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SOME HERMITE -HADAMARD TYPE INTEGRAL INEQUALITIES FOR TWICE DIFFERENTIABLE MAPPINGS VIA FRACTIONAL INTEGRALS

MEHMET ZEKI SARIKAYA AND HÜSEYIN BUDAK

Abstract. In this paper, a general integral identity for fractional integrals is derived. By use this identity, we establish new some generelized inequalities of the Hermite-Hadamard's type for functions whose absolute values of derivatives are convex.

1. Introduction

The following definition for convex functions is well known in the mathematical literature:

The function $f:[a,b]\subset\mathbb{R}\rightarrow\mathbb{R}$, is said to be convex if the following inequality holds

$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. We say that f is concave if $(-f)$ is convex.

Many inequalities have been established for convex functions but the most famous inequality is the Hermite-Hadamard's inequality, due to its rich geometrical significance and applications(see, e.g.,[14, p.137], [8]). These inequalities state that if $f:I\rightarrow\mathbb{R}$ is a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$, then

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [1, 2, 8, 10, 11, 14],[16]-[18],[23],[24]) and the references cited therein.

The following lemma was proved for twice differentiable mappings in [8]:

Lemma 1. *Let $f:I\subset\mathbb{R}\rightarrow\mathbb{R}$ be a twice differentiable mapping on I^o, $a,b\in I$ with $a<b$ and f'' of integrable on $[a,b]$, the following equality holds:*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx=\frac{(b-a)^2}{2}\int_0^1 t(1-t)f''(ta+(1-t)b)\,dt. \tag{1.2}$$

In [10], by using Lemma 1, Hussain et al. proved some inequalities related to Hermite-Hadamard's inequality for s-convex functions:

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Theorem 1. *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be twice differentiable mapping on I° such that $f'' \in L_1[a, b]$, where $a, b \in I$ with $a < b$. If $|f''|$ is $s-$convex on $[a, b]$ for some fixed $s \in [0, 1]$ and $q \geq 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{2\times 6^{\frac{1}{p}}}\left[\frac{|f''(a)|^q+|f''(b)|^q}{(s+2)(s+3)}\right]^{\frac{1}{q}}, \tag{1.3}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Remark 1. *If we take $s = 1$ in (1.3), then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{12}\left[\frac{|f''(a)|^q+|f''(b)|^q}{2}\right]^{\frac{1}{q}}. \tag{1.4}$$

In [16], Sarikaya and Aktan gave the following inequalities:

Theorem 2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° with $f'' \in L_1[a, b]$. If $|f''|$ is a convex on $[a, b]$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{f(a)+f(b)}{2}\right| \leq \frac{(b-a)^2}{12}\left[\frac{|f''(a)|+|f''(b)|}{2}\right]. \tag{1.5}$$

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult [9, 12, 13, 15].

Definition 1. *Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.

Meanwhile, Sarikaya et al.[20] presented the following important integral identity including the first-order derivative of f to establish many interesting Hermite-Hadamard type inequalities for convexity functions via Riemann-Liouville fractional integrals of the order $\alpha > 0$.

Lemma 2. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $f' \in L[a, b]$, then the following equality for fractional integrals holds:*

(1.6)

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)\right] = \frac{b-a}{2}\int_0^1\left[(1-t)^\alpha-t^\alpha\right]f'(ta+(1-t)b)\,dt.$$

It is remarkable that Sarikaya et al.[20] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 3. *Let* $f : [a,b] \to \mathbb{R}$ *be a positive function with* $0 \leq a < b$ *and* $f \in L_1[a,b]$. *If* f *is a convex function on* $[a,b]$*, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.7}$$

with $\alpha > 0$.

On the other hand, in [22], Wang et al. extended Lemma 2 to the case of including a twice differentiable function involving Riemann-Liouville fractional integrals as follows:

Lemma 3. *Let* $f : [a,b] \to \mathbb{R}$ *be a twice differentiable mapping on* (a,b) *with* $0 \leq a < b$. *If* $f'' \in L[a,b]$, *then the following equality for fractional integrals holds:*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right] \\ &= \frac{(b-a)^2}{2(\alpha+1)}\int_0^1 \left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right] f''\left[ta+(1-t)b\right]dt. \end{aligned} \tag{1.8}$$

For some recent results connected with fractional integral inequalities see ([3, 4, 5, 6, 7],[19],[21],[22],[25])

The aim of this paper is to establish generalized Hermite-Hadamard type integral inequalities for Riemann-Liouville fractional integral and some other integral inequalities using the generalized identity is obtained for fractional integrals.

2. Main Results

Lemma 4. *Let* $f : [a,b] \to \mathbb{R}$ *be twice differentiable mapping on* (a,b) *with* $0 \leq a < b$. *If* $f'' \in L[a,b]$, *then the following equality for fractional integrals holds:*

$$\begin{aligned} &\frac{(\alpha+1)\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]}{(1-2\lambda)^2(b-a)^2} - \frac{\Gamma(\alpha+2)}{(1-2\lambda)^{\alpha+2}(b-a)^{\alpha+2}} \\ &\times\left[J_{(\lambda b+(1-\lambda)a)^+}^{\alpha}f(\lambda a+(1-\lambda)b)+J_{(\lambda a+(1-\lambda)b)^-}^{\alpha}f(\lambda b+(1-\lambda)a)\right] \\ &= \int_0^1 \left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right] f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt \end{aligned} \tag{2.1}$$

where $\lambda \in [0,1] \setminus \{\frac{1}{2}\}$ *and* $\alpha > 0$.

Proof. Denote

$$\begin{aligned} I &= \int_0^1 \left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right] f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt. \\ &= \int_0^1 f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt \\ &\quad -\int_0^1 (1-t)^{\alpha+1} f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt \\ &\quad -\int_0^1 t^{\alpha+1} f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt \\ &= I_1-I_2-I_3. \end{aligned} \tag{2.2}$$

Calculating I_1, I_2 and I_3, we have

$$\begin{aligned} I_1 &= \int_0^1 f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt \\ &= \frac{1}{(1-2\lambda)(b-a)}\left[f'(\lambda a+(1-\lambda)b)-f'(\lambda b+(1-\lambda)a)\right], \end{aligned} \tag{2.3}$$

$$\begin{aligned} I_2 &= \int_0^1 (1-t)^{\alpha+1} f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt \\ &= \frac{(1-t)^{\alpha+1}}{(1-2\lambda)(b-a)} f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\Big|_0^1 \\ &\quad +\frac{\alpha+1}{(1-2\lambda)(b-a)}\int_0^1 (1-t)^{\alpha+1} f'\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt \\ &= -\frac{f'(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)}+\frac{\alpha+1}{(1-2\lambda)(b-a)}\left[-\frac{f(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)}+\frac{\alpha}{(1-2\lambda)(b-a)}\right. \\ &\quad \left.\times\int_0^1 (1-t)^{\alpha-1} f\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt\right] \\ &= -\frac{f'(\lambda b+(1-\lambda)a)}{(1-2\lambda)(b-a)}-\frac{(\alpha+1)f(\lambda b+(1-\lambda)a)}{(1-2\lambda)^2(b-a)^2}+\frac{\alpha(\alpha+1)}{(1-2\lambda)^2(b-a)^2} \\ &\quad \times\int_0^1 (1-t)^{\alpha-1} f\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt \end{aligned} \tag{2.4}$$

and similarly

$$
\begin{aligned}
I_3 &= \int_0^1 t^{\alpha+1} f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt \\
&= \frac{f'(\lambda a+(1-\lambda)b)}{(1-2\lambda)(b-a)} - \frac{(\alpha+1)f(\lambda a+(1-\lambda)b)}{(1-2\lambda)^2(b-a)^2} + \frac{\alpha(\alpha+1)}{(1-2\lambda)^2(b-a)^2} \\
&\quad \times \int_0^1 t^{\alpha-1} f\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt. \qquad (2.5)
\end{aligned}
$$

Using (2.3), (2.4) and (2.5) in (2.2), it follows that

$$
\begin{aligned}
I &= I_1 - I_2 - I_3 \\
&= \frac{(\alpha+1)\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]}{(1-2\lambda)^2(b-a)^2} - \frac{\alpha(\alpha+1)}{(1-2\lambda)^2(b-a)^2} \\
&\quad \times \left[\int_0^1 (1-t)^{\alpha-1} f\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt\right. \\
&\quad \left. + \int_0^1 t^{\alpha-1} f\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]dt\right] \\
&= \frac{(\alpha+1)\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]}{(1-2\lambda)^2(b-a)^2} - \frac{\alpha(\alpha+1)}{(1-2\lambda)^{\alpha+2}(b-a)^{\alpha+2}} \\
&\quad \times \left[\int_{\lambda b+(1-\lambda)a}^{\lambda a+(1-\lambda)b} \left[(\lambda a+(1-\lambda)b)-x\right] f(x)dx + \int_{\lambda b+(1-\lambda)a}^{\lambda a+(1-\lambda)b} \left[x-(\lambda b+(1-\lambda)a)\right] f(x)dx\right] \\
&= \frac{(\alpha+1)\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]}{(1-2\lambda)^2(b-a)^2} - \frac{\Gamma(\alpha+2)}{(1-2\lambda)^{\alpha+2}(b-a)^{\alpha+2}} \\
&\quad \times \left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^+} f(\lambda a+(1-\lambda)b) + J^{\alpha}_{(\lambda a+(1-\lambda)b)^-} f(\lambda b+(1-\lambda)a)\right].
\end{aligned}
$$

□

Corollary 1. *Under the some assumptions of Lemma 4 with $\lambda = 0$, we have*

$$
\begin{aligned}
&\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{b^-} f(a) + J^{\alpha}_{a^+} f(b)\right] \\
&= \frac{(b-a)^2}{2(\alpha+1)} \int_0^1 \left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right] f''\left[tb+(1-t)a\right]dt.
\end{aligned}
$$

Remark 2. *If we take $\lambda = 1$ in Lemma 4, we obtain*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{(-1)^{\alpha+2}\, 2(b-a)^{\alpha}} \left[J^{\alpha}_{a^-} f(b) + J^{\alpha}_{b^+} f(a)\right]$$

$$= \frac{(b-a)^2}{2(\alpha+1)} \int_0^1 \left[1-(1-t)^{\alpha+1} - t^{\alpha+1}\right] f''\left[ta+(1-t)b\right] dt. \tag{2.6}$$

By using $J^{\alpha}_{b^+} f(a) + J^{\alpha}_{a^-} f(b) = (-1)^{\alpha}\left[J^{\alpha}_{a^+} f(b) + J^{\alpha}_{b^-} f(a)\right]$ in (2.6), the identity (2.6) reduce to the identity (1.8) which prove by Wang et al. in [22].

Remark 3. *If we take $\alpha = 1$ in (2.6), then the equality in (2.6) becomes to the equality (1.2).*

Theorem 4. *$f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b) with $0 \leq a < b$. If $|f''|^q$ is convex on $[a,b]$ for some fixed $q \geq 1$ then following inequality for fractional holds:*

$$\left| \frac{(\alpha+1)\left[f(\lambda a+(1-\lambda)b) + f(\lambda b+(1-\lambda)a)\right]}{(1-2\lambda)^2(b-a)^2} - \frac{\Gamma(\alpha+2)}{(1-2\lambda)^{\alpha+2}(b-a)^{\alpha+2}} \right.$$
$$\left. \times \left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^+} f(\lambda a+(1-\lambda)b) + J^{\alpha}_{(\lambda a+(1-\lambda)b)^-} f(\lambda b+(1-\lambda)a)\right] \right|$$

$$\leq \frac{\alpha}{\alpha+2}\left[\frac{|f''(\lambda a+(1-\lambda)b)|^q + |f''(\lambda b+(1-\lambda)a)|^q}{2}\right]^{\frac{1}{q}} \tag{2.7}$$

where $\lambda \in [0,1] \setminus \{\frac{1}{2}\}$ and $\alpha > 0$.

Proof. Firstly, we suppose that $q = 1$. From Lemma 4, we have

$$\left| \frac{(\alpha+1)\left[f(\lambda a+(1-\lambda)b) + f(\lambda b+(1-\lambda)a)\right]}{(1-2\lambda)^2(b-a)^2} - \frac{\Gamma(\alpha+2)}{(1-2\lambda)^{\alpha+2}(b-a)^{\alpha+2}} \right.$$
$$\left. \times \left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^+} f(\lambda a+(1-\lambda)b) + J^{\alpha}_{(\lambda a+(1-\lambda)b)^-} f(\lambda b+(1-\lambda)a)\right] \right|$$

$$\leq \int_0^1 \left[1-(1-t)^{\alpha+1} - t^{\alpha+1}\right] \left|f''\left[t(\lambda a+(1-\lambda)b) + (1-t)(\lambda b+(1-\lambda)a)\right]\right| dt.$$

$$\leq \int_0^1 \left[1-(1-t)^{\alpha+1} - t^{\alpha+1}\right] \left[t\left|f''(\lambda a+(1-\lambda)b)\right| + (1-t)\left|f''(\lambda b+(1-\lambda)a)\right|\right] dt$$

$$= \left|f''(\lambda a+(1-\lambda)b)\right| \int_0^1 \left[t-(1-t)^{\alpha+1} t - t^{\alpha+2}\right] dt$$
$$+\left|f''(\lambda b+(1-\lambda)a)\right| \int_0^1 \left[(1-t)-(1-t)^{\alpha+2} - t^{\alpha+1}(1-t)\right] dt$$

$$= \frac{\alpha}{(\alpha+2)}\left[\frac{|f''(\lambda a+(1-\lambda)b)| + |f''(\lambda b+(1-\lambda)a)|}{2}\right].$$

Here we use $(1-t)^{\alpha+1}+t^{\alpha+1}<1$ for any $t\in[0,1]$ and the convexity of $|f''|$.

Secondly, we suppose that $q>1$. Using Lemma 4 and the power mean equality for q, we have

$$\begin{aligned}
&\int_0^1 \left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]|f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]|\,dt. \\
\leq\ & \left(\int_0^1 \left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 \left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]|f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]|^q\,dt\right)^{\frac{1}{q}}. \qquad (2.8)
\end{aligned}$$

Using Lemma 4, (2.8) and the convexity of $|f''|^q$, we have

$$\begin{aligned}
&\Bigg|\frac{(\alpha+1)\left[f\left(\lambda a+(1-\lambda)b\right)+f\left(\lambda b+(1-\lambda)a\right)\right]}{(1-2\lambda)^2(b-a)^2}-\frac{\Gamma(\alpha+2)}{(1-2\lambda)^{\alpha+2}(b-a)^{\alpha+2}} \\
&\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^+}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(\lambda b+(1-\lambda)a)\right]\Bigg| \\
\leq\ & \left(\int_0^1 \left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 \left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]|f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]|^q\,dt\right) \\
\leq\ & \left(\int_0^1 \left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 \left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]\left[t\,|f''(\lambda a+(1-\lambda)b)|^q+(1-t)\,|f''(\lambda b+(1-\lambda)a)|^q\right]dt\right)^{\frac{1}{q}} \\
=\ & \left(\left[t+\frac{(1-t)^{\alpha+2}}{\alpha+2}-\frac{t^{\alpha+2}}{\alpha+2}\right]\Bigg|_0^1\right)^{1-\frac{1}{q}} \\
&\times\left(\frac{\alpha}{\alpha+2}\left[\frac{|f''(\lambda a+(1-\lambda)b)|^q+|f''(\lambda b+(1-\lambda)a)|^q}{2}\right]\right)^{\frac{1}{q}} \\
=\ & \left(\frac{\alpha}{\alpha+2}\right)^{1-\frac{1}{q}}\left(\frac{\alpha}{\alpha+2}\right)^{\frac{1}{q}}\left[\frac{|f''(\lambda a+(1-\lambda)b)|^q+|f''(\lambda b+(1-\lambda)a)|^q}{2}\right]^{\frac{1}{q}} \\
=\ & \frac{\alpha}{\alpha+2}\left[\frac{|f''(\lambda a+(1-\lambda)b)|^q+|f''(\lambda b+(1-\lambda)a)|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}$$

□

Corollary 2. *Under the same assumptions of Theorem 4 with $\lambda = 0$, then we take*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{b-}^{\alpha}f(a)+J_{a+}^{\alpha}f(b)\right]\right| \leq \frac{\alpha(b-a)^2}{2(\alpha+1)(\alpha+2)}\left[\frac{|f''(a)|^q+|f''(b)|^q}{2}\right]^{\frac{1}{q}}. \tag{2.9}$$

Remark 4. *If we take $\lambda = 1$ in Theorem 4, then we obtain*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{(-1)^{\alpha+2}2(b-a)^{\alpha}}\left[J_{b+}^{\alpha}f(a)+J_{a-}^{\alpha}f(b)\right]\right| \leq \frac{\alpha(b-a)^2}{2(\alpha+1)(\alpha+2)}\left[\frac{|f''(a)|^q+|f''(b)|^q}{2}\right]^{\frac{1}{q}}. \tag{2.10}$$

By using $J_{b+}^{\alpha}f(a)+J_{a-}^{\alpha}f(b)=(-1)^{\alpha}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]$ in (2.10), the inequality (2.10) reduces to the inequality (2.9).

Remark 5. *If we take $\alpha = 1$ in (2.9), then the inequality in (2.9) becomes to the inequality (1.4).*

Remark 6. *If we take $\alpha = 1$ and $q = 1$ in (2.9), then the inequality in (2.9) becomes to the inequality (1.5).*

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b) with $0 \leq a < b$. If $|f''|^q$ is convex on $[a,b]$ for some fixed $q > 1$, then following inequality for fractional holds:*

$$\begin{aligned} &\left|\frac{(\alpha+1)\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]}{(1-2\lambda)^2(b-a)^2}-\frac{\Gamma(\alpha+2)}{(1-2\lambda)^{\alpha+2}(b-a)^{\alpha+2}}\right. \\ &\left.\times\left[J_{(\lambda b+(1-\lambda)a)^+}^{\alpha}f(\lambda a+(1-\lambda)b)+J_{(\lambda a+(1-\lambda)b)^-}^{\alpha}f(\lambda b+(1-\lambda)a)\right]\right| \\ \leq\ &\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{|f''(\lambda a+(1-\lambda)b)|^q+|f''(\lambda b+(1-\lambda)a)|^q}{2}\right)^{\frac{1}{q}} \end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\lambda \in [0,1]\backslash\{\frac{1}{2}\}$ and $\alpha > 0$.

Proof. From Lemma 4 and using the well-known Hölder's inequality and convexity of $|f''|^q$, we have

$$\begin{aligned}
&\left|\frac{(\alpha+1)\left[f\left(\lambda a+(1-\lambda)b\right)+f\left(\lambda b+(1-\lambda)a\right)\right]}{(1-2\lambda)^2(b-a)^2}-\frac{\Gamma\left(\alpha+2\right)}{(1-2\lambda)^{\alpha+2}(b-a)^{\alpha+2}}\right.\\
&\left.\times\left[J^{\alpha}_{(\lambda b+(1-\lambda)a)^+}f(\lambda a+(1-\lambda)b)+J^{\alpha}_{(\lambda a+(1-\lambda)b)^-}f(\lambda b+(1-\lambda)a)\right]\right|\\
\leq\ &\int_0^1\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]\left|f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|dt\\
\leq\ &\left(\int_0^1\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]^p dt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1\left|f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|^q dt\right)^{\frac{1}{q}}\\
\leq\ &\left(\int_0^1\left[1-(1-t)^{p(\alpha+1)}-t^{p(\alpha+1)}\right]dt\right)^{\frac{1}{p}}\\
&\times\left(|f''(\lambda a+(1-\lambda)b)|^q\int_0^1 t\,dt+|f''(\lambda b+(1-\lambda)a)|^q\int_0^1(1-t)dt\right)^{\frac{1}{q}}\\
=\ &\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{|f''(\lambda a+(1-\lambda)b)|^q+|f''(\lambda b+(1-\lambda)a)|^q}{2}\right)^{\frac{1}{q}} \qquad (2.11)
\end{aligned}$$

Here we use

$$\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]^p\leq 1-(1-t)^{p(\alpha+1)}-t^{p(\alpha+1)}$$

for any $t\in[0,1]$ which follows from

$$(A-B)^p\leq A^p-B^p,$$

for any $A>B\geq 0$ and $p\geq 1$. which completes the proof. □

Corollary 3. *Under the same assumptions of Theorem 5 with $\lambda=0$, then we have*

$$\begin{aligned}
&\left|\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{\Gamma\left(\alpha+1\right)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(b)+J^{\alpha}_{b^-}f(a)\right]\right| \qquad (2.12)\\
\leq\ &\frac{(b-a)^2}{2\left(\alpha+1\right)}\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

Remark 7. *If we take $\lambda=1$ in Theorem 5, then we obtain*

$$\begin{aligned}
&\left|\frac{f\left(a\right)+f\left(b\right)}{2}-\frac{\Gamma\left(\alpha+1\right)}{(-1)^{\alpha+2}2(b-a)^{\alpha}}\left[J^{\alpha}_{b^+}f(a)+J^{\alpha}_{a^-}f(b)\right]\right|\\
\leq\ &\frac{(b-a)^2}{2\left(\alpha+1\right)}\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}. \qquad (2.13)
\end{aligned}$$

By using $J_{b^+}^{\alpha}f(a)+J_{a^-}^{\alpha}f(b)=(-1)^{\alpha}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]$ *in (2.13), the inequality (2.13) reduces to the inequality (2.12).*

Theorem 6. *Let* $f:[a,b]\to\mathbb{R}$ *be a twice differentiable mapping on* (a,b) *with* $0\leq a<b$. *If* $|f''|^q$ *is convex on* $[a,b]$, *for some fixed* $q\geq 1$, *then following inequality for fractional hold:*

$$\left|\frac{(\alpha+1)\left[f\left(\lambda a+(1-\lambda)b\right)+f\left(\lambda b+(1-\lambda)a\right)\right]}{(1-2\lambda)^2(b-a)^2}-\frac{\Gamma(\alpha+2)}{(1-2\lambda)^{\alpha+2}(b-a)^{\alpha+2}}\right.$$
$$\left.\times\left[J_{(\lambda b+(1-\lambda)a)^+}^{\alpha}f(\lambda a+(1-\lambda)b)+J_{(\lambda a+(1-\lambda)b)^-}^{\alpha}f(\lambda b+(1-\lambda)a)\right]\right|$$
$$\leq\left(\frac{q(\alpha+1)-1}{q(\alpha+1)+1}\right)^{\frac{1}{q}}\left(\frac{|f''(\lambda a+(1-\lambda)b)|^q+|f''(\lambda b+(1-\lambda)a)|^q}{2}\right)^{\frac{1}{q}}$$

where $\lambda\in[0,1]\setminus\{\frac{1}{2}\}$ *and* $\alpha>0$.

Proof. From Lemma 4 and using well-known Hölder's inequality, we have

$$\left|\frac{(\alpha+1)\left[f\left(\lambda a+(1-\lambda)b\right)+f\left(\lambda b+(1-\lambda)a\right)\right]}{(1-2\lambda)^2(b-a)^2}-\frac{\Gamma(\alpha+2)}{(1-2\lambda)^{\alpha+2}(b-a)^{\alpha+2}}\right.$$
$$\left.\times\left[J_{(\lambda b+(1-\lambda)a)^+}^{\alpha}f(\lambda a+(1-\lambda)b)+J_{(\lambda a+(1-\lambda)b)^-}^{\alpha}f(\lambda b+(1-\lambda)a)\right]\right|$$
$$\leq\int_0^1\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]\left|f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|dt$$
$$\leq\left(\int_0^1 1dt\right)^{\frac{1}{p}}\left(\int_0^1\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]^q\left|f''\left[t(\lambda a+(1-\lambda)b)+(1-t)(\lambda b+(1-\lambda)a)\right]\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq\left(\int_0^1\left[1-(1-t)^{q(\alpha+1)}-t^{q(\alpha+1)}\right]\left[t\left|f''(\lambda a+(1-\lambda)b)\right|^q+(1-t)\left|f''(\lambda b+(1-\lambda)a)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$=\left(\left|f''(\lambda a+(1-\lambda)b)\right|^q\int_0^1\left[t-(1-t)^{q(\alpha+1)}t-t^{q(\alpha+1)+1}\right]dt\right.$$
$$\left.+\left|f''(\lambda b+(1-\lambda)a)\right|^q\int_0^1\left[(1-t)-(1-t)^{q(\alpha+1)+1}-t^{q(\alpha+1)}(1-t)\right]\right)^{\frac{1}{q}}$$
$$=\left(\frac{q(\alpha+1)-1}{q(\alpha+1)+1}\right)^{\frac{1}{q}}\left(\frac{|f''(\lambda a+(1-\lambda)b)|^q+|f''(\lambda b+(1-\lambda)a)|^q}{2}\right)^{\frac{1}{q}}.$$

Here we use

$$\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]^q\leq 1-(1-t)^{q(\alpha+1)}-t^{q(\alpha+1)}$$

for any $t\in[0,1]$ which follows from

$$(A-B)^q\leq A^q-B^q,$$

for any $A > B \geq 0$ and $q \geq 1$ which completes the proof. □

Corollary 4. *Under the same assumptions of Theorem 6 with $\lambda = 0$, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a)\right]\right| \tag{2.14}$$

$$\leq \frac{(b-a)^{2}}{2(\alpha+1)}\left(\frac{q(\alpha+1)-1}{q(\alpha+1)+1}\right)^{\frac{1}{q}}\left(\frac{|f''(a|^{q}+|f''(b)|^{q}}{2}\right)^{\frac{1}{q}}.$$

Corollary 5. *If we take $\lambda = 1$ in Theorem 6, then we obtain*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{(-1)^{\alpha+2}2(b-a)^{\alpha}}\left[J_{b^{+}}^{\alpha}f(a)+J_{a^{-}}^{\alpha}f(b)\right]\right|$$

$$\leq \frac{(b-a)^{2}}{2(\alpha+1)}\left(\frac{q(\alpha+1)-1}{q(\alpha+1)+1}\right)^{\frac{1}{q}}\left(\frac{|f''(a)|^{q}+|f''(b)|^{q}}{2}\right)^{\frac{1}{q}}. \tag{2.15}$$

By using $J_{b^{+}}^{\alpha}f(a)+J_{a^{-}}^{\alpha}f(b)=(-1)^{\alpha}\left[J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a)\right]$ in (2.15), the inequality (2.15) reduces to the inequality (2.14).

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Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY

E-mail address: sarikayamz@gmail.com

E-mail address: hsyn.budak@gmail.com