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SOME GENERALIZATION OF INTEGRAL INEQUALITIES FOR TWICE DIFFERENTIABLE MAPPINGS INVOLVING FRACTIONAL INTEGRALS

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Abstract. In this paper, a general integral identity involving Riemann-Liouville fractional integrals is derived. By use this identity, we establish new some generalized inequalities of the Hermite-Hadamard's type for functions whose absolute values of derivatives are convex.

1. Introduction

The following definition for convex functions is well known in the mathematical literature:

The function $f:[a,b]\subset\mathbb{R}\rightarrow\mathbb{R}$, is said to be convex if the following inequality holds

$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. We say that f is concave if $(-f)$ is convex.

Many inequalities have been established for convex functions but the most famous inequality is the Hermite-Hadamard's inequality, due to its rich geometrical significance and applications(see, e.g.,[12, p.137], [6]). These inequalities state that if $f:I\rightarrow\mathbb{R}$ is a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$, then

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [6, 8, 9, 12],[14]-[16],[22],[23]) and the references cited therein.

In [16], Sarikaya et. al. established inequalities for twice differentiable convex mappings which are connected with Hadamard's inequality, and they used the following lemma to prove their results:

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Lemma 1. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If $f'' \in L_1[a,b]$, then*

$$\begin{aligned}(1.2)\qquad &\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \\ &= \frac{(b-a)^2}{2}\int_0^1 m(t)\left[f''(ta+(1-t)b) + f''(tb+(1-t)a)\right]dt,\end{aligned}$$

where

$$m(t) := \begin{cases} t^2 & , t \in [0, \frac{1}{2}) \\ (1-t)^2 & , t \in [\frac{1}{2}, 1]. \end{cases}$$

Also, the main inequalities in [16], pointed out as follows:

Theorem 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° with $f'' \in L_1[a,b]$. If $|f''|$ is convex on $[a,b]$, then*

$$(1.3)\qquad \left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{24}\left[\frac{|f''(a)| + |f''(b)|}{2}\right].$$

Theorem 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L_1[a,b]$ where $a, b \in I$, $a < b$. If $|f''|^q$ is convex on $[a,b]$, $q > 1$, then*

$$(1.4)\qquad \left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{8(2p+1)^{1/p}}\left[\frac{|f''(a)|^q + |f''(b)|^q}{2}\right]^{1/q}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult [7, 10, 11, 13].

Definition 1. *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0}f(x) = J_{b-}^{0}f(x) = f(x)$.

Meanwhile, Sarikaya et al.[19] presented the following important integral identity including the first-order derivative of f to establish many interesting Hermite-Hadamard type inequalities for convexity functions via Riemann-Liouville fractional integrals of the order $\alpha > 0$.

Lemma 2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $0\leq a<b$. If $f'\in L[a,b]$, then the following equality for fractional integrals holds:*

$$\begin{aligned}&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]-f\left(\frac{a+b}{2}\right)\\ &= \frac{b-a}{4}\left\{\int_0^1 t^{\alpha}f'\left(\frac{t}{2}a+\frac{2-t}{2}b\right)dt-\int_0^1 t^{\alpha}f'\left(\frac{2-t}{2}a+\frac{t}{2}b\right)dt\right\}\end{aligned} \tag{1.5}$$

with $\alpha>0$.

It is remarkable that Sarikaya et al.[19] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\leq a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)+}f(b)+J^{\alpha}_{\left(\frac{a+b}{2}\right)-}f(a)\right]\leq\frac{f(a)+f(b)}{2} \tag{1.6}$$

with $\alpha>0$.

For some recent results connected with fractional integral inequalities see ([1, 2, 3, 4, 5],[17],[18],[20],[21],[24])

In this paper, we expand the Lemma 2 to the case of including a twice differentiable function involving Riemann-Liouville fractional integrals and some other integral inequalities using the generalized identity is obtained for fractional integrals.

2. Main Results

For our results, we give the following important fractional integrtal identity:

Lemma 3. *Let $f:[a,b]\to\mathbb{R}$ be twice differentiable mapping on (a,b) with $0\leq a<b$. If $f''\in L[a,b]$, then the following equality for fractional integrals holds:*

$$\begin{aligned}&(\alpha+1)(1-\lambda)^{\alpha}\lambda^{\alpha}f(\lambda a+(1-\lambda)b)\\ &-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[\lambda^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(a)+(1-\lambda)^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)^+}f(b)\right]\\ =&-(b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}\left\{(1-\lambda)\int_0^1 t^{\alpha+1}f''\left[t(\lambda a+(1-\lambda)b)+(1-t)a\right]dt\right.\\ &\left.+\lambda\int_0^1(1-t)^{\alpha+1}f''\left[tb+(1-t)(\lambda a+(1-\lambda)b)\right]dt\right\}\end{aligned} \tag{2.1}$$

where $\lambda\in(0,1)$ and $\alpha>0$.

Proof. Integrating by parts

$$
\begin{aligned}
&\int_0^1 t^{\alpha+1} f''\left[t(\lambda a+(1-\lambda)b)+(1-t)a\right]dt \\
=& \left.\frac{t^{\alpha+1} f'\left[t(\lambda a+(1-\lambda)b)+(1-t)a\right]}{(1-\lambda)(b-a)}\right|_0^1 \\
&-\frac{\alpha+1}{(1-\lambda)(b-a)}\int_0^1 t^{\alpha} f'\left[t(\lambda a+(1-\lambda)b)+(1-t)a\right]dt \\
=& \frac{f'\left(\lambda a+(1-\lambda)b\right)}{(1-\lambda)(b-a)}-\frac{\alpha+1}{(1-\lambda)(b-a)} \\
&\times\left[\frac{f\left(\lambda a+(1-\lambda)b\right)}{(1-\lambda)(b-a)}-\frac{\alpha}{(1-\lambda)(b-a)}\int_0^1 t^{\alpha-1} f\left[t(\lambda a+(1-\lambda)b)+(1-t)a\right]dt\right] \\
=& \frac{f'\left(\lambda a+(1-\lambda)b\right)}{(1-\lambda)(b-a)}-\frac{(\alpha+1)\, f\left(\lambda a+(1-\lambda)b\right)}{(1-\lambda)^2(b-a)^2} \\
&+\frac{(\alpha+1)\,\alpha}{(1-\lambda)^{\alpha+2}(b-a)^{\alpha+2}}\int_a^{\lambda a+(1-\lambda)b}(x-a)^{\alpha-1} f(x)dx
\end{aligned}
$$

$$
\begin{aligned}
=& \frac{f'\left(\lambda a+(1-\lambda)b\right)}{(1-\lambda)(b-a)}-\frac{(\alpha+1)\, f\left(\lambda a+(1-\lambda)b\right)}{(1-\lambda)^2(b-a)^2} \\
&+\frac{(\alpha+1)\,\Gamma(\alpha+1)}{(1-\lambda)^{\alpha+2}(b-a)^{\alpha+2}} J^{\alpha}_{(\lambda a+(1-\lambda)b)^-} f(a)
\end{aligned}
$$

that is,

$$
\begin{aligned}
&-\int_0^1 t^{\alpha+1} f''\left[t(\lambda a+(1-\lambda)b)+(1-t)a\right]dt \\
=& -\frac{f'\left(\lambda a+(1-\lambda)b\right)}{(1-\lambda)(b-a)}+\frac{(\alpha+1)\, f\left(\lambda a+(1-\lambda)b\right)}{(1-\lambda)^2(b-a)^2} \\
&-\frac{(\alpha+1)\,\Gamma(\alpha+1)}{(1-\lambda)^{\alpha+2}(b-a)^{\alpha+2}} J^{\alpha}_{(\lambda a+(1-\lambda)b)^-} f(a) \qquad (2.2)
\end{aligned}
$$

and similarly we have

$$
\begin{aligned}
& -\int_0^1 (1-t)^{\alpha+1} f''\left[tb+(1-t)(\lambda a+(1-\lambda)b)\right]dt \\
= \; & \frac{f'(\lambda a+(1-\lambda)b)}{\lambda(b-a)} + \frac{(\alpha+1)f(\lambda a+(1-\lambda)b)}{\lambda^2(b-a)^2} \\
& -\frac{(\alpha+1)\alpha}{\lambda^{\alpha+2}(b-a)^{\alpha+2}} \int_{\lambda a+(1-\lambda)b}^{b} (b-x)^{\alpha-1} f(x)dx \\
= \; & \frac{f'(\lambda a+(1-\lambda)b)}{\lambda(b-a)} + \frac{(\alpha+1)f(\lambda a+(1-\lambda)b)}{\lambda^2(b-a)^2} \\
& -\frac{(\alpha+1)\Gamma(\alpha+1)}{\lambda^{\alpha+2}(b-a)^{\alpha+2}} J^{\alpha}_{(\lambda a+(1-\lambda)b)^+} f(b). \qquad (2.3)
\end{aligned}
$$

Adding (2.2) and (2.3) we have (2.1). This completes the proof. □

Corollary 1. *Under the assumptions Lemma 3 with $\lambda = \frac{1}{2}$, then it follows that*

$$
\begin{aligned}
& \frac{-(b-a)^2}{8}\left\{\int_0^1 t^{\alpha+1} f''\left[t\left(\frac{a+b}{2}\right)+(1-t)a\right]dt + \int_0^1 (1-t)^{\alpha+1} f''\left[tb+(1-t)\frac{a+b}{2}\right]dt\right\} \\
= \; & (\alpha+1) f\left(\frac{a+b}{2}\right) - \frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha}2^{1-\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b)\right].
\end{aligned}
$$

Remark 1. *If we choose $\alpha = 1$ in Corollary 1, we have*

$$
\begin{aligned}
& f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \\
= \; & \frac{-(b-a)^2}{16}\left\{\int_0^1 t^2 f''\left[t\left(\frac{a+b}{2}\right)+(1-t)a\right]dt + \int_0^1 (1-t)^2 f''\left[tb+(1-t)\frac{a+b}{2}\right]dt\right\}.
\end{aligned}
$$

Theorem 4. *Let f:$[a,b] \to \mathbb{R}$ be twice differentiable mapping on (a,b) with $0 \le a < b$. If $|f''|^q$, $q \ge 1$ is convex on $[a,b]$, then the following inequality for fractional integrals holds:*

$$
\begin{aligned}
& \Bigg|(\alpha+1)(1-\lambda)^{\alpha}\lambda^{\alpha} f(\lambda a+(1-\lambda)b) - \frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha}} \\
& \times\left[\lambda^{\alpha+1} J^{\alpha}_{(\lambda a+(1-\lambda)b)^-} f(a) + (1-\lambda)^{\alpha+1} J^{\alpha}_{(\lambda a+(1-\lambda)b)^+} f(b)\right]\Bigg| \\
\le \; & \frac{(b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}}{(\alpha+2)^{1-\frac{1}{q}}}\left\{(1-\lambda)\left(\frac{(\alpha+2)|f''(\lambda a+(1-\lambda)b)|^q+|f''(a)|^q}{\alpha+3}\right)^{\frac{1}{q}}\right. \\
& \left. +\lambda\left(\frac{(\alpha+2)|f''(\lambda a+(1-\lambda)b)|^q+|f''(b)|^q}{\alpha+3}\right)^{\frac{1}{q}}\right\}. \qquad (2.4)
\end{aligned}
$$

where $\lambda \in (0,1)$ *and* $\alpha > 0$.

Proof. Firstly, we suppose that $q = 1$. Using Lemma 3 and convexity of $|f''|^q$, we find that

$$\begin{aligned}
&\Bigg|(\alpha+1)(1-\lambda)^{\alpha}\lambda^{\alpha} f(\lambda a+(1-\lambda)b) - \frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha}} \\
&\times \left[\lambda^{\alpha+1} J^{\alpha}_{(\lambda a+(1-\lambda)b)^-} f(a) + (1-\lambda)^{\alpha+1} J^{\alpha}_{(\lambda a+(1-\lambda)b)^+} f(b)\right]\Bigg| \\
\leq\ & (b-a)^2 (1-\lambda)^{\alpha+1}\lambda^{\alpha+1} \Bigg\{(1-\lambda)\int_0^1 t^{\alpha+1}\left|f''\left[t(\lambda a+(1-\lambda)b)+(1-t)a\right]\right| dt \\
&+\lambda \int_0^1 (1-t)^{\alpha+1}\left|f''\left[tb+(1-t)(\lambda a+(1-\lambda)b)\right]\right| dt\Bigg\} \\
\leq\ & (b-a)^2 (1-\lambda)^{\alpha+1}\lambda^{\alpha+1} \Bigg\{(1-\lambda)\int_0^1 t^{\alpha+1}\left[t\left|f''(\lambda a+(1-\lambda)b)\right| + (1-t)\left|f''(a)\right|\right] dt \\
&+\lambda \int_0^1 (1-t)^{\alpha+1}\left[t\left|f''(b)\right| + (1-t)\left|f''(\lambda a+(1-\lambda)b)\right|\right] dt\Bigg\} \\
=\ & \frac{(b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}}{\alpha+2}\Bigg\{(1-\lambda)\left(\frac{(\alpha+2)\left|f''(\lambda a+(1-\lambda)b)\right| + \left|f''(a)\right|}{\alpha+3}\right) \\
&+\lambda\left(\frac{(\alpha+2)\left|f''(\lambda a+(1-\lambda)b)\right| + \left|f''(b)\right|^q}{\alpha+3}\right)\Bigg\}.
\end{aligned}$$

Secondly, we suppose that $q > 1$. Using Lemma 3 and power mean inequality, we have

$$\begin{aligned}
&\Bigg\{(1-\lambda)\int_0^1 t^{\alpha+1} f''\left[t(\lambda a+(1-\lambda)b)+(1-t)a\right] dt \\
&+\lambda\int_0^1 (1-t)^{\alpha+1} f''\left[tb+(1-t)(\lambda a+(1-\lambda)b)\right] dt\Bigg\} \\
\leq\ & (1-\lambda)\left(\int_0^1 t^{\alpha+1}\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha+1}\left|f''\left[t(\lambda a+(1-\lambda)b)+(1-t)a\right]\right|^q dt\right)^{\frac{1}{q}} \\
&+\lambda\left(\int_0^1 (1-t)^{\alpha+1}\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)^{\alpha+1}\left|f''\left[tb+(1-t)(\lambda a+(1-\lambda)b)\right]\right|^q dt\right)^{\frac{1}{q}}. \qquad (2.5)
\end{aligned}$$

Hence, using convexity of $|f''|^q$ and (2.5) we obtain

$$\begin{aligned}
&\Bigg|(\alpha+1)(1-\lambda)^{\alpha}\lambda^{\alpha} f(\lambda a+(1-\lambda)b)-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha}} \\
&\times\left[\lambda^{\alpha+1} J^{\alpha}_{(\lambda a+(1-\lambda)b)^-} f(a)+(1-\lambda)^{\alpha+1} J^{\alpha}_{(\lambda a+(1-\lambda)b)^+} f(b)\right]\Bigg| \\
\leq\ & \frac{(b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}}{(\alpha+2)^{1-\frac{1}{q}}}\left\{(1-\lambda)\left(\int_0^1 t^{\alpha+1}\left[t\left|f''(\lambda a+(1-\lambda)b)\right|+(1-t)\left|f''(a)\right|\right]dt\right)^{\frac{1}{q}}\right. \\
&\left.+\lambda\left(\int_0^1 (1-t)^{\alpha+1}\left[t\left|f''(b)\right|+(1-t)\left|f''(\lambda a+(1-\lambda)b)\right|\right]dt\right)^{\frac{1}{q}}\right\} \\
\leq\ & \frac{(b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}}{(\alpha+2)^{1-\frac{1}{q}}}\left\{(1-\lambda)\left(\frac{(\alpha+2)\left|f''(\lambda a+(1-\lambda)b)\right|+\left|f''(a)\right|}{(\alpha+2)(\alpha+3)}\right)^{\frac{1}{q}}\right. \\
&\left.+\lambda\left(\frac{(\alpha+2)\left|f''(\lambda a+(1-\lambda)b)\right|+\left|f''(b)\right|^q}{(\alpha+2)(\alpha+3)}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

This completes the proof. □

Corollary 2. *Under assumption Theorem 4 with $\lambda=\frac{1}{2}$, we obtain*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}2^{1-\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b)\right]\right| \\
\leq\ & \frac{(b-a)^2}{8(\alpha+1)(\alpha+2)^{1-\frac{1}{q}}}\left\{\left(\frac{(\alpha+2)\left|f''\left(\frac{a+b}{2}\right)\right|^q+\left|f''(a)\right|^q}{\alpha+3}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{(\alpha+2)\left|f''\left(\frac{a+b}{2}\right)\right|^q+\left|f''(b)\right|^q}{\alpha+3}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Remark 2. *If we choose $\alpha=1$ in Corollary 2, we have*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{(b-a)^2}{16\times 3^{1-\frac{1}{q}}}\left\{\left(\frac{3\left|f''\left(\frac{a+b}{2}\right)\right|^q+\left|f''(a)\right|^q}{4}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{3\left|f''\left(\frac{a+b}{2}\right)\right|^q+\left|f''(b)\right|^q}{4}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Theorem 5. *Let f:$[a,b]\to\mathbb{R}$ be twice differentiable mapping on (a,b) with $0\leq a<b$. If $|f''|^q$ is convex on $[a,b]$ for same fixed $q>1$, then the following inequality for* *fractional integrals holds:*

$$\begin{aligned} & \left|(\alpha+1)(1-\lambda)^{\alpha}\lambda^{\alpha}f(\lambda a+(1-\lambda)b)-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha}}\right. \\ & \left.\times\left[\lambda^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(a)+(1-\lambda)^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)^+}f(b)\right]\right| \\ \leq & \frac{(b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}}{(p(\alpha+1)+1)^{\frac{1}{p}}}\left\{(1-\lambda)\left(\frac{|f''(\lambda a+(1-\lambda)b)|^q+|f''(a)|^q}{2}\right)^{\frac{1}{q}}\right. \\ & \left.+\lambda\left(\frac{|f''(\lambda a+(1-\lambda)b)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}\right\}. \end{aligned} \tag{2.6}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\lambda\in(0,1)$ *and* $\alpha>0$.

Proof. Using Lemma 3, convexity of $|f''|^q$ well-known Hölder's inequality, we have

$$\begin{aligned} & \left|(\alpha+1)(1-\lambda)^{\alpha}\lambda^{\alpha}f(\lambda a+(1-\lambda)b)-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha}}\right. \\ & \left.\times\left[\lambda^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(a)+(1-\lambda)^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)^+}f(b)\right]\right| \\ \leq & (b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}\left\{(1-\lambda)\left(\int_0^1 t^{p(\alpha+1)}\right)^{\frac{1}{p}}\left(\int_0^1 |f''[t(\lambda a+(1-\lambda)b)+(1-t)a]|^q\,dt\right)^{\frac{1}{q}}\right. \\ & \left.+\lambda\left(\int_0^1 (1-t)^{p(\alpha+1)}\right)^{\frac{1}{p}}\left(\int_0^1 |f''[tb+(1-t)(\lambda a+(1-\lambda)b)]|^q\,dt\right)^{\frac{1}{q}}\right\} \\ \leq & (b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}\left\{(1-\lambda)\frac{1}{(p(\alpha+1)+1)^{\frac{1}{p}}}\left(\int_0^1\left[t|f''(\lambda a+(1-\lambda)b)|^q+(1-t)|f''(a)|^q\right]dt\right)^{\frac{1}{q}}\right. \\ & \left.+\lambda\frac{1}{(p(\alpha+1)+1)^{\frac{1}{p}}}\left(\int_0^1\left[t|f''(b)|^q+(1-t)|f''(\lambda a+(1-\lambda)b)|^q\right]dt\right)^{\frac{1}{q}}\right\} \\ = & \frac{(b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}}{(p(\alpha+1)+1)^{\frac{1}{p}}}\left\{(1-\lambda)\left(\frac{|f''(\lambda a+(1-\lambda)b)|^q+|f''(a)|^q}{2}\right)^{\frac{1}{q}}\right. \\ & \left.+\lambda\left(\frac{|f''(\lambda a+(1-\lambda)b)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

□

Corollary 3. *Under assumption Theorem 5 with $\lambda = \frac{1}{2}$, we obtain*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha} 2^{1-\alpha}} \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b) \right] \right| \\
\leq\ & \frac{(b-a)^2}{8(\alpha+1)\left(p(\alpha+1)+1\right)^{\frac{1}{p}}} \left\{ \left(\frac{\left|f''\left(\frac{a+b}{2}\right)\right|^q + |f''(a)|^q}{2} \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\frac{\left|f''\left(\frac{a+b}{2}\right)\right|^q + |f''(b)|^q}{2} \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

Remark 3. *If we choose $\alpha = 1$ in Corollary 3, we have*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \\
\leq\ & \frac{(b-a)^2}{16(2p+1)^{\frac{1}{p}}} \left\{ \left(\frac{\left|f''\left(\frac{a+b}{2}\right)\right|^q + |f''(a)|^q}{2} \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\frac{\left|f''\left(\frac{a+b}{2}\right)\right|^q + |f''(b)|^q}{2} \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

Theorem 6. *Let $f:[a,b] \to \mathbb{R}$ be twice differentiable mapping on (a,b) with $0 \leq a < b$. If $|f''|^q$ is convex on $[a,b]$ for same fixed $q > 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left| (\alpha+1)(1-\lambda)^{\alpha}\lambda^{\alpha} f(\lambda a + (1-\lambda)b) - \frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha}} \right. \\
&\left. \times \left[\lambda^{\alpha+1} J^{\alpha}_{(\lambda a+(1-\lambda)b)^-} f(a) + (1-\lambda)^{\alpha+1} J^{\alpha}_{(\lambda a+(1-\lambda)b)^+} f(b) \right] \right| \\
\leq\ & (b-a)^2 (1-\lambda)^{\alpha+1}\lambda^{\alpha+1} \left\{ (1-\lambda) \left(\frac{(q(\alpha+1)+1)|f''(\lambda a+(1-\lambda)b)|^q + |f''(a)|^q}{(q(\alpha+1)+1)(q(\alpha+1)+2)} \right)^{\frac{1}{q}} \right. \\
&\left. +\lambda \left(\frac{(q(\alpha+1)+1)|f''(\lambda a+(1-\lambda)b)|^q + |f''(b)|^q}{(q(\alpha+1)+1)(q(\alpha+1)+2)} \right)^{\frac{1}{q}} \right\}.
\end{aligned} \tag{2.7}$$

where $\lambda \in (0,1)$ and $\alpha > 0$.

Proof. Using Lemma 3, convexity of $|f''|^q$ well-known Hölder's inequality, we have

$$\begin{aligned}
&\Bigg|(\alpha+1)(1-\lambda)^{\alpha}\lambda^{\alpha}f(\lambda a+(1-\lambda)b)-\frac{(\alpha+1)\Gamma(\alpha+1)}{(b-a)^{\alpha}}\\
&\times\left[\lambda^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)^{-}}f(a)+(1-\lambda)^{\alpha+1}J^{\alpha}_{(\lambda a+(1-\lambda)b)^{+}}f(b)\right]\Bigg|\\
\leq\;& (b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}\Bigg\{(1-\lambda)\left(\int_0^1 1^p\right)^{\frac{1}{p}}\left(\int_0^1 t^{q(\alpha+1)}\left|f''\left[t(\lambda a+(1-\lambda)b)+(1-t)a\right]\right|^q dt\right)^{\frac{1}{q}}\\
&+\lambda\left(\int_0^1 1^p\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)^{q(\alpha+1)}\left|f''\left[tb+(1-t)(\lambda a+(1-\lambda)b)\right]\right|^q dt\right)^{\frac{1}{q}}\Bigg\}\\
\leq\;& (b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}\Bigg\{(1-\lambda)\left(\int_0^1 t^{q(\alpha+1)}\left[t\left|f''(\lambda a+(1-\lambda)b)\right|^q+(1-t)\left|f''(a)\right|^q\right]dt\right)^{\frac{1}{q}}\\
&+\lambda\left(\int_0^1 (1-t)^{q(\alpha+1)}\left[t\left|f''(b)\right|^q+(1-t)\left|f''(\lambda a+(1-\lambda)b)\right|^q\right]dt\right)^{\frac{1}{q}}\Bigg\}\\
=\;& (b-a)^2(1-\lambda)^{\alpha+1}\lambda^{\alpha+1}\Bigg\{(1-\lambda)\left(\frac{(q(\alpha+1)+1)\left|f''(\lambda a+(1-\lambda)b)\right|^q+\left|f''(a)\right|^q}{(q(\alpha+1)+1)(q(\alpha+1)+2)}\right)^{\frac{1}{q}}\\
&+\lambda\left(\frac{(q(\alpha+1)+1)\left|f''(\lambda a+(1-\lambda)b)\right|^q+\left|f''(b)\right|^q}{(q(\alpha+1)+1)(q(\alpha+1)+2)}\right)^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

□

Corollary 4. *Under assumption Theorem 6 with $\lambda=\frac{1}{2}$, we obtain*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}2^{1-\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)\right]\right|\\
\leq\;&\frac{(b-a)^2}{8(\alpha+1)}\Bigg\{\left(\frac{(q(\alpha+1)+1)\left|f''\left(\frac{a+b}{2}\right)\right|^q+\left|f''(a)\right|^q}{(q(\alpha+1)+1)(q(\alpha+1)+2)}\right)^{\frac{1}{q}}\\
&+\left(\frac{(q(\alpha+1)+1)\left|f''\left(\frac{a+b}{2}\right)\right|^q+\left|f''(b)\right|^q}{(q(\alpha+1)+1)(q(\alpha+1)+2)}\right)^{\frac{1}{q}}\Bigg\}.
\end{aligned}$$

Remark 4. *If we choose $\alpha = 1$ in Corollary 4, we have*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right|$$

$$\leq \frac{(b-a)^2}{16}\left\{ \left(\frac{(2q+1)\left|f''\left(\frac{a+b}{2}\right)\right|^q + |f''(a)|^q}{(2q+1)(2q+2)} \right)^{\frac{1}{q}} + \left(\frac{(2q+1)\left|f''\left(\frac{a+b}{2}\right)\right|^q + |f''(b)|^q}{(2q+1)(2q+2)} \right)^{\frac{1}{q}} \right\}.$$

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