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On an inequality of Grüss type via variant of Pompeiu's mean value theorem

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Abstract

The main of this paper is to establish an Grüss type inequality by using a mean value theorem.

1 Introduction

In 1935, G. Grüss [4] proved the following inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{b-a}\int_a^b f(x)dx\frac{1}{b-a}\int_a^b g(x)dx\right|\leq\frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma),$$

provided that f and g are two integrable function on $[a,b]$ satisfying the condition

$$\varphi\leq f(x)\leq\Phi \text{ and } \gamma\leq g(x)\leq\Gamma \text{ for all } x\in[a,b].$$

The constant $\frac{1}{4}$ is best possible.

In 1882, P. L. Čebyšev [2] gave the following inequality:

$$|T(f,g)|\leq\frac{1}{12}(b-a)^2\left\|f'\right\|_\infty\left\|g'\right\|_\infty,$$

where $f,g:[a,b]\to\mathbb{R}$ are absolutely continuous function, whose first derivatives f' and g' are bounded,

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)$$

and $\|.\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|p\|_\infty=ess\sup_{t\in[a,b]}|p(t)|$.

For a differentiable function $f : [a,b] \to \mathbb{R},\ a{\cdot}b > 0$, Pachpatte has in [6] proved, using Pompeiu's mean value theorem [9], the following Grüss type inequality:

$$\left| \int_a^b f(t)g(t)dt - \frac{1}{b^2-a^2}\left(\int_a^b f(t)dt.\int_a^b tg(t)dt + \int_a^b g(t)dt.\int_a^b tf(t)dt \right) \right|$$

$$\leq \|f-\ell f'\|_\infty \int_a^b |g(t)| \left| \frac{1}{2} - \frac{t}{a+b} \right| dt + \|g-\ell g'\|_\infty \int_a^b |f(t)| \left| \frac{1}{2} - \frac{t}{a+b} \right| dt$$

where $\ell(t) = t,\ t \in [a,b]$.

In [7], Pecaric and Ungar proved a general estimate with the p-norm, $1 < p < \infty$, which will for $p = \infty$ give the Pachpatte [6] result.

The interested reader is also referred to ([1], [3], [5]-[10]) for integral inequalities by using Pompeiu's mean value theorem.

In this paper, we establish some new integral inequalities similar to that of the Grüss type integral inequality via Pompeiu's mean value theorem.

2 Main Results

First we give the following notations used to simplify the details of presentation

$$\begin{aligned} F(u,v) &= uvf_{uv}(u,v) - uf_u(u,v) - vf_v(u,v) + f(u,v) \\ G(u,v) &= vg_{uv}(u,v) - ug_u(u,v) - vg_v(u,v) + g(u,v), \end{aligned}$$

$$\begin{aligned}
&A(x,y,p) \\
&= (b-a)^{\frac{1}{p}-1}(d-c)^{\frac{1}{p}-1} \\
&\left\{ \left(\frac{a^{2-q}-x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q}-a^{1+q}x^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \left(\frac{c^{2-q}-y^{2-q}}{(1-2q)(2-q)} + \frac{y^{2-q}-c^{1+q}y^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \right. \\
&+ \left(\frac{a^{2-q}-x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q}-a^{1+q}x^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \left(\frac{d^{2-q}-y^{2-q}}{(1-2q)(2-q)} + \frac{y^{2-q}-d^{1+q}y^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \\
&+ \left(\frac{b^{2-q}-x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q}-b^{1+q}x^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \left(\frac{c^{2-q}-y^{2-q}}{(1-2q)(2-q)} + \frac{y^{2-q}-c^{1+q}y^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \\
&\left. + \left(\frac{b^{2-q}-x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q}-b^{1+q}x^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \left(\frac{d^{2-q}-y^{2-q}}{(1-2q)(2-q)} + \frac{y^{2-q}-d^{1+q}y^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \right\},
\end{aligned}$$

To prove our theorems, we need the following lemma:

Lemma 1 *$f:\Delta=[a,b]\times[c,d]\rightarrow\mathbb{R}$ be an absolutely continuous function such that the partial derivative of order 2 exists for all $(t,s)\in\Delta$ with $0<a<b,\ 0<c<d$. Then for any $(t,s),(x,y)\in\Delta$, we have*

$$stf(x,y)-ytf(x,s)-xsf(t,y)+xyf(t,s)=xyst\int_t^x\int_s^y F(u,v)\frac{dvdu}{u^2v^2}.$$

Proof. Define $\Psi:\left[\frac{1}{b},\frac{1}{a}\right]\times\left[\frac{1}{d},\frac{1}{c}\right]\rightarrow\mathbb{R}$ by $\Psi(t,s):=tsf\left(\frac{1}{t},\frac{1}{s}\right)$. The function Ψ is continuously differentiable on $\left(\frac{1}{b},\frac{1}{a}\right)\times\left(\frac{1}{d},\frac{1}{c}\right)$, and for all $(x_1,y_1),(x_2,y_2)\in\left[\frac{1}{b},\frac{1}{a}\right]\times\left[\frac{1}{d},\frac{1}{c}\right]$, we get

$$\begin{aligned}
&\Psi(x_1,y_1)-\Psi(x_1,y_2)-\Psi(x_2,y_1)+\Psi(x_2,y_2)\\
&=\int_{x_2}^{x_1}\int_{y_2}^{y_1}\frac{\partial^2\Psi(t,s)}{\partial t\partial s}dsdt\\
&=\int_{x_2}^{x_1}\int_{y_2}^{y_1}\left[f\left(\frac{1}{t},\frac{1}{s}\right)-\frac{1}{s}f_s\left(\frac{1}{t},\frac{1}{s}\right)-\frac{1}{t}f_t\left(\frac{1}{t},\frac{1}{s}\right)+\frac{1}{ts}f_{ts}\left(\frac{1}{t},\frac{1}{s}\right)\right]dsdt.
\end{aligned}$$

Using the change of the variable in last integrals with $u=\frac{1}{t}$ and $v=\frac{1}{s}$, we get

$$\begin{aligned}
&\Psi(x_1,y_1)-\Psi(x_1,y_2)-\Psi(x_2,y_1)+\Psi(x_2,y_2) \qquad (1)\\
&=\int_{\frac{1}{x_2}}^{\frac{1}{x_1}}\int_{\frac{1}{y_2}}^{\frac{1}{y_1}}\left[f(u,v)-vf_v(u,v)-uf_u(u,v)+uvf_{uv}(u,v)\right]\frac{dvdu}{u^2v^2}.
\end{aligned}$$

Denote $x_1=\frac{1}{x},\ x_2=\frac{1}{t},\ y_1=\frac{1}{y}$ and $y_2=\frac{1}{s}$. Then for all $(x,y),(t,s)\in[a,b]\times[c,d]$ from (1), we have

$$\begin{aligned}
&\frac{1}{xy}f(x,y)-\frac{1}{xs}f(x,s)-\frac{1}{ty}f(t,y)+\frac{1}{ts}f(t,s)\\
&=\int_x^t\int_y^s F(u,v)\frac{dvdu}{u^2v^2}
\end{aligned}$$

which gives (2) and completes the proof. ■

Theorem 2 *$f:\Delta\rightarrow\mathbb{R}$ be an absolutely continuous function such that the partial derivative of order 2 exists for all $(t,s)\in\Delta$ with $0<a<$*

$b,\ 0<c<d$. *Then for* $\frac{1}{p}+\frac{1}{q}=1$ *with* $1\le p,q\le\infty$ *any* $(t,s),(x,y)\in\Delta$, *we have*

$$
\begin{aligned}
&\Bigg|\int_a^b\int_c^d f(x,y)g(x,y)dydx \qquad (2)\\
&-\frac{1}{(d^2-c^2)}\int_a^b\int_c^d\int_c^d\left[f(x,s)g(x,y)+g(x,s)f(x,y)\right]ydsdydx\\
&-\frac{1}{(b^2-a^2)}\int_a^b\int_c^d\int_a^b\left[f(t,y)g(x,y)+g(t,y)f(x,y)\right]xdtdydx\\
&+\frac{2}{(b^2-a^2)(d^2-c^2)}\left(\int_a^b\int_c^d xyg(x,y)dydx\right)\left(\int_a^b\int_c^d f(t,s)dsdt\right)\\
&+\frac{2}{(b^2-a^2)(d^2-c^2)}\left(\int_a^b\int_c^d xyf(x,y)dydx\right)\left(\int_a^b\int_c^d g(t,s)dsdt\right)\Bigg|\\
\le&\frac{2(b-a)^{\frac{1}{p}}(d-c)^{\frac{1}{p}}}{(b^2-a^2)(d^2-c^2)}\Bigg(\left\|l_1f_{uv}-l_2f_u-l_3f_v+f\right\|_p\int_a^b\int_c^d xy\left|g(x,y)\right|A(x,y,p)dydx\\
&+\left\|l_1g_{uv}-l_2g_u-l_3g_v+g\right\|_p\int_a^b\int_c^d xy\left|f(x,y)\right|A(x,y,p)dydx\Bigg)
\end{aligned}
$$

where $l_1(x,y)=xy,\ l_2(x,\cdot)=x$ *and* $l_3(\cdot,y)=y$ *for all* $(x,y)\in\Delta$.

Proof. From Lemma 1, we have

$$stf(x,y)-ytf(x,s)-xsf(t,y)+xyf(t,s)=xyst\int_t^x\int_s^y F(u,v)\frac{dvdu}{u^2v^2},$$

$$stg(x,y)-ytg(x,s)-xsg(t,y)+xyg(t,s)=xyst\int_t^x\int_s^y G(u,v)\frac{dvdu}{u^2v^2}.$$

Multiplying these identities by $g(x,y)$ and $f(x,y)$ respectively, and adding the results gives

$$
\begin{aligned}
&2tsf(x,y)g(x,y)-ty\left[f(x,s)g(x,y)+g(x,s)f(x,y)\right]\\
&-xs\left[f(t,y)g(x,y)+g(t,y)f(x,y)\right]+xy\left[f(t,s)g(x,y)+g(t,s)f(x,y)\right]\\
=&\,xtysg(x,y)\int_t^x\int_s^y F(u,v)\frac{dvdu}{u^2v^2}+xtysf(x,y)\int_t^x\int_s^y G(u,v)\frac{dvdu}{u^2v^2}
\end{aligned}
$$

By integrating with respect to (t, s) on $[a, b] \times [c, d]$, we get

$$\frac{(b^2-a^2)(d^2-c^2)}{2} f(x,y)g(x,y) - \frac{(b^2-a^2)}{2} y \int_c^d [f(x,s)g(x,y) + g(x,s)f(x,y)]\, ds$$
$$-\frac{(d^2-c^2)}{2} x \int_a^b [f(t,y)g(x,y) + g(t,y)f(x,y)]\, dt$$
$$+xy \int_a^b \int_c^d [f(t,s)g(x,y) + g(t,s)f(x,y)]\, dsdt$$
$$= xyg(x,y) \int_a^b \int_c^d st \left[\int_t^x \int_s^y F(u,v) \frac{dvdu}{u^2v^2} \right] dsdt$$
$$+xyf(x,y) \int_a^b \int_c^d st \left[\int_t^x \int_s^y G(u,v) \frac{dvdu}{u^2v^2} \right] dsdt$$

and therefore by integrating with respect to (x, y) on $[a, b] \times [c, d]$, and taking the modulus, we obtain

$$\left| \frac{(b^2-a^2)(d^2-c^2)}{2} \int_a^b \int_c^d f(x,y)g(x,y)dydx \right. \tag{3}$$
$$-\frac{(b^2-a^2)}{2} \int_a^b \int_c^d \int_c^d [f(x,s)g(x,y) + g(x,s)f(x,y)]\, ydsdydx$$
$$-\frac{(d^2-c^2)}{2} \int_a^b \int_c^d \int_a^b [f(t,y)g(x,y) + g(t,y)f(x,y)]\, xdtdydx$$
$$+\left(\int_a^b \int_c^d xyg(x,y)dydx \right) \left(\int_a^b \int_c^d f(t,s)dsdt \right)$$
$$\left. +\left(\int_a^b \int_c^d xyf(x,y)dydx \right) \left(\int_a^b \int_c^d g(t,s)dsdt \right) \right|$$

$$\leq \left| \int_a^b \int_c^d xyg(x,y) \left(\int_a^b \int_c^d st \left[\int_t^x \int_s^y F(u,v) \frac{dvdu}{u^2v^2} \right] dsdt \right) dydx \right|$$

$$+ \left| \int_a^b \int_c^d xyf(x,y) \left(\int_a^b \int_c^d st \left[\int_t^x \int_s^y G(u,v) \frac{dvdu}{u^2v^2} \right] dsdt \right) dydx \right|.$$

For the first added in the right-hand side of (3), we have

$$\left| \int_a^b \int_c^d xyg(x,y) \left(\int_a^b \int_c^d st \left[\int_t^x \int_s^y F(u,v) \frac{dvdu}{u^2v^2} \right] dsdt \right) dydx \right| \tag{4}$$

$$\leq \int_a^b \int_c^d |xyg(x,y)| \left| \int_a^b \int_c^d st \left[\int_t^x \int_s^y F(u,v) \frac{dvdu}{u^2v^2} \right] dsdt \right| dydx$$

and for the second factor under the integral on (x,y) (i.e the outher most integral) in (4), we have

$$\left| \int_a^b \int_c^d st \left[\int_t^x \int_s^y F(u,v) \frac{dvdu}{u^2v^2} \right] dsdt \right| \tag{5}$$

$$\leq \int_a^b \int_c^d \left| \int_t^x \int_s^y |F(u,v)| \, dvdu \right| dsdt$$

$$= \int_a^x \int_c^y \left| \int_x^t \int_y^s \left| F(u,v) \frac{ts}{u^2v^2} \right| dvdu \right| dsdt$$

$$+ \int_a^x \int_y^d \left| \int_x^t \int_y^s \left| F(u,v) \frac{ts}{u^2v^2} \right| dvdu \right| dsdt$$

$$+ \int_x^b \int_c^y \left| \int_x^t \int_y^s \left| F(u,v) \frac{ts}{u^2v^2} \right| dvdu \right| dsdt$$

$$+ \int_x^b \int_y^d \left| \int_x^t \int_y^s \left| F(u,v) \frac{ts}{u^2v^2} \right| dvdu \right| dsdt.$$

By using Hölder's inequality, the sum in the last line (5) is

$$\leq \left(\int_a^x\int_c^y \left(\int_t^x\int_s^y |F(u,v)|^p\,dvdu\right) dsdt\right)^{\frac{1}{p}} \left(\int_a^x\int_c^y \left(\int_t^x\int_s^y \frac{t^q s^q dvdu}{u^{2q}v^{2q}}\right) dsdt\right)^{\frac{1}{q}} \quad (6)$$

$$+ \left(\int_a^x\int_y^d \left(\int_t^x\int_y^s |F(u,v)|^p\,dvdu\right) dsdt\right)^{\frac{1}{p}} \left(\int_a^x\int_y^d \left(\int_t^x\int_y^s \frac{t^q s^q dvdu}{u^{2q}v^{2q}}\right) dsdt\right)^{\frac{1}{q}}$$

$$+ \left(\int_x^b\int_c^y \left(\int_x^t\int_s^y |F(u,v)|^p\,dvdu\right) dsdt\right)^{\frac{1}{p}} \left(\int_x^b\int_c^y \left(\int_x^t\int_s^y \frac{t^q s^q dvdu}{u^{2q}v^{2q}}\right) dsdt\right)^{\frac{1}{q}}$$

$$+ \left(\int_x^b\int_y^d \left(\int_x^t\int_y^s |F(u,v))|^p\,dvdu\right) dsdt\right)^{\frac{1}{p}} \left(\int_x^b\int_y^d \left(\int_x^t\int_y^s \frac{t^q s^q dvdu}{u^{2q}v^{2q}}\right) dsdt\right)^{\frac{1}{q}}$$

$$\leq \left(\int_a^b\int_c^d \left(\int_a^b\int_c^d |F(u,v)|^p\,dvdu\right) dsdt\right)^{\frac{1}{p}}$$

$$\left\{\left(\int_a^x\int_c^y \left(\int_t^x\int_s^y \frac{t^q s^q dvdu}{u^{2q}v^{2q}}\right) dsdt\right)^{\frac{1}{q}} + \left(\int_a^x\int_y^d \left(\int_t^x\int_y^s \frac{t^q s^q dvdu}{u^{2q}v^{2q}}\right) dsdt\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\int_x^b\int_c^y \left(\int_x^t\int_s^y \frac{t^q s^q dvdu}{u^{2q}v^{2q}}\right) dsdt\right)^{\frac{1}{q}} + \left(\int_x^b\int_y^d \left(\int_x^t\int_y^s \frac{t^q s^q dvdu}{u^{2q}v^{2q}}\right) dsdt\right)^{\frac{1}{q}}\right\}$$

The first factor in (6) equals

$$\left(\int_a^b\int_c^d \left(\int_a^b\int_c^d |F(u,v)|^p\,dvdu\right) dsdt\right)^{\frac{1}{p}}$$

$$= (b-a)^{\frac{1}{p}}\,(d-c)^{\frac{1}{p}}\,\|l_1 f_{uv} - l_2 f_u - l_3 f_v + f\|_p\,.$$

and by Lemma, the second factor equals $A(x,y,p)$. Plugging into (5) shows that for the first added on the right-hand side in (4), we get.

$$\left|\int_a^b\int_c^d xyg(x,y)\left(\int_a^b\int_c^d st\left[\int_t^x\int_s^y F(u,v)\frac{dvdu}{u^2v^2}\right]dsdt\right)dydx\right| \tag{7}$$

$$\leq \int_a^b\int_c^d |xyg(x,y)|\,(b-a)^{\frac{1}{p}}(d-c)^{\frac{1}{p}}\left\|l_1 f_{uv}-l_2 f_u - l_3 f_v + f\right\|_p .A(x,y,p)dydx$$

$$=(b-a)^{\frac{1}{p}}(d-c)^{\frac{1}{p}}\left\|l_1 f_{uv}-l_2 f_u - l_3 f_v + f\right\|_p \int_a^b\int_c^d xy\,|g(x,y)|\,A(x,y,p)dydx.$$

An analogous inequality holds for the second added in (3), so putting these two inequalities into (3) and dividing $\frac{(b^2-a^2)(d^2-c^2)}{2}$ gives the required inequality (2), proving the theorem. ■

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