

# On an inequality of Ostrowski type via variant of Pompeiu's mean value theorem

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## Abstract

The main of this paper is to establish an Ostrowski type inequality for two variables functions by using a mean value theorem.

## 1 Introduction

The inequality of Ostrowski [7] gives us an estimate for the deviation of the values of a smooth function from its mean value. More precisely, if  $f : [a, b] \rightarrow \mathbb{R}$  is a differentiable function with bounded derivative, then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty$$

for every  $x \in [a, b]$ . Moreover the constant  $1/4$  is the best possible. For a differentiable function  $f : [a, b] \rightarrow \mathbb{R}$ ,  $a \cdot b > 0$ , Dragomir has in [2] proved, using Pompeiu's mean value theorem [6], the following Ostrowski type inequality:

$$\left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq D(x) \|f - \ell f'\|_\infty$$

where  $\ell(t) = t$ ,  $t \in [a, b]$ , and

$$D(x) = \frac{(b-a)}{|x|} \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right].$$

In [4], Pecaric and Ungar proved a general estimate with the  $p$ -norm,  $1 < p < \infty$ , which will for  $p = 1$  give the Dragomir [2] result. The

interested reader is also referred to ([1]-[5], [8]) for integral inequalities by using Pompeiu's mean value theorem. In this paper, we establish some new integral inequalities similar to that of the Ostrowski type integral inequality for two variables functions via Pompeiu's mean value theorem.

## 2 Main Results

First we give the following notations used to simplify the details of presentation

$$\begin{aligned} F(u, v) &= uvf_{uv}(u, v) - uf_u(u, v) - vf_v(u, v) + f(u, v) \\ G(u, v) &= v g_{uv}(u, v) - u g_u(u, v) - v g_v(u, v) + g(u, v), \end{aligned}$$

and

$$\begin{aligned} &PU(x, y, p) \\ &= (b-a)^{\frac{1}{p}-1} (d-c)^{\frac{1}{p}-1} \\ &\left\{ \left( \frac{a^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - a^{1+q}x^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \left( \frac{c^{2-q} - y^{2-q}}{(1-2q)(2-q)} + \frac{y^{2-q} - c^{1+q}y^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \right. \\ &+ \left( \frac{a^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - a^{1+q}x^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \left( \frac{d^{2-q} - y^{2-q}}{(1-2q)(2-q)} + \frac{y^{2-q} - d^{1+q}y^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \\ &+ \left( \frac{b^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - b^{1+q}x^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \left( \frac{c^{2-q} - y^{2-q}}{(1-2q)(2-q)} + \frac{y^{2-q} - c^{1+q}y^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \\ &\left. + \left( \frac{b^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - b^{1+q}x^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \left( \frac{d^{2-q} - y^{2-q}}{(1-2q)(2-q)} + \frac{y^{2-q} - d^{1+q}y^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \right\}, \end{aligned}$$

To prove our theorems, we need the following lemma:

**Lemma 1**  *$f : \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R}$  be an absolutely continuous function such that the partial derivative of order 2 exists for all  $(t, s) \in \Delta$  with  $0 < a < b$ ,  $0 < c < d$ . Then for any  $(t, s), (x, y) \in \Delta$ , we have*

$$stf(x, y) - ytf(x, s) - xsf(t, y) + xyf(t, s) = xyst \int_t^x \int_s^y F(u, v) \frac{dvdu}{u^2v^2}.$$

**Proof.** Define  $\Psi : \left[\frac{1}{b}, \frac{1}{a}\right] \times \left[\frac{1}{d}, \frac{1}{c}\right] \rightarrow \mathbb{R}$  by  $\Psi(t, s) := tsf\left(\frac{1}{t}, \frac{1}{s}\right)$ . The function  $\Psi$  is continuously differentiable on  $\left(\frac{1}{b}, \frac{1}{a}\right) \times \left(\frac{1}{d}, \frac{1}{c}\right)$ , and for all

$(x_1, y_1), (x_2, y_2) \in \left[\frac{1}{b}, \frac{1}{a}\right] \times \left[\frac{1}{d}, \frac{1}{c}\right]$ , we get

$$\begin{aligned} & \Psi(x_1, y_1) - \Psi(x_1, y_2) - \Psi(x_2, y_1) + \Psi(x_2, y_2) \\ &= \int_{x_2}^{x_1} \int_{y_2}^{y_1} \frac{\partial^2 \Psi(t, s)}{\partial t \partial s} ds dt \\ &= \int_{x_2}^{x_1} \int_{y_2}^{y_1} \left[ f\left(\frac{1}{t}, \frac{1}{s}\right) - \frac{1}{s} f_s\left(\frac{1}{t}, \frac{1}{s}\right) - \frac{1}{t} f_t\left(\frac{1}{t}, \frac{1}{s}\right) + \frac{1}{ts} f_{ts}\left(\frac{1}{t}, \frac{1}{s}\right) \right] ds dt. \end{aligned}$$

Using the change of the variable in last integrals with  $u = \frac{1}{t}$  and  $v = \frac{1}{s}$ , we get

$$\begin{aligned} & \Psi(x_1, y_1) - \Psi(x_1, y_2) - \Psi(x_2, y_1) + \Psi(x_2, y_2) \tag{1} \\ &= \int_{\frac{1}{x_2}}^{\frac{1}{x_1}} \int_{\frac{1}{y_2}}^{\frac{1}{y_1}} [f(u, v) - v f_v(u, v) - u f_u(u, v) + uv f_{uv}(u, v)] \frac{dv du}{u^2 v^2}. \end{aligned}$$

Denote  $x_1 = \frac{1}{x}$ ,  $x_2 = \frac{1}{t}$ ,  $y_1 = \frac{1}{y}$  and  $y_2 = \frac{1}{s}$ . Then for all  $(x, y), (t, s) \in [a, b] \times [c, d]$  from (1), we have

$$\frac{1}{xy} f(x, y) - \frac{1}{xs} f(x, s) - \frac{1}{ty} f(t, y) + \frac{1}{ts} f(t, s) = \int_x^t \int_y^s F(u, v) \frac{dv du}{u^2 v^2}$$

which gives (2) and completes the proof. ■

**Theorem 2**  $f : \Delta \rightarrow \mathbb{R}$  be an absolutely continuous function such that the partial derivative of order 2 exists for all  $(t, s) \in \Delta$  with  $0 < a < b$ ,  $0 < c < d$ . Then for  $\frac{1}{p} + \frac{1}{q} = 1$  with  $1 \leq p, q \leq \infty$  any  $(t, s), (x, y) \in \Delta$ , we have

$$\begin{aligned} & \left| \frac{(b^2 - a^2)(d^2 - c^2)}{4} \frac{f(x, y)}{xy} - \frac{(b^2 - a^2)}{2x} \int_c^d f(x, s) ds \right. \\ & \quad \left. - \frac{(d^2 - c^2)}{2y} \int_a^b f(t, y) dt + \int_a^b \int_c^d f(t, s) ds dt \right| \\ & \leq PU(x, y, p) \|l_1 f_{uv} - r l_2 f_u - l_3 f_v + f\|_p \end{aligned} \tag{2}$$

where  $l_1(x, y) = xy$ ,  $l_2(x, \cdot) = x$  and  $l_3(\cdot, y) = y$  for all  $(x, y) \in \Delta$ .

**Proof.** From Lemma 1, we have

$$stf(x, y) - ytf(x, s) - xsf(t, y) + xyf(t, s) = xyst \int_t^x \int_s^y F(u, v) \frac{dvdu}{u^2v^2}. \quad (3)$$

Integrating with respect to  $(t, s)$  on  $[a, b] \times [c, d]$  and dividing by  $xy$ , we get

$$\begin{aligned} & \frac{(b^2 - a^2)(d^2 - c^2)}{4} \frac{f(x, y)}{xy} - \frac{(b^2 - a^2)}{2x} \int_c^d f(x, s) ds \\ & - \frac{(d^2 - c^2)}{2y} \int_a^b f(t, y) dt + \int_a^b \int_c^d f(t, s) ds dt \\ & = \int_a^b \int_c^d st \left[ \int_t^x \int_s^y F(u, v) \frac{dvdu}{u^2v^2} \right] ds dt \end{aligned}$$

and therefore

$$\begin{aligned} & \left| \frac{(b^2 - a^2)(d^2 - c^2)}{4} \frac{f(x, y)}{xy} - \frac{(b^2 - a^2)}{2x} \int_c^d f(x, s) ds \right. \\ & \left. - \frac{(d^2 - c^2)}{2y} \int_a^b f(t, y) dt + \int_a^b \int_c^d f(t, s) ds dt \right| \\ & \leq \int_a^b \int_c^d \left| \int_t^x \int_s^y |F(u, v)| \frac{tsdvdu}{u^2v^2} \right| ds dt \\ & = \int_a^x \int_c^y \left| \int_x^t \int_y^s |F(u, v)| \frac{tsdvdu}{u^2v^2} \right| ds dt + \int_a^x \int_y^d \left| \int_x^t \int_y^s |F(u, v)| \frac{tsdvdu}{u^2v^2} \right| ds dt \\ & \quad + \int_x^b \int_c^y \left| \int_x^t \int_y^s |F(u, v)| \frac{tsdvdu}{u^2v^2} \right| ds dt + \int_x^b \int_y^d \left| \int_x^t \int_y^s |F(u, v)| \frac{tsdvdu}{u^2v^2} \right| ds dt. \end{aligned} \quad (4)$$

Firstly, we will consider the case  $1 < p, q < \infty$ . By using Hölder's inequality, the sum in the last line (4) is

$$\begin{aligned}
&\leq \left( \int_a^x \int_c^y \left( \int_t^x \int_s^y |F(u, v)|^p dvdu \right) dsdt \right)^{\frac{1}{p}} \left( \int_a^x \int_c^y \left( \int_t^x \int_s^y \frac{t^q s^q dvdu}{u^{2q} v^{2q}} \right) dsdt \right)^{\frac{1}{q}} \quad (5) \\
&+ \left( \int_a^x \int_y^d \left( \int_t^x \int_y^s |F(u, v)|^p dvdu \right) dsdt \right)^{\frac{1}{p}} \left( \int_a^x \int_y^d \left( \int_t^x \int_y^s \frac{t^q s^q dvdu}{u^{2q} v^{2q}} \right) dsdt \right)^{\frac{1}{q}} \\
&+ \left( \int_x^b \int_c^y \left( \int_x^t \int_s^y |F(u, v)|^p dvdu \right) dsdt \right)^{\frac{1}{p}} \left( \int_x^b \int_c^y \left( \int_x^t \int_s^y \frac{t^q s^q dvdu}{u^{2q} v^{2q}} \right) dsdt \right)^{\frac{1}{q}} \\
&+ \left( \int_x^b \int_y^d \left( \int_x^t \int_y^s |F(u, v)|^p dvdu \right) dsdt \right)^{\frac{1}{p}} \left( \int_x^b \int_y^d \left( \int_x^t \int_y^s \frac{t^q s^q dvdu}{u^{2q} v^{2q}} \right) dsdt \right)^{\frac{1}{q}} \\
&\leq \left( \int_a^b \int_c^d \left( \int_a^b \int_c^d |F(u, v)|^p dvdu \right) dsdt \right)^{\frac{1}{p}} \\
&\left\{ \left( \int_a^x \int_c^y \left( \int_t^x \int_s^y \frac{t^q s^q dvdu}{u^{2q} v^{2q}} \right) dsdt \right)^{\frac{1}{q}} + \left( \int_a^x \int_y^d \left( \int_t^x \int_y^s \frac{t^q s^q dvdu}{u^{2q} v^{2q}} \right) dsdt \right)^{\frac{1}{q}} \right. \\
&\left. + \left( \int_x^b \int_c^y \left( \int_x^t \int_s^y \frac{t^q s^q dvdu}{u^{2q} v^{2q}} \right) dsdt \right)^{\frac{1}{q}} + \left( \int_x^b \int_y^d \left( \int_x^t \int_y^s \frac{t^q s^q dvdu}{u^{2q} v^{2q}} \right) dsdt \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

The first factor in (5) equals

$$\begin{aligned}
&\left( \int_a^b \int_c^d \left( \int_a^b \int_c^d |F(u, v)|^p dvdu \right) dsdt \right)^{\frac{1}{p}} \\
&= (b-a)^{\frac{1}{p}} (d-c)^{\frac{1}{p}} \|l_1 f_{uv} - r l_2 f_u - l_3 f_v + f\|_p. \quad (6)
\end{aligned}$$

and for the second factor, for  $p, q \neq 2$ , we get

$$\begin{aligned}
& \left\{ \left( \int_a^x \int_c^y \left( \int_t^x \int_s^y \frac{t^q s^q dvdu}{u^{2q} v^{2q}} \right) dsdt \right)^{\frac{1}{q}} + \left( \int_a^x \int_y^d \left( \int_t^x \int_y^s \frac{t^q s^q dvdu}{u^{2q} v^{2q}} \right) dsdt \right)^{\frac{1}{q}} \right. \\
& \left. + \left( \int_x^b \int_c^y \left( \int_x^t \int_s^y \frac{t^q s^q dvdu}{u^{2q} v^{2q}} \right) dsdt \right)^{\frac{1}{q}} + \left( \int_x^b \int_y^d \left( \int_x^t \int_y^s \frac{t^q s^q dvdu}{u^{2q} v^{2q}} \right) dsdt \right)^{\frac{1}{q}} \right\} \\
&= \left( \frac{a^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - a^{1+q} x^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \left( \frac{c^{2-q} - y^{2-q}}{(1-2q)(2-q)} + \frac{y^{2-q} - c^{1+q} y^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \\
&+ \left( \frac{a^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - a^{1+q} x^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \left( \frac{d^{2-q} - y^{2-q}}{(1-2q)(2-q)} + \frac{y^{2-q} - d^{1+q} y^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \\
&+ \left( \frac{b^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - b^{1+q} x^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \left( \frac{c^{2-q} - y^{2-q}}{(1-2q)(2-q)} + \frac{y^{2-q} - c^{1+q} y^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \\
&+ \left( \frac{b^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - b^{1+q} x^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} \left( \frac{d^{2-q} - y^{2-q}}{(1-2q)(2-q)} + \frac{y^{2-q} - d^{1+q} y^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}}.
\end{aligned} \tag{7}$$

For  $p = q = 2$ , instead of (7), we obtain

$$\begin{aligned}
& \left\{ \left( \int_a^x \int_c^y \left( \int_t^x \int_s^y \frac{t^2 s^2 dvdu}{u^4 v^4} \right) dsdt \right)^{\frac{1}{2}} + \left( \int_a^x \int_y^d \left( \int_t^x \int_y^s \frac{t^2 s^2 dvdu}{u^4 v^4} \right) dsdt \right)^{\frac{1}{2}} \right. \\
& \left. + \left( \int_x^b \int_c^y \left( \int_x^t \int_s^y \frac{t^2 s^2 dvdu}{u^4 v^4} \right) dsdt \right)^{\frac{1}{2}} + \left( \int_x^b \int_y^d \left( \int_x^t \int_y^s \frac{t^2 s^2 dvdu}{u^4 v^4} \right) dsdt \right)^{\frac{1}{2}} \right\} \\
&= \frac{1}{9} \left[ \left( \ln \left( \frac{x}{a} \right)^3 + \frac{a^3}{x^3} - 1 \right)^{\frac{1}{2}} + \left( \ln \left( \frac{x}{b} \right)^3 + \frac{b^3}{x^3} - 1 \right)^{\frac{1}{2}} \right] \\
&\quad \times \left[ \left( \ln \left( \frac{y}{c} \right)^3 + \frac{c^3}{y^3} - 1 \right)^{\frac{1}{2}} + \left( \ln \left( \frac{y}{d} \right)^3 + \frac{d^3}{y^3} - 1 \right)^{\frac{1}{2}} \right]
\end{aligned} \tag{8}$$

which is easily shown to be equal to the limit of the right hand side of (7) for  $q \rightarrow 2$ , i.e.

$$\begin{aligned}
\lim_{q \rightarrow 2} PU(x, y, p) &= \frac{1}{9(b-a)^{\frac{1}{q}}(d-c)^{\frac{1}{q}}} \left\{ \left( \ln \left( \frac{x}{a} \right)^3 + \frac{a^3}{x^3} - 1 \right)^{\frac{1}{2}} \left( \ln \left( \frac{x}{b} \right)^3 + \frac{b^3}{x^3} - 1 \right)^{\frac{1}{2}} \right. \\
&\quad \left. + \left( \ln \left( \frac{y}{c} \right)^3 + \frac{c^3}{y^3} - 1 \right)^{\frac{1}{2}} \left( \ln \left( \frac{y}{d} \right)^3 + \frac{d^3}{y^3} - 1 \right)^{\frac{1}{2}} \right\}.
\end{aligned}$$

Now, consider the case  $p = \infty$ ,  $q = 1$ . Then, the last line in (4) is

$$\begin{aligned}
&\leq \sup_{(u,v) \in [a,b] \times [c,d]} |F(u,v)| \tag{9} \\
&\times \left\{ \int_a^x \int_c^y \left( \int_t^x \int_s^y \frac{tsdvdu}{u^2v^2} \right) dsdt + \int_a^x \int_y^d \left( \int_t^x \int_y^s \frac{tsdvdu}{u^2v^2} \right) dsdt \right. \\
&\left. + \int_x^b \int_c^y \left( \int_x^t \int_s^y \frac{tsdvdu}{u^2v^2} \right) dsdt + \int_x^b \int_y^d \left( \int_x^t \int_y^s \frac{tsdvdu}{u^2v^2} \right) dsdt \right\} \\
&= \|l_1 f_{uv} - l_2 f_u - l_3 f_v + f\|_\infty \left( \frac{a^2 + b^2}{2x} + x - a - b \right) \left( \frac{c^2 + d^2}{2y} + y - c - d \right).
\end{aligned}$$

Putting (9) into (4) and dividing by  $(b-a)(d-c)$  gives

$$\begin{aligned}
&\left| \frac{(a+b)(c+d)}{4} \frac{f(x,y)}{xy} - \frac{a+b}{2x(d^2c^2)} \int_c^d f(x,s) ds \right. \\
&\left. - \frac{c+d}{2y(b^2-a^2)} \int_a^b f(t,y) dt + \frac{1}{(b^2-a^2)(d^2c^2)} \int_a^b \int_c^d f(t,s) dsdt \right| \\
&\leq \frac{1}{(b-a)(d-c)} \|l_1 f_{uv} - l_2 f_u - l_3 f_v + f\|_\infty \\
&\times \left( \frac{a^2 + b^2}{2x} + x - a - b \right) \left( \frac{c^2 + d^2}{2y} + y - c - d \right).
\end{aligned}$$

Finally, we consider the case  $p = 1$ ,  $q = \infty$ . then, the last line of (4) is

$$\begin{aligned}
&\leq \int_a^x \int_c^y \left( \int_t^x \int_s^y |F(u,v)| \max_{\substack{(u,v) \in [t,x] \times [s,y] \\ (t,s) \in [a,x] \times [c,y]}} \frac{ts}{u^2v^2} dvdu \right) dsdt \tag{10} \\
&+ \int_a^x \int_y^d \left( \int_x^t \int_y^s |F(u,v)| \max_{\substack{(u,v) \in [t,x] \times [y,s] \\ (t,s) \in [a,x] \times [c,y]}} \frac{ts}{u^2v^2} dvdu \right) dsdt \\
&+ \int_x^b \int_c^y \left( \int_x^t \int_s^y |F(u,v)| \max_{\substack{(u,v) \in [x,t] \times [s,y] \\ (t,s) \in [a,x] \times [c,y]}} \frac{ts}{u^2v^2} dvdu \right) dsdt \\
&+ \int_x^b \int_y^d \left( \int_x^t \int_y^s |F(u,v)| \max_{\substack{(u,v) \in [x,t] \times [y,s] \\ (t,s) \in [a,x] \times [c,y]}} \frac{ts}{u^2v^2} dvdu \right) dsdt
\end{aligned}$$

$$\begin{aligned}
&\leq \int_a^b \int_c^d \left( \int_a^b \int_c^d |F(u, v)| \, dv du \right) ds dt. \left( \frac{1}{a} + \frac{b}{x^2} \right) \left( \frac{1}{c} + \frac{d}{y^2} \right) \\
&= (b-a)(d-c) \left( \frac{1}{a} + \frac{b}{x^2} \right) \left( \frac{1}{c} + \frac{d}{y^2} \right) \|l_1 f_{uv} - l_2 f_u - l_3 f_v + f\|_1
\end{aligned}$$

Appending (10) to (4) and dividing by  $(b-a)(d-c)$  gives

$$\begin{aligned}
&\left| \frac{(a+b)(c+d)}{4} \frac{f(x, y)}{xy} - \frac{a+b}{2x(d^2-c^2)} \int_c^d f(x, s) ds \right. \\
&\quad \left. - \frac{c+d}{2y(b^2-a^2)} \int_a^b f(t, y) dt + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(t, s) ds dt \right| \\
&\leq \left( \frac{1}{a} + \frac{b}{x^2} \right) \left( \frac{1}{c} + \frac{d}{y^2} \right) \|l_1 f_{uv} - l_2 f_u - l_3 f_v + f\|_1.
\end{aligned} \tag{11}$$

It is not too difficult to show that

$$\lim_{p \rightarrow 1} PU(x, y, p) = \left( \frac{1}{a} + \frac{b}{x^2} \right) \left( \frac{1}{c} + \frac{d}{y^2} \right) \tag{12}$$

so (11) proves formula (2) for  $p = 1$ ,  $q = \infty$ , proving the theorem. ■

**Theorem 3**  *$f : \Delta \rightarrow \mathbb{R}$  be an absolutely continuous function such that the partial derivative of order 2 exists for all  $(t, s) \in \Delta$  with  $0 < a < b$ ,  $0 < c < d$ , and let  $w : \Delta \rightarrow \mathbb{R}$  be a nonnegative integrable function. Then for  $\frac{1}{p} + \frac{1}{q} = 1$  with  $1 \leq p, q \leq \infty$  any  $(t, s), (x, y) \in \Delta$ , we have*

$$\begin{aligned}
&\left| \frac{f(x, y)}{xy} \int_a^b \int_c^d w(t, s) ds dt - \frac{1}{x} \int_a^b \int_c^d w(t, s) f(x, s) ds dt \right. \\
&\quad \left. - \frac{1}{y} \int_a^b \int_c^d w(t, s) f(t, y) dt + \int_a^b \int_c^d w(t, s) f(t, s) ds dt \right| \\
&\leq \frac{(b-a)^{\frac{1}{p}} (d-c)^{\frac{1}{p}}}{(1-2q)^{\frac{1}{q}}} \|l_1 f_{uv} - r l_2 f_u - l_3 f_v + f\|_p
\end{aligned} \tag{13}$$



$$\begin{aligned}
& \times \left\{ \left( \int_a^x \int_c^y [x^{1-2q}t^q - t^{1-q}] [y^{1-2q}s^q - s^{1-q}] w^q(t, s) dsdt \right)^{\frac{1}{q}} \right. \\
& + \left( \int_a^x \int_y^d [x^{1-2q}t^q - t^{1-q}] [s^{1-q} - y^{1-2q}s^q] w^q(t, s) dsdt \right)^{\frac{1}{q}} \\
& + \left( \int_x^b \int_c^y [t^{1-q} - x^{1-2q}t^q] [y^{1-2q}s^q - s^{1-q}] w^q(t, s) dsdt \right)^{\frac{1}{q}} \\
& \left. + \left( \int_x^b \int_y^d [t^{1-q} - x^{1-2q}t^q] [s^{1-q} - y^{1-2q}s^q] w^q(t, s) dsdt \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

**Proof.** Multiplying (3) by  $\frac{w(t,s)}{xy}$  and integrating with respect to  $(t, s)$  on  $[a, b] \times [c, d]$ , we have

$$\begin{aligned}
& \frac{f(x, y)}{xy} \int_a^b \int_c^d w(t, s) dsdt - \frac{1}{x} \int_a^b \int_c^d w(t, s) f(x, s) dsdt \\
& - \frac{1}{y} \int_a^b \int_c^d w(t, s) f(t, y) dt + \int_a^b \int_c^d w(t, s) f(t, s) dsdt \\
& = \int_a^b \int_c^d stw(t, s) \left[ \int_t^x \int_s^y [F(u, v)] \frac{dvdu}{u^2v^2} \right] dsdt
\end{aligned}$$

and as in the proof of Theorem 2, we get

$$\begin{aligned}
& \left| \frac{f(x, y)}{xy} \int_a^b \int_c^d w(t, s) dsdt - \frac{1}{x} \int_a^b \int_c^d w(t, s) f(x, s) dsdt \right. \\
& \left. - \frac{1}{y} \int_a^b \int_c^d w(t, s) f(t, y) dt + \int_a^b \int_c^d w(t, s) f(t, s) dsdt \right| \\
& \leq \int_a^b \int_c^d \left| \int_t^x \int_s^y |F(u, v)| \frac{tsw(t, s) dvdu}{u^2v^2} \right| dsdt
\end{aligned}$$

$$\begin{aligned}
&= \int_a^x \int_c^y \left( \int_t^x \int_s^y |F(u, v)| \frac{tsw(t, s) dvdu}{u^2 v^2} \right) dsdt \\
&\quad + \int_a^x \int_y^d \left( \int_t^x \int_y^s |F(u, v)| \frac{tsw(t, s) dvdu}{u^2 v^2} \right) dsdt \\
&\quad + \int_x^b \int_c^y \left( \int_x^t \int_s^y |F(u, v)| \frac{tsw(t, s) dvdu}{u^2 v^2} \right) dsdt \\
&\quad + \int_x^b \int_y^d \left( \int_x^t \int_y^s |F(u, v)| \frac{tsw(t, s) dvdu}{u^2 v^2} \right) dsdt \\
&\leq \left[ \int_a^x \int_c^y \left( \int_t^x \int_s^y |F(u, v)|^p dvdu \right) dsdt \right]^{\frac{1}{p}} \left[ \int_a^x \int_c^y \left( \int_t^x \int_s^y \frac{t^q s^q w^q(t, s) dvdu}{u^{2q} v^{2q}} \right) dsdt \right]^{\frac{1}{q}} \\
&\quad + \left[ \int_a^x \int_y^d \left( \int_t^x \int_y^s |F(u, v)|^p dvdu \right) dsdt \right]^{\frac{1}{p}} \left[ \int_a^x \int_y^d \left( \int_t^x \int_y^s \frac{t^q s^q w^q(t, s) dvdu}{u^{2q} v^{2q}} \right) dsdt \right]^{\frac{1}{q}} \\
&\quad + \left[ \int_x^b \int_c^y \left( \int_x^t \int_s^y |F(u, v)|^p dvdu \right) dsdt \right]^{\frac{1}{p}} \left[ \int_x^b \int_c^y \left( \int_x^t \int_s^y \frac{t^q s^q w^q(t, s) dvdu}{u^{2q} v^{2q}} \right) dsdt \right]^{\frac{1}{q}} \\
&\quad + \left[ \int_x^b \int_y^d \left( \int_x^t \int_y^s |F(u, v)|^p dvdu \right) dsdt \right]^{\frac{1}{p}} \left[ \int_x^b \int_y^d \left( \int_x^t \int_y^s \frac{t^q s^q w^q(t, s) dvdu}{u^{2q} v^{2q}} \right) dsdt \right]^{\frac{1}{q}} \\
&\leq \left[ \int_a^x \int_c^y \left( \int_a^b \int_c^d |F(u, v)|^p dvdu \right) dsdt \right]^{\frac{1}{p}} \left\{ \left[ \int_a^x \int_c^y \left( \int_t^x \int_s^y \frac{t^q s^q w^q(t, s) dvdu}{u^{2q} v^{2q}} \right) dsdt \right]^{\frac{1}{q}} \right. \\
&\quad + \left[ \int_a^x \int_y^d \left( \int_t^x \int_y^s \frac{t^q s^q w^q(t, s) dvdu}{u^{2q} v^{2q}} \right) dsdt \right]^{\frac{1}{q}} + \left[ \int_x^b \int_c^y \left( \int_x^t \int_s^y \frac{t^q s^q w^q(t, s) dvdu}{u^{2q} v^{2q}} \right) dsdt \right]^{\frac{1}{q}} \\
&\quad \left. + \left[ \int_x^b \int_y^d \left( \int_x^t \int_y^s \frac{t^q s^q w^q(t, s) dvdu}{u^{2q} v^{2q}} \right) dsdt \right]^{\frac{1}{q}} \right\}
\end{aligned}$$

which gives (13). ■

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