

**GENERALIZED HERMITE -HADAMARD TYPE INTEGRAL
INEQUALITIES FOR s -CONVEX FUNCTIONS VIA
FRACTIONAL INTEGRALS**

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ABSTRACT. In this paper, we have established Hermite-Hadamard type inequalities for the class of functions whose derivatives in absolute value at certain powers are s -convex functions by using fractional integrals depending on a parameter.

1. INTRODUCTION

Definition 1. *The function $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$, is said to be convex if the following inequality holds*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. We say that f is concave if $(-f)$ is convex.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g., [18, p.137], [12]). These inequalities state that if $f : I \rightarrow \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [1, 2, 12, 13, 18]) and the references cited therein.

Definition 2. [4] *Let s be a real numbers, $s \in (0, 1]$. A function $f : [0, \infty) \rightarrow [0, \infty)$ is said to be s -convex (in the second sense), or that f belongs to the class K_s^2 , if f*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y)$$

for all $x, y \in [0, \infty)$ and $\lambda \in [0, 1]$.

An s -convex function was introduced in Breckner's paper [4] and a number of properties and connections with s -convexity in the first sense are discussed in paper [11]. Of course, s -convexity means just convexity when $s = 1$.

In [10], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s -convex functions in the second sense.

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Theorem 1. [10] *Suppose that $f : [0, \infty) \rightarrow [0, \infty)$ is an s -convex function in the second sense, where $s \in (0, 1)$, and let $a, b \in [0, \infty)$, $a < b$. If $f \in L^1([a, b])$, then the following inequalities hold:*

$$(1.2) \quad 2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{s+1}.$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.2).

Meanwhile, Sarikaya et al.[21] presented the following important integral identity including the first-order derivative of f to establish many interesting Hermite-Hadamard type inequalities for convexity functions via Riemann-Liouville fractional integrals of the order $\alpha > 0$.

Lemma 1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $f' \in L[a, b]$, then the following equality for fractional integrals holds:*

$$(1.3) \quad \begin{aligned} & \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \\ &= \frac{b-a}{2} \int_0^1 [(1-t)^\alpha - t^\alpha] f'(ta + (1-t)b) dt. \end{aligned}$$

It is remarkable that Sarikaya et al.[21] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 2. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$(1.4) \quad f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \leq \frac{f(a) + f(b)}{2}$$

with $\alpha > 0$.

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult [14, 15, 17, 19].

Definition 3. *Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

For some recent results connected with fractional integral inequalities see ([3, 5, 6, 7, 8, 9],[16],[20],[22],[23],[24])

The aim of this paper is to establish generalized Hermite-Hadamard type integral inequalities for the class of functions whose derivatives in absolute value at certain powers are s -convex functions by using Riemann-Liouville fractional integral

and some other integral inequalities. The results presented in this paper provide extensions of those given in earlier works.

2. MAIN RESULTS

For our results, we give the following important fractional integral identity [22]:

Lemma 2. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $0 \leq a < b$. If $f' \in L[a, b]$, then the following equality for fractional integrals hold:*

$$\begin{aligned}
 & -\frac{f(\lambda a + (1-\lambda)b) + f(\lambda b + (1-\lambda)a)}{(1-2\lambda)(b-a)} + \frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}} \\
 & \times \left[J_{(\lambda b + (1-\lambda)a)^+}^\alpha f(\lambda a + (1-\lambda)b) + J_{(\lambda a + (1-\lambda)b)^-}^\alpha f(\lambda b + (1-\lambda)a) \right] \\
 (2.1) \quad & = \int_0^1 [(1-t)^\alpha - t^\alpha] f' [t(\lambda a + (1-\lambda)b) + (1-t)(\lambda b + (1-\lambda)a)] dt
 \end{aligned}$$

where $\lambda \in [0, 1] \setminus \{\frac{1}{2}\}$ and $\alpha > 0$.

Theorem 3. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $0 \leq a < b$. If $|f'|^q$, $q \geq 1$ is s -convex on $[a, b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
 & \left| \frac{f(\lambda a + (1-\lambda)b) + f(\lambda b + (1-\lambda)a)}{(1-2\lambda)(b-a)} - \frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}} \right. \\
 & \times \left. \left[J_{(\lambda b + (1-\lambda)a)^+}^\alpha f(\lambda a + (1-\lambda)b) + J_{(\lambda a + (1-\lambda)b)^-}^\alpha f(\lambda b + (1-\lambda)a) \right] \right| \\
 (2.2) \quad & \leq \left(\frac{2}{\alpha+1} \left[1 - \frac{1}{2^\alpha} \right] \right)^{1-\frac{1}{q}} \left[|f'(\lambda a + (1-\lambda)b)|^q + |f'(\lambda b + (1-\lambda)a)|^q \right]^{\frac{1}{q}} \\
 & \times \left[B_{1 \setminus 2}(s+1, \alpha+1) - \frac{1}{2^{\alpha+s}(\alpha+s+1)} + \frac{1}{\alpha+s+1} - B_{1/2}(\alpha+1, s+1) \right]^{\frac{1}{q}}
 \end{aligned}$$

where $\lambda \in [0, 1] \setminus \{\frac{1}{2}\}$, $\alpha > 0$, and B_x is the incomplete beta function defined as follows

$$(2.3) \quad B_x(m, n) = \int_0^x t^{m-1}(1-t)^{n-1}, \quad m, n > 0, \quad 0 < x \leq 1.$$

Proof. Firstly, we suppose that $q = 1$. Using Lemma 2 and s -convexity of $|f'|^q$, we find that

$$\begin{aligned}
& \left| \frac{f(\lambda a + (1-\lambda)b) + f(\lambda b + (1-\lambda)a)}{(1-2\lambda)(b-a)} - \frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}} \right. \\
& \quad \left. \times \left[J_{(\lambda b+(1-\lambda)a)^+}^\alpha f(\lambda a + (1-\lambda)b) + J_{(\lambda a+(1-\lambda)b)^-}^\alpha f(\lambda b + (1-\lambda)a) \right] \right| \\
& \leq \int_0^1 |(1-t)^\alpha - t^\alpha| |f'[t(\lambda a + (1-\lambda)b) + (1-t)(\lambda b + (1-\lambda)a)]| dt \\
& \leq \int_0^1 |(1-t)^\alpha - t^\alpha| [t^s |f'(\lambda a + (1-\lambda)b)| + (1-t)^s |f'(\lambda b + (1-\lambda)a)|] dt \\
& = |f'(\lambda a + (1-\lambda)b)| \left[\int_0^{\frac{1}{2}} [(1-t)^\alpha \cdot t^s - t^{\alpha+s}] dt + \int_{\frac{1}{2}}^1 [t^{\alpha+s} - (1-t)^\alpha \cdot t^s] dt \right] \\
& \quad + |f'(\lambda b + (1-\lambda)a)| \left[\int_0^{\frac{1}{2}} [(1-t)^{\alpha+s} - t^\alpha (1-t)^s] dt + \int_{\frac{1}{2}}^1 [t^\alpha (1-t)^s - (1-t)^{\alpha+s}] dt \right] \\
& = \left[B_{1/2}(s+1, \alpha+1) - B_{1/2}(\alpha+1, s+1) - \frac{1}{2^{\alpha+s}(\alpha+s+1)} + \frac{1}{\alpha+s+1} \right] \\
& \quad \times [|f'(\lambda a + (1-\lambda)b)| + |f'(\lambda b + (1-\lambda)a)|]
\end{aligned}$$

Secondly, we suppose that $q > 1$. Using Lemma 2 and power mean inequality, we obtain

$$\begin{aligned}
& \int_0^1 |(1-t)^\alpha - t^\alpha| |f'[t(\lambda a + (1-\lambda)b) + (1-t)(\lambda b + (1-\lambda)a)]| dt \\
& \leq \left(\int_0^1 |(1-t)^\alpha - t^\alpha| dt \right)^{1-\frac{1}{q}} \\
(2.4) \quad & \times \left(\int_0^1 |(1-t)^\alpha - t^\alpha| |f'[t(\lambda a + (1-\lambda)b) + (1-t)(\lambda b + (1-\lambda)a)]|^q dt \right)^{\frac{1}{q}}.
\end{aligned}$$

Hence, using s -convexity of $|f'|^q$ and (2.4) we obtain

$$\begin{aligned}
 & \left| \frac{f(\lambda a + (1-\lambda)b) + f(\lambda b + (1-\lambda)a)}{(1-2\lambda)(b-a)} - \frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}} \right. \\
 & \quad \left. \times \left[J_{(\lambda b+(1-\lambda)a)^+}^\alpha f(\lambda a + (1-\lambda)b) + J_{(\lambda a+(1-\lambda)b)^-}^\alpha f(\lambda b + (1-\lambda)a) \right] \right| \\
 & \leq \left(\int_0^1 |(1-t)^\alpha - t^\alpha| dt \right)^{1-\frac{1}{q}} \\
 & \quad \times \left(\int_0^1 |(1-t)^\alpha - t^\alpha| |f'[t(\lambda a + (1-\lambda)b) + (1-t)(\lambda b + (1-\lambda)a)]|^q dt \right)^{\frac{1}{q}} \\
 & \leq \left(\int_0^{\frac{1}{2}} [(1-t)^\alpha - t^\alpha] dt + \int_{\frac{1}{2}}^1 [t^\alpha - (1-t)^\alpha] dt \right)^{1-\frac{1}{q}} \\
 & \quad \times \left(\int_0^1 |(1-t)^\alpha - t^\alpha| [t^s |f'(\lambda a + (1-\lambda)b)|^q + (1-t)^s |f'(\lambda b + (1-\lambda)a)|^q] dt \right)^{\frac{1}{q}} \\
 & \leq \left(\frac{2}{\alpha+1} \left[1 - \frac{1}{2^\alpha} \right] \right)^{1-\frac{1}{q}} [|f'(\lambda a + (1-\lambda)b)|^q + |f'(\lambda b + (1-\lambda)a)|^q]^{\frac{1}{q}} \\
 & \quad \left[B_{1/2}(s+1, \alpha+1) - B_{1/2}(\alpha+1, s+1) - \frac{1}{2^{\alpha+s}(\alpha+s+1)} + \frac{1}{\alpha+s+1} \right]^{\frac{1}{q}}.
 \end{aligned}$$

This completes the proof. \square

Remark 1. If we take $s = 1$ in Theorem 3, the Theorem 3 reduces the Theorem 3 which is proved by Sarikaya and Budak in [22].

Remark 2 (Trapezoid Inequality). If we take $s = 1$, $\alpha = 1$ and $\lambda = 0$ (or $\lambda = 1$) in Theorem 3, we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{8} 2^{\frac{q-1}{q}} [|f'(a)|^q + |f'(b)|^q]^{\frac{1}{q}}$$

where $q \geq 1$. Choosing $q = 1$ in last inequality, it follows that

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{8} [|f'(a)| + |f'(b)|]$$

which are proved Dragomir and Agarwal in [13].

Theorem 4. Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $0 \leq a < b$. If $|f'|^q$ s -convex on $[a, b]$ for some fixed $q > 1$, then the following inequality for

fractional integrals holds:

$$\begin{aligned} & \left| \frac{f(\lambda a + (1-\lambda)b) + f(\lambda b + (1-\lambda)a)}{(1-2\lambda)(b-a)} - \frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}} \right. \\ & \quad \left. \times \left[J_{(\lambda b+(1-\lambda)a)^+}^\alpha f(\lambda a + (1-\lambda)b) + J_{(\lambda a+(1-\lambda)b)^-}^\alpha f(\lambda b + (1-\lambda)a) \right] \right| \\ & \leq \left(\frac{2}{\alpha p + 1} \right)^{\frac{1}{p}} \left(1 - \frac{1}{2^{\alpha p}} \right)^{\frac{1}{p}} \left(\frac{|f'(\lambda a + (1-\lambda)b)|^q + |f'(\lambda b + (1-\lambda)a)|^q}{s+1} \right)^{\frac{1}{q}} \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > 0$ and $\lambda \in [0, 1] \setminus \{\frac{1}{2}\}$.

Proof. Proof: Using Lemma 2, s -convexity of $|f|^q$ and well-known Hölder's inequality, we obtain

$$\begin{aligned} & \left| \frac{f(\lambda a + (1-\lambda)b) + f(\lambda b + (1-\lambda)a)}{(1-2\lambda)(b-a)} - \frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}} \right. \\ & \quad \left. \times \left[J_{(\lambda b+(1-\lambda)a)^+}^\alpha f(\lambda a + (1-\lambda)b) + J_{(\lambda a+(1-\lambda)b)^-}^\alpha f(\lambda b + (1-\lambda)a) \right] \right| \\ & \leq \int_0^1 |(1-t)^\alpha - t^\alpha| |f' [t(\lambda a + (1-\lambda)b) + (1-t)(\lambda b + (1-\lambda)a)]| dt \\ & \leq \left(\int_0^1 |(1-t)^\alpha - t^\alpha|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f' [t(\lambda a + (1-\lambda)b) + (1-t)(\lambda b + (1-\lambda)a)]|^q dt \right)^{\frac{1}{q}} \\ & \leq \left(\int_0^{\frac{1}{2}} [(1-t)^\alpha - t^\alpha]^p dt + \int_{\frac{1}{2}}^1 [t^\alpha - (1-t)^\alpha]^p dt \right)^{\frac{1}{p}} \\ & \quad \times \left(\int_0^1 t^s |f'(\lambda a + (1-\lambda)b)|^q + (1-t)^s |f'(\lambda b + (1-\lambda)a)|^q dt \right)^{\frac{1}{q}} \\ & = \left(\frac{2}{\alpha p + 1} \right)^{\frac{1}{p}} \left(1 - \frac{1}{2^{\alpha p}} \right)^{\frac{1}{p}} \left(\frac{|f'(\lambda a + (1-\lambda)b)|^q + |f'(\lambda b + (1-\lambda)a)|^q}{s+1} \right)^{\frac{1}{q}}. \end{aligned}$$

Here, we use

$$(c-d)^p \leq c^p - d^p,$$

for any $c > d \geq 0$ and $p \geq 1$. □

Remark 3. If we take $s = 1$ in Theorem 4, the Theorem 4 reduces the Theorem 4 which is proved by Sarikaya and Budak in [22].

Remark 4 (Trapezoid Inequality). *If we take $s = 1$, $\alpha = 1$ and $\lambda = 0$ (or $\lambda = 1$) in Theorem 4, we have*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2} \left[\frac{2}{p+1} \left(1 - \frac{1}{2^p} \right) \right]^{\frac{1}{p}} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}$$

which are proved Dragomir and Agarwal in [13].

Theorem 5. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $0 \leq a < b$. If $|f'|^q$ s -convex on $[a, b]$ for some fixed $q \geq 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} & \left| \frac{f(\lambda a + (1-\lambda)b) + f(\lambda b + (1-\lambda)a)}{(1-2\lambda)(b-a)} - \frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}} \right. \\ & \quad \times \left. \left[J_{(\lambda b + (1-\lambda)a)^+}^\alpha f(\lambda a + (1-\lambda)b) + J_{(\lambda a + (1-\lambda)b)^-}^\alpha f(\lambda b + (1-\lambda)a) \right] \right| \\ & \leq \left[\frac{1}{q\alpha+1} \left(1 - \frac{1}{2^{q\alpha+1}} \right) \right]^{\frac{1}{q}} \left(\frac{|f'(\lambda a + (1-\lambda)b)|^q + |f'(\lambda b + (1-\lambda)a)|^q}{s+1} \right)^{\frac{1}{q}} \end{aligned}$$

where $\lambda \in [0, 1] \setminus \{\frac{1}{2}\}$ and $\alpha > 0$.

Proof. Using Lemma 2, s -convexity of $|f'|^q$, and well-known Hölder's inequality, we have

$$\begin{aligned} & \left| \frac{f(\lambda a + (1-\lambda)b) + f(\lambda b + (1-\lambda)a)}{(1-2\lambda)(b-a)} - \frac{\Gamma(\alpha+1)}{(1-2\lambda)^{\alpha+1}(b-a)^{\alpha+1}} \right. \\ & \quad \times \left. \left[J_{(\lambda b + (1-\lambda)a)^+}^\alpha f(\lambda a + (1-\lambda)b) + J_{(\lambda a + (1-\lambda)b)^-}^\alpha f(\lambda b + (1-\lambda)a) \right] \right| \\ & \leq \int_0^1 |(1-t)^\alpha - t^\alpha| |f' [t(\lambda a + (1-\lambda)b) + (1-t)(\lambda b + (1-\lambda)a)]| dt \\ & \leq \left(\int_0^1 |(1-t)^\alpha - t^\alpha|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f' [t(\lambda a + (1-\lambda)b) + (1-t)(\lambda b + (1-\lambda)a)]|^q dt \right)^{\frac{1}{q}} \\ & = \left(\int_0^{\frac{1}{2}} [(1-t)^\alpha - t^\alpha]^p dt + \int_{\frac{1}{2}}^1 [t^\alpha - (1-t)^\alpha]^p dt \right)^{\frac{1}{p}} \\ & \quad \times \left(\int_0^1 [t^s |f'(\lambda a + (1-\lambda)b)|^q + (1-t)^s |f'(\lambda b + (1-\lambda)a)|^q] dt \right)^{\frac{1}{q}} \\ & \leq \left(\int_0^{\frac{1}{2}} [(1-t)^{\alpha p} - t^{\alpha p}] dt + \int_{\frac{1}{2}}^1 [t^{\alpha p} - (1-t)^{\alpha p}] dt \right)^{\frac{1}{p}} \\ & \quad \times \left(\frac{|f'(\lambda a + (1-\lambda)b)|^q + |f'(\lambda b + (1-\lambda)a)|^q}{s+1} \right)^{\frac{1}{q}}. \end{aligned}$$

Here, we use $(c-d)^p \leq c^p - d^p$, for any $c > d \geq 0$ and $q \geq 1$. \square

Remark 5. If we take $s = 1$ in Theorem 5, the Theorem 5 reduces the Theorem 5 which is proved by Sarikaya and Budak in [22].

Remark 6 (Trapezoid Inequality). If we take $s = 1$, $\alpha = 1$ and $\lambda = 0$ (or $\lambda = 1$) in Theorem 5, we have

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{2} \left[\frac{1}{q+1} \left(1 - \frac{1}{2^{q+1}} \right) \right]^{\frac{1}{q}} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}. \end{aligned}$$

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