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Some new integral inequalities via variant of Pompeiu's mean value theorem

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Abstract

The main of this paper is to establish an inequality providing some better bounds for integral mean by using a mean value theorem. Our results are generalization the results of Farooq et. al in [8].

1 Introduction

The inequality of Ostrowski [7] gives us an estimate for the deviation of the values of a smooth function from its mean value. More precisely, if $f : [a, b] \to \mathbb{R}$ is a differentiable function with bounded derivative, then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

for every $x \in [a, b]$. Moreover the constant $1/4$ is the best possible.

For a differentiable function $f : [a, b] \to \mathbb{R}$, $a{\cdot}b > 0$, Dragomir has in [2] proved, using Pompeiu's mean value theorem [5], the following Ostrowski type inequality:

$$\left| \frac{a+b}{2} . \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq D(x) \|f - \ell f'\|_\infty$$

where $\ell(t) = t,\ t \in [a, b]$, and

$$D(x) = \frac{(b-a)}{|x|} \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right].$$

In [4], Pecaric and Ungar proved a general estimate with the p-norm, $1 < p < \infty$, which will for $p = 1$ give the Dragomir [2] result.

In [8], for a twice differentiable function $f : [a, b] \to \mathbb{R}$, $a \cdot b > 0$ Farooq et. al gave the following integral inequality:

$$\left| \frac{a+b}{2} \left(\frac{2f(x)}{3x} - \frac{f'(x)}{2} \right) + \frac{1}{3} \left(\frac{bf(b) - af(a)}{b-a} \right) - \frac{1}{b-a} \int_a^b f(t)dt \right|$$
$$\leq \frac{(b-a)}{3|x|} \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] \left\| 2lf - 2\ell f' + \ell^2 f'' \right\|_\infty$$

where $\ell(t) = t, \ t \in [a, b]$.

The interested reader is also referred to ([2]-[4], [6], [8], [9]) for integral inequalities by using Pompeiu's mean value theorem.

In this paper, we establish an general form with the p-norm, $1 \leq p \leq \infty$, which will give the Farooq et. al result for $p = \infty$. Our results are generalization the results of Farooq et. al in [8].

2 Main Results

Before stating the main results, we will give the following lemma proved by Pecaric and Ungar in [4]:

Lemma 1 *For $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p, q \leq \infty$, and $0 < a \leq x \leq b$, denote*

$$A(x,q) := \left(\int_a^x \left(\int_t^x \frac{t^q du}{u^{2q}} \right) dt \right)^{\frac{1}{q}} + \left(\int_x^b \left(\int_x^t \frac{t^q du}{u^{2q}} \right) dt \right)^{\frac{1}{q}} \tag{1}$$

where for $p = 1$, i.e. $q = \infty$, the integrals are to be interpreted as the ∞-norms, i.e. as maxima of the function $(u,t) \mapsto \frac{1}{u^2}$ on the corresponding domains of integration. Then,

$$A(x,q) = \left(\frac{a^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - a^{1+q}x^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}}$$
$$+ \left(\frac{b^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - b^{1+q}x^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}},$$

for $1 < p, q < \infty$, $p, q \neq 2$;

$$A(x,2) = \frac{1}{3} \left[\left(\ln \left(\frac{x}{a} \right)^3 + \frac{a^3}{x^3} - 1 \right)^{\frac{1}{2}} + \left(\ln \left(\frac{x}{b} \right)^3 + \frac{b^3}{x^3} - 1 \right)^{\frac{1}{2}} \right] = \lim_{q \to 2} A(x,q);$$

$$A(x,\infty)=\frac{a^2+b^2}{2x}+x-a-b=\lim_{q\to\infty}A(x,q);$$
$$A(x,1)=\frac{1}{a}+\frac{b}{x^2}=\lim_{q\to 1}A(x,1).$$

To prove our theorems, we need the following lemma:

Lemma 2 *$f:[a,b]\to\mathbb{R}$ be continuous function on $[a,b]$ and twice order differentiable function on (a,b) with $0<a<b$. Then for any $t,x\in[a,b]$, we have*

$$tf(x)-xf(t)+xt\frac{f'(t)-f'(x)}{2}=\frac{xt}{2}\int_x^t\left[2uf(u)-2uf'(u)+u^2f''(u)\right]\frac{1}{u^2}du.$$

Proof. Define $\Psi:\left[\frac{1}{b},\frac{1}{a}\right]\to\mathbb{R}$ by $\Psi(t):=t^2f\left(\frac{1}{t}\right)$. The function Ψ is continuously differentiable on $\left(\frac{1}{b},\frac{1}{a}\right)$, and for all $x_1,x_2\in\left[\frac{1}{b},\frac{1}{a}\right]$, we get

$$\Psi(x_1)-\Psi(x_2)=\int_{x_2}^{x_1}\Psi''(t)dt$$
$$=\int_{x_2}^{x_1}\left[2tf\left(\frac{1}{t}\right)-\frac{2}{t}f'\left(\frac{1}{t}\right)+\frac{1}{t^2}f''\left(\frac{1}{t}\right)\right]dt.$$

Using the change of the variable in last integrals with $u=\frac{1}{t}$, we get

$$\Psi(x_1)-\Psi(x_2)=-\int_{\frac{1}{x_2}}^{\frac{1}{x_1}}\left[2uf(u)-2uf'(u)+u^2f''(u)\right]\frac{1}{u^2}du. \qquad (2)$$

Denote $x_1=\frac{1}{x}$ and $x_2=\frac{1}{t}$. Then for all $x,t\in[a,b]$ from (2), we have

$$\frac{2}{x}f(x)-f'(x)-\frac{2}{t}f(t)+f'(t)=\int_x^t\left[2uf(u)-2uf'(u)+u^2f''(u)\right]\frac{1}{u^2}du$$

which gives (3) and completes the proof. ■

Theorem 3 *$f:[a,b]\to\mathbb{R}$ be continuous function on $[a,b]$ and twice order differentiable function on (a,b) with $0<a<b$. Then for $\frac{1}{p}+\frac{1}{q}=1$,*

with $1 \le p, q \le \infty,$ *and all* $x \in [a,b]$, *we have*

$$\left| \frac{a+b}{2}\left(\frac{2f(x)}{3x}-\frac{f'(x)}{2}\right)+\frac{1}{3}\left(\frac{bf(b)-af(a)}{b-a}\right)-\frac{1}{b-a}\int_a^b f(t)dt \right| \quad (3)$$
$$\le \frac{(b-a)^{\frac{1}{p}}}{3}\left\|2lf-2\ell f'+\ell^2 f''\right\|_p A(x,q)$$

where $l(t)=t,\ t\in[a,b]$.

Proof. From Lemma 2, we have

$$tf(x)-xf(t)+xt\frac{f'(t)-f'(x)}{2}=\frac{xt}{2}\int_x^t\left[2uf(u)-2uf'(u)+u^2f''(u)\right]\frac{1}{u^2}du. \quad (4)$$

Integrating with respect to t on $[a,b]$ and dividing by $\frac{3x}{2}$, we get

$$\frac{(b^2-a^2)}{2}\left(\frac{2f(x)}{3x}-\frac{f'(x)}{2}\right)+\frac{bf(b)-af(a)}{3}-\int_a^b f(t)dt$$
$$=\int_a^b\frac{t}{3}\left(\int_x^t\left[2uf(u)-2uf'(u)+u^2f''(u)\right]\frac{1}{u^2}du\right)dt$$

and therefore

$$\left|\frac{(b^2-a^2)}{2}\left(\frac{2f(x)}{3x}-\frac{f'(x)}{2}\right)+\frac{bf(b)-af(a)}{3}-\int_a^b f(t)dt\right| \quad (5)$$
$$\le\int_a^b\left(\left|\int_x^t\left|\left[2uf(u)-2uf'(u)+u^2f''(u)\right]\frac{t}{3u^2}\right|du\right|\right)dt$$
$$=\int_a^x\left(\left|\int_x^t\left|\left[2uf(u)-2uf'(u)+u^2f''(u)\right]\frac{t}{3u^2}\right|du\right|\right)dt$$
$$+\int_x^b\left(\left|\int_x^t\left|\left[2uf(u)-2uf'(u)+u^2f''(u)\right]\frac{t}{3u^2}\right|du\right|\right)dt.$$

Firstly, we consider the case $1 < p, q < \infty$. By using Hölder's inequality, the sum in the last line (5) is

$$\leq \left(\int_a^x \left(\int_t^x \left| 2uf(u) - 2uf'(u) + u^2 f''(u) \right|^p du \right) dt \right)^{\frac{1}{p}} \left(\int_a^x \left(\int_t^x \frac{t^q du}{u^{2q} 3^q} \right) dt \right)^{\frac{1}{q}} \tag{6}$$

$$+ \left(\int_x^b \left(\int_x^t \left| 2uf(u) - 2uf'(u) + u^2 f''(u) \right|^p du \right) dt \right)^{\frac{1}{p}} \left(\int_x^b \left(\int_x^t \frac{t^q du}{u^{2q} 3^q} \right) dt \right)^{\frac{1}{q}}$$

$$\leq \frac{1}{3} \left(\int_a^b \left(\int_a^b \left| 2uf(u) - 2uf'(u) + u^2 f''(u) \right|^p du \right) dt \right)^{\frac{1}{p}}$$

$$\times \left[\left(\int_a^x \left(\int_t^x \frac{t^q du}{u^{2q}} \right) dt \right)^{\frac{1}{q}} + \left(\int_x^b \left(\int_x^t \frac{t^q du}{u^{2q}} \right) dt \right)^{\frac{1}{q}} \right].$$

The first factor in (6) equals

$$\left(\int_a^b \left(\int_a^b \left| 2uf(u) - 2uf'(u) + u^2 f''(u) \right|^p du \right) dt \right)^{\frac{1}{p}}$$

$$= (b-a)^{\frac{1}{p}} \left\| 2lf - 2\ell f' + \ell^2 f'' \right\|_p . \tag{7}$$

and, by Lemma 1, the second factor equals $A(x,q)$. Thus, putting (7) into (5) and dividing $b-a$ gives the required inequality (3). ■

Theorem 4 *$f : \Delta \to \mathbb{R}$ be an absolutely continuous function such that the partial derivative of order 2 exists for all $(t,s) \in \Delta$ with $0 < a < b$, $0 < c < d$, and let $w : \Delta \to \mathbb{R}$ be a nonnegative integrable function. Then for $\frac{1}{p} + \frac{1}{q} = 1$ with $1 \leq p, q \leq \infty$ any $(t,s), (x,y) \in \Delta$, we have*

$$\left| \left(\frac{2f(x) - xf(x)}{2x} \right) \int_a^b tw(t)\, dt - \int_a^b w(t) f(t) dt + \frac{1}{2} \int_a^b tw(t) f'(t) dt \right| \tag{8}$$

$$\leq \frac{(b-a)^{\frac{1}{p}}}{(1-2q)^{\frac{1}{q}}} \left\| 2lf - 2\ell f' + \ell^2 f'' \right\|_p$$

$$\times \left[\left(\int_a^x \left[x^{1-2q} t^q - t^{1-q} \right] w^q(t,s)\, dsdt \right)^{\frac{1}{q}} + \left(\int_x^b \left[t^{1-q} - x^{1-2q} t^q \right] w^q(t,s)\, dsdt \right)^{\frac{1}{q}} \right].$$

Proof. Multiplying (4) by $\frac{w(t)}{x}$ and integrating with respect to t on $[a,b]$, we have

$$\left(\frac{2f(x)-xf(x)}{2x}\right)\int_a^b tw(t)\,dt-\int_a^b w(t)\,f(t)dt+\frac{1}{2}\int_a^b tw(t)\,f'(t)dt$$
$$=\frac{1}{2}\int_a^b tw(t)\left(\int_x^t\left[2uf(u)-2uf'(u)+u^2f''(u)\right]\frac{1}{u^2}du\right)dt$$

and as in the proof of Theorem 3, we get

$$\left|\left(\frac{2f(x)-xf(x)}{2x}\right)\int_a^b tw(t)\,dt-\int_a^b w(t)\,f(t)dt+\frac{1}{2}\int_a^b tw(t)\,f'(t)dt\right|$$
$$\leq\frac{1}{2}\int_a^b\left|\int_x^t\left|2uf(u)-2uf'(u)+u^2f''(u)\right|\frac{tw(t)}{u^2}du\right|dt$$
$$=\int_a^x\left(\int_t^x\left|2uf(u)-2uf'(u)+u^2f''(u)\right|\frac{tw(t)}{u^2}du\right)dt$$
$$+\int_x^b\left(\int_x^t\left|2uf(u)-2uf'(u)+u^2f''(u)\right|\frac{tw(t)}{u^2v^2}du\right)dt$$
$$\leq\left[\int_a^x\int_c^y\left(\int_t^x\int_s^y\left|2uf(u)-2uf'(u)+u^2f''(u)\right|^p du\right)dsdt\right]^{\frac{1}{p}}\left[\int_a^x\left(\int_t^x\frac{t^qw^q(t)\,du}{u^{2q}}\right)dt\right]^{\frac{1}{q}}$$
$$+\left[\int_x^b\left(\int_x^t\left|2uf(u)-2uf'(u)+u^2f''(u)\right|^p du\right)dt\right]^{\frac{1}{p}}\left[\int_x^b\left(\int_x^t\frac{t^qw^q(t)\,du}{u^{2q}}\right)dt\right]^{\frac{1}{q}}$$
$$\leq\left[\int_a^b\left(\int_a^b\left|2uf(u)-2uf'(u)+u^2f''(u)\right|^p du\right)dt\right]^{\frac{1}{p}}$$
$$\times\left(\left[\int_a^x\left(\int_t^x\frac{t^qw^q(t)\,du}{u^{2q}}\right)dt\right]^{\frac{1}{q}}+\left[\int_x^b\left(\int_x^t\frac{t^qw^q(t)\,du}{u^{2q}}\right)dt\right]^{\frac{1}{q}}\right)$$

which gives (8). ∎

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