ & \frac{1}{4}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[\frac{1}{2}+\left|\frac{y-\frac{c+d}{2}}{d-c}\right|\right] \\
&\times\left[(b-a)(d-c)|f_{\lambda\alpha}(x,y)|+(b-a)\|f_{\lambda\alpha}(x,.)\|_1+(d-c)\|f_{\lambda\alpha}(.,y)\|_1+\|f_{\lambda\alpha}\|_1\right].
\end{aligned}$$

This completes the proof. □

Corollary 1. *With the assumptions of Theorem 3 with $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ we have the inequality*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},s\right)ds\right. \\
&\left.-\frac{1}{b-a}\int_a^b f\left(t,\frac{c+d}{2}\right)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right| \\
\leq\ & \frac{(b-a)(d-c)}{64} \\
&\times\left\|\left|f_{\lambda\alpha}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|+\left|f_{\lambda\alpha}\left(\frac{a+b}{2},s\right)\right|+\left|f_{\lambda\alpha}\left(t,\frac{c+d}{2}\right)\right|+|f_{\lambda\alpha}(t,s)|\right\|_\infty,
\end{aligned}$$

provided $f_{\lambda\alpha}\in L_\infty(\Delta)$.

If $f_{\lambda\alpha}\in L_p(\Delta)$ $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},s\right)ds\right. \\
&\left.-\frac{1}{b-a}\int_a^b f\left(t,\frac{c+d}{2}\right)dt+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\right| \\
\leq\ & \frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{16(q+1)^{\frac{2}{q}}} \\
&\times\left\|\left|f_{\lambda\alpha}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|^p+\left|f_{\lambda\alpha}\left(\frac{a+b}{2},.\right)\right|^p+\left|f_{\lambda\alpha}\left(.,\frac{c+d}{2}\right)\right|^p+|f_{\lambda\alpha}|^p\right\|_p.
\end{aligned}$$

If $f_{\lambda\alpha} \in L_1(\Delta)$, then

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},s\right)ds \right. \\
&\left. -\frac{1}{b-a}\int_a^b f\left(t,\frac{c+d}{2}\right)dt + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \right| \\
\leq\ & \frac{1}{16}\left[(b-a)(d-c)\left|f_{\lambda\alpha}\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right| + (b-a)\left\|f_{\lambda\alpha}\left(\frac{a+b}{2},.\right)\right\|_1 \right. \\
&\left. +(d-c)\left\|f_{\lambda\alpha}\left(.,\frac{c+d}{2}\right)\right\|_1 + \|f_{\lambda\alpha}\|_1\right].
\end{aligned}$$

Theorem 4. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b,\ c < d$ and $\left|\frac{\partial^2 f}{\partial\lambda\partial\alpha}\right|^p = |f_{\lambda\alpha}|^p,\ p > 1$ is a convex function on the co-ordinates on Δ.*

(i) If $f_{\lambda\alpha} \in L_\infty(\Delta)$ then for any $(x,y) \in \Delta$,

$$\begin{aligned}
&\left| f(x,y) - \frac{1}{d-c}\int_c^d f(x,s)ds - \frac{1}{b-a}\int_a^b f(t,y)dt + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \right| \\
\leq\ & \frac{(b-a)(d-c)}{4^{\frac{1}{p}}}\left[|f_{\lambda\alpha}(x,y)|^p + \|f_{\lambda\alpha}(x,.)\|_\infty^p + \|f_{\lambda\alpha}(.,y)\|_\infty^p + \|f_{\lambda\alpha}\|_\infty^p\right]^{\frac{1}{p}} \\
&\times \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{4} + \left(\frac{y-\frac{c+d}{2}}{d-c}\right)^2\right].
\end{aligned}$$

(ii) If $f_{\lambda\alpha} \in L_p(\Delta)$ $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$ then for any $(x,y) \in \Delta$,

$$\begin{aligned}
&\left| f(x,y) - \frac{1}{d-c}\int_c^d f(x,s)ds - \frac{1}{b-a}\int_a^b f(t,y)dt + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \right| \\
\leq\ & \frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{4^{\frac{1}{p}}(q+1)^{\frac{2}{q}}}\left[\left(\frac{b-x}{b-a}\right)^{q+1} + \left(\frac{x-a}{b-a}\right)^{q+1}\right]^{\frac{1}{q}}\left[\left(\frac{d-y}{d-c}\right)^{q+1} + \left(\frac{y-c}{d-c}\right)^{q+1}\right]^{\frac{1}{q}} \\
&\times \left[(b-a)(d-c)|f_{\lambda\alpha}(x,y)|^p + \|f_{\lambda\alpha}(x,.)\|_p^p + \|f_{\lambda\alpha}(.,y)\|_p^p + \|f_{\lambda\alpha}\|_p^p\right]^{\frac{1}{p}}.
\end{aligned}$$

(iii) If $f_{\lambda\alpha} \in L_p(\Delta)$, $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$ then for any $(x,y) \in \Delta$,

$$\begin{aligned}
&\left| f(x,y) - \frac{1}{d-c}\int_c^d f(x,s)ds - \frac{1}{b-a}\int_a^b f(t,y)dt + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \right| \\
\leq\ & \frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{4^{\frac{1}{p}}}\left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[\frac{1}{2} + \left|\frac{y-\frac{c+d}{2}}{d-c}\right|\right] \\
&\times \left[(b-a)(d-c)|f_{\lambda\alpha}(x,y)|^p + \|f_{\lambda\alpha}(x,.)\|_p^p + \|f_{\lambda\alpha}(.,y)\|_p^p + \|f_{\lambda\alpha}\|_p^p\right]^{\frac{1}{p}}.
\end{aligned}$$

Proof. As in the proof of Theorem 3 we have

$$\begin{aligned}
&\left| f(x,y) - \frac{1}{d-c}\int_c^d f(x,s)ds - \frac{1}{b-a}\int_a^b f(t,y)dt + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \right| \\
\leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d |x-t|\,|y-s| \int_0^1\int_0^1 \left| f_{\lambda\alpha}(\lambda x + (1-\lambda)t, \alpha y + (1-\alpha)s) \right| d\alpha d\lambda dsdt
\end{aligned}$$

for any $(x,y) \in \Delta$. By Hölder's inequality, we get

$$\begin{aligned}
&\int_0^1\int_0^1 \left| f_{\lambda\alpha}(\lambda x + (1-\lambda)t, \alpha y + (1-\alpha)s) \right| d\alpha d\lambda \\
\leq\ & \left(\int_0^1\int_0^1 1^q d\alpha d\lambda \right)^{\frac{1}{q}} \left(\int_0^1\int_0^1 \left| f_{\lambda\alpha}(\lambda x + (1-\lambda)t, \alpha y + (1-\alpha)s) \right|^p d\alpha d\lambda \right)^{\frac{1}{p}}
\end{aligned}$$

for any for any $(x,y) \in \Delta$, where $\frac{1}{p}+\frac{1}{q}=1$, $p>1$.

Since $|f_{\lambda\alpha}|^p$ is a convex function on the co-ordinates on Δ, then

$$\begin{aligned}
&\left(\int_0^1\int_0^1 \left| f_{\lambda\alpha}(\lambda x + (1-\lambda)t, \alpha y + (1-\alpha)s) \right|^p d\alpha d\lambda \right)^{\frac{1}{p}} \\
\leq\ & \left(\frac{1}{4}\left[|f_{\lambda\alpha}(x,y)|^p + |f_{\lambda\alpha}(x,s)|^p + |f_{\lambda\alpha}(t,y)|^p + |f_{\lambda\alpha}(t,s)|^p \right] \right)^{\frac{1}{p}}
\end{aligned}$$

for any $(x,y) \in \Delta$. Therefore

$$\begin{aligned}
&\left| f(x,y) - \frac{1}{d-c}\int_c^d f(x,s)ds - \frac{1}{b-a}\int_a^b f(t,y)dt + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \right| \\
\leq\ & \frac{1}{4^{\frac{1}{p}}(b-a)(d-c)}\int_a^b\int_c^d |x-t|\,|y-s| \\
& \times \left(\left[|f_{\lambda\alpha}(x,y)|^p + |f_{\lambda\alpha}(x,s)|^p + |f_{\lambda\alpha}(t,y)|^p + |f_{\lambda\alpha}(t,s)|^p \right] \right)^{\frac{1}{p}} dsdt. \qquad (2.3)
\end{aligned}$$

(i). Now, if $f_{\lambda\alpha} \in L_\infty(\Delta)$ then

$$\begin{aligned}
&\left| f(x,y) - \frac{1}{d-c}\int_c^d f(x,s)ds - \frac{1}{b-a}\int_a^b f(t,y)dt + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \right| \\
\leq\ & \frac{1}{4^{\frac{1}{p}}(b-a)(d-c)} ess \sup_{(t,s)\in\Delta} \left[|f_{\lambda\alpha}(x,y)|^p + |f_{\lambda\alpha}(x,s)|^p + |f_{\lambda\alpha}(t,y)|^p + |f_{\lambda\alpha}(t,s)|^p \right]^{\frac{1}{p}} \\
& \times \int_a^b\int_c^d |x-t|\,|y-s|\,dsdt \\
=\ & \frac{1}{4^{\frac{1}{p}}(b-a)(d-c)} \left[|f_{\lambda\alpha}(x,y)|^p + \|f_{\lambda\alpha}(x,.)\|_\infty^p + f_{\lambda\alpha}(.,y)_\infty^p + \|f_{\lambda\alpha}\|_\infty^p \right]^{\frac{1}{p}} \\
& \times \left(\frac{(x-a)^2+(b-x)^2}{2} \right) \left(\frac{(y-c)^2+(d-y)^2}{2} \right)
\end{aligned}$$

for any $(x,y) \in \Delta$.

(ii). If $f_{\lambda\alpha} \in L_p(\Delta)$ $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, by using Hölder's inequality in (2.3), we have

$$\begin{aligned}
&\left| f(x,y) - \frac{1}{d-c}\int_c^d f(x,s)ds - \frac{1}{b-a}\int_a^b f(t,y)dt + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \right| \\
\leq\ & \frac{1}{4^{\frac{1}{p}}(b-a)(d-c)} \left(\int_a^b\int_c^d |x-t|^q\,|y-s|^q\,dsdt \right)^{\frac{1}{q}} \\
& \times \left(\int_a^b\int_c^d \left[|f_{\lambda\alpha}(x,y)|^p + |f_{\lambda\alpha}(x,s)|^p + |f_{\lambda\alpha}(t,y)|^p + |f_{\lambda\alpha}(t,s)|^p \right] dsdt \right)^{\frac{1}{p}} \\
\leq\ & \frac{1}{4^{\frac{1}{p}}(b-a)(d-c)} \left(\frac{(x-a)^2+(b-x)^2}{2} \right) \left(\frac{(y-c)^2+(d-y)^2}{2} \right) \\
& \times \left[(b-a)(d-c)\,|f_{\lambda\alpha}(x,y)|^p + \|f_{\lambda\alpha}(x,.)\|_p^p + \|f_{\lambda\alpha}(.,y)\|_p^p + \|f_{\lambda\alpha}\|_p^p \right]^{\frac{1}{p}} \\
=\ & \frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{4^{\frac{1}{p}}(q+1)^{\frac{2}{q}}} \left[\left(\frac{b-x}{b-a}\right)^{q+1} + \left(\frac{x-a}{b-a}\right)^{q+1} \right]^{\frac{1}{q}} \left[\left(\frac{d-y}{d-c}\right)^{q+1} + \left(\frac{y-c}{d-c}\right)^{q+1} \right]^{\frac{1}{q}} \\
& \times \left[(b-a)(d-c)\,|f_{\lambda\alpha}(x,y)|^p + \|f_{\lambda\alpha}(x,.)\|_p^p + \|f_{\lambda\alpha}(.,y)\|_p^p + \|f_{\lambda\alpha}\|_p^p \right]^{\frac{1}{p}}
\end{aligned}$$

for any $(x,y) \in \Delta$.

(iii) If $f_{\lambda\alpha} \in L_p(\Delta)$ $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then, by Hölders inequality we have

$$\left| f(x,y) - \frac{1}{d-c} \int_c^d f(x,s)ds - \frac{1}{b-a} \int_a^b f(t,y)dt + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt \right|$$

$$\begin{aligned}
&\leq \frac{1}{4^{\frac{1}{p}}(b-a)(d-c)} \int_a^b \int_c^d |x-t|\,|y-s| \\
&\quad \times \left[|f_{\lambda\alpha}(x,y)|^p + |f_{\lambda\alpha}(x,s)|^p + |f_{\lambda\alpha}(t,y)|^p + |f_{\lambda\alpha}(t,s)|^p \right]^{\frac{1}{p}} dsdt \\
&= \frac{1}{4^{\frac{1}{p}}(b-a)(d-c)} \sup_{t \in [a,b]} |x-t| \sup_{s \in [c,d]} |y-s| \left(\int_a^b \int_c^d 1^q dsdt \right)^{\frac{1}{q}} \\
&\quad \times \left(\int_a^b \int_c^d \left[|f_{\lambda\alpha}(x,y)|^p + |f_{\lambda\alpha}(x,s)|^p + |f_{\lambda\alpha}(t,y)|^p + |f_{\lambda\alpha}(t,s)|^p \right] dsdt \right)^{\frac{1}{p}} \\
&\leq \frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{4^{\frac{1}{p}}(b-a)(d-c)} \max\{x-a, b-x\} \max\{y-c, d-y\} \\
&\quad \times \left[(b-a)(d-c)\,|f_{\lambda\alpha}(x,y)|^p + \|f_{\lambda\alpha}(x,.)\|_p^p + \|f_{\lambda\alpha}(.,y)\|_p^p + \|f_{\lambda\alpha}\|_p^p \right]^{\frac{1}{p}} \\
&= \frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{4^{\frac{1}{p}}} \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \left[\frac{1}{2} + \left| \frac{y - \frac{c+d}{2}}{d-c} \right| \right] \\
&\quad \times \left[(b-a)(d-c)\,|f_{\lambda\alpha}(x,y)|^p + \|f_{\lambda\alpha}(x,.)\|_p^p + \|f_{\lambda\alpha}(.,y)\|_p^p + \|f_{\lambda\alpha}\|_p^p \right]^{\frac{1}{p}}.
\end{aligned}$$

□

Corollary 2. *With the assumptions of Theorem 4 with $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ we have the inequality*

$$\begin{aligned}
&\Bigg| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, s\right) ds \\
&- \frac{1}{b-a} \int_a^b f\left(t, \frac{c+d}{2}\right) dt + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t,s)dsdt \Bigg| \\
&\leq \frac{(b-a)(d-c)}{4^{2+\frac{1}{p}}} \\
&\quad \times \left[\left| f_{\lambda\alpha}\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right|^p + \left\| f_{\lambda\alpha}\left(\frac{a+b}{2}, .\right) \right\|_\infty^p + \left\| f_{\lambda\alpha}\left(., \frac{c+d}{2}\right) \right\|_\infty^p + \|f_{\lambda\alpha}\|_\infty^p \right]^{\frac{1}{p}},
\end{aligned}$$

where $f_{\lambda\alpha} \in L_\infty(\Delta)$.

If $f_{\lambda\alpha} \in L_p(\Delta)$ $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ *then we have*

$$\left| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, s\right) ds \right.$$
$$\left. -\frac{1}{b-a}\int_a^b f\left(t, \frac{c+d}{2}\right) dt + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \right|$$
$$\leq \frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{4^{1+\frac{1}{p}}(q+1)^{\frac{2}{q}}}\left[(b-a)(d-c)\left|f_{\lambda\alpha}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|^p\right.$$
$$\left. + \left\|f_{\lambda\alpha}\left(\frac{a+b}{2}, .\right)\right\|_p^p + \left\|f_{\lambda\alpha}\left(., \frac{c+d}{2}\right)\right\|_p^p + \|f_{\lambda\alpha}\|_p^p\right]^{\frac{1}{p}}.$$

If $f_{\lambda\alpha} \in L_p(\Delta)$ $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, *then we have,*

$$\left| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, s\right) ds \right.$$
$$\left. -\frac{1}{b-a}\int_a^b f\left(t, \frac{c+d}{2}\right) dt + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt \right|$$
$$\leq \frac{(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}}{4^{1+\frac{1}{p}}}\left[(b-a)(d-c)\left|f_{\lambda\alpha}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right|^p\right.$$
$$\left. + \left\|f_{\lambda\alpha}\left(\frac{a+b}{2}, .\right)\right\|_p^p + \left\|f_{\lambda\alpha}\left(., \frac{c+d}{2}\right)\right\|_p^p + \|f_{\lambda\alpha}\|_p^p\right]^{\frac{1}{p}}.$$

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