

## ON GENERALIZATION INTEGRAL INEQUALITIES FOR FRACTIONAL INTEGRALS

MEHMET ZEKI SARIKAYA AND HATICE YALDIZ

ABSTRACT. In this paper, some generalization integral inequalities of Hermite-Hadamard type for functions whose derivatives are convex in modulus are given by using fractional integrals.

### 1. INTRODUCTION

**Definition 1.** *The function  $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ , is said to be convex if the following inequality holds*

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all  $x, y \in [a, b]$  and  $\lambda \in [0, 1]$ . We say that  $f$  is concave if  $(-f)$  is convex.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see, e.g., [9], [16, p.137]). These inequalities state that if  $f : I \rightarrow \mathbb{R}$  is a convex function on the interval  $I$  of real numbers and  $a, b \in I$  with  $a < b$ , then

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.$$

Both inequalities hold in the reversed direction if  $f$  is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [1, 2, 9, 10, 12, 16]) and the references cited therein.

In [17], Pearce and Pečarić proved the following theorem:

**Theorem 1.** *Let  $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ , be a differentiable mapping on  $I^\circ$ ,  $a, b \in I^\circ$  with  $a < b$ . If the mapping  $|f'|^q$  is convex on  $[a, b]$  for some  $q \geq 1$ , then*

$$(1.2) \quad \left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{4} \left( \frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}.$$

Also, in [12], Kirmacı obtained the following inequality for differentiable mappings which are connected with Hermite-Hadamard's inequality:

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**Theorem 2.** Let  $f : I^\circ \subset \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$ ,  $a, b \in I^\circ$  with  $a < b$ . If the mapping  $|f'|$  is convex on  $[a, b]$ , then we have

$$(1.3) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{8} (|f'(a)| + |f'(b)|).$$

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult [11, 13, 15].

**Definition 2.** Let  $f \in L_1[a, b]$ . The Riemann-Liouville integrals  $J_{a+}^\alpha f$  and  $J_{b-}^\alpha f$  of order  $\alpha > 0$  with  $a \geq 0$  are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b$$

respectively. Here,  $\Gamma(\alpha)$  is the Gamma function and  $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$ .

Meanwhile, Sarikaya et al.[18] presented the following important integral identity including the first-order derivative of  $f$  to establish many interesting Hermite-Hadamard type inequalities for convexity functions via Riemann-Liouville fractional integrals of the order  $\alpha > 0$ .

**Lemma 1.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  with  $a < b$ . If  $f' \in L[a, b]$ , then the following equality for fractional integrals holds:

$$(1.4) \quad \begin{aligned} & \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ J_{\left(\frac{a+b}{2}\right)+}^\alpha f(b) + J_{\left(\frac{a+b}{2}\right)-}^\alpha f(a) \right] - f\left(\frac{a+b}{2}\right) \\ &= \frac{b-a}{4} \left\{ \int_0^1 t^\alpha f' \left( \frac{t}{2}a + \frac{2-t}{2}b \right) dt - \int_0^1 t^\alpha f' \left( \frac{2-t}{2}a + \frac{t}{2}b \right) dt \right\} \end{aligned}$$

with  $\alpha > 0$ .

It is remarkable that Sarikaya et al.[18] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

**Theorem 3.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a positive function with  $0 \leq a < b$  and  $f \in L_1[a, b]$ . If  $f$  is a convex function on  $[a, b]$ , then the following inequalities for fractional integrals hold:

$$(1.5) \quad f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ J_{\left(\frac{a+b}{2}\right)+}^\alpha f(b) + J_{\left(\frac{a+b}{2}\right)-}^\alpha f(a) \right] \leq \frac{f(a) + f(b)}{2}$$

with  $\alpha > 0$ .

For some recent results connected with fractional integral inequalities see ([3]-[8],[14],[19],[21],[22])

The aim of this paper is to establish generalized Hermite-Hadamard type integral inequalities for Riemann-Liouville fractional integral and some other integral inequalities using the generalized identity is obtained for fractional integrals. The results presented in this paper provide extensions of those given in earlier works.

## 2. MAIN RESULTS

We give an important fractional integral identity for differentiable convex functions:

**Lemma 2.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  with  $0 \leq a < b$ . If  $f' \in L[a, b]$ , then the following equality for fractional integrals holds:*

$$(2.1) \quad \begin{aligned} & [\lambda^{\alpha+1}(1-\lambda)^\alpha + \lambda^\alpha(1-\lambda)^{\alpha+1}] f(\lambda a + (1-\lambda)b) \\ & - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ \lambda^{\alpha+1} J_{(\lambda a+(1-\lambda)b)^-}^\alpha f(a) + (1-\lambda)^{\alpha+1} J_{(\lambda a+(1-\lambda)b)^+}^\alpha f(b) \right] \\ & = \lambda^{\alpha+1}(1-\lambda)^{\alpha+1} (b-a) \left\{ \int_0^1 t^\alpha f' [t(\lambda a + (1-\lambda)b) + (1-t)a] dt \right. \\ & \quad \left. - \int_0^1 (1-t)^\alpha f' [tb + (1-t)(\lambda a + (1-\lambda)b)] dt \right\} \end{aligned}$$

where  $\lambda \in (0, 1)$  and  $\alpha > 0$ .

*Proof.* Integrating by parts

$$\begin{aligned} & \int_0^1 t^\alpha f' [t(\lambda a + (1-\lambda)b) + (1-t)a] dt \\ & = \frac{t^\alpha f [t(\lambda a + (1-\lambda)b) + (1-t)a]}{(1-\lambda)(b-a)} \Big|_0^1 \\ & \quad - \frac{\alpha}{(1-\lambda)(b-a)} \int_0^1 t^{\alpha-1} f [t(\lambda a + (1-\lambda)b) + (1-t)a] dt \\ & = \frac{f(\lambda a + (1-\lambda)b)}{(1-\lambda)(b-a)} - \frac{\alpha}{(1-\lambda)(b-a)} \int_0^1 t^{\alpha-1} f [t(\lambda a + (1-\lambda)b) + (1-t)a] dt \\ & = \frac{f(\lambda a + (1-\lambda)b)}{(1-\lambda)(b-a)} - \frac{\alpha}{(1-\lambda)^{\alpha+1}(b-a)^{\alpha+1}} \int_a^{\lambda a+(1-\lambda)b} (x-a)^{\alpha-1} f(x) dx \\ & = \frac{f(\lambda a + (1-\lambda)b)}{(1-\lambda)(b-a)} - \frac{\Gamma(\alpha+1)}{(1-\lambda)^{\alpha+1}(b-a)^{\alpha+1}} J_{(\lambda a+(1-\lambda)b)^-}^\alpha f(a) \end{aligned}$$

that is

$$(2.2) \quad \begin{aligned} & \int_0^1 t^\alpha f' [t(\lambda a + (1-\lambda)b) + (1-t)a] dt \\ & = \frac{f(\lambda a + (1-\lambda)b)}{(1-\lambda)(b-a)} - \frac{\Gamma(\alpha+1)}{(1-\lambda)^{\alpha+1}(b-a)^{\alpha+1}} J_{(\lambda a+(1-\lambda)b)^-}^\alpha f(a) \end{aligned}$$

and similarly we get

$$\begin{aligned}
& - \int_0^1 (1-t)^\alpha f' [tb + (1-t)(\lambda a + (1-\lambda)b)] dt \\
= & \frac{f(\lambda a + (1-\lambda)b)}{\lambda(b-a)} - \frac{\alpha}{\lambda^{\alpha+1}(b-a)^{\alpha+1}} \int_{\lambda a + (1-\lambda)b}^b (b-x)^{\alpha-1} f(x) dx \\
= & \frac{f(\lambda a + (1-\lambda)b)}{\lambda(b-a)} - \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}(b-a)^{\alpha+1}} J_{(\lambda a + (1-\lambda)b)^+}^\alpha f(b)
\end{aligned}$$

i.e

$$\begin{aligned}
(2.3) \quad & - \int_0^1 (1-t)^\alpha f' [tb + (1-t)(\lambda a + (1-\lambda)b)] dt \\
= & \frac{f(\lambda a + (1-\lambda)b)}{\lambda(b-a)} - \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}(b-a)^{\alpha+1}} J_{(\lambda a + (1-\lambda)b)^+}^\alpha f(b)
\end{aligned}$$

Adding (2.2) and (2.3), we obtain (2.1). This completes the proof.  $\square$

**Remark 1.** If we take  $\lambda = \frac{1}{2}$  in Lemma 2, then it follows that

$$\begin{aligned}
(2.4) \quad & f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^\alpha} \left[ J_{\left(\frac{a+b}{2}\right)^-}^\alpha f(a) + J_{\left(\frac{a+b}{2}\right)^+}^\alpha f(b) \right] \\
= & \frac{(b-a)}{4} \left\{ \int_0^1 t^\alpha f' \left( t \frac{a+b}{2} + (1-t)a \right) dt - \int_0^1 (1-t)^\alpha f' \left( tb + (1-t) \frac{a+b}{2} \right) dt \right\}
\end{aligned}$$

If we choose  $\alpha = 1$  in (2.4), it follows that

$$\begin{aligned}
& f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \\
= & \frac{(b-a)}{4} \left\{ \int_0^1 t f' \left( t \frac{a+b}{2} + (1-t)a \right) dt - \int_0^1 (1-t) f' \left( tb + (1-t) \frac{a+b}{2} \right) dt \right\}.
\end{aligned}$$

**Theorem 4.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  with  $0 \leq a < b$ . If  $|f'|^q, q \geq 1$  is convex on  $[a, b]$ , then the following inequality for fractional integrals holds:

$$\begin{aligned}
& |[\lambda^{\alpha+1}(1-\lambda)^\alpha + \lambda^\alpha(1-\lambda)^{\alpha+1}] f(\lambda a + (1-\lambda)b) \\
& - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ \lambda^{\alpha+1} J_{(\lambda a+(1-\lambda)b)^-}^\alpha f(a) + (1-\lambda)^{\alpha+1} J_{(\lambda a+(1-\lambda)b)^+}^\alpha f(b) \right] \Big| \\
\leq & \frac{\lambda^{\alpha+1}(1-\lambda)^{\alpha+1}(b-a)}{(\alpha+1)} \left\{ \left( \frac{(\alpha+1)|f'(\lambda a + (1-\lambda)b|^q + |f'(a)|^q)}{(\alpha+2)} \right)^{\frac{1}{q}} \right. \\
(2.5) \quad & \left. + \left( \frac{(\alpha+1)|f'(\lambda a + (1-\lambda)b|^q + |f'(b)|^q)}{(\alpha+2)} \right)^{\frac{1}{q}} \right\}
\end{aligned}$$

where  $\lambda \in (0, 1)$  and  $\alpha > 0$ .

*Proof.* Firstly, we suppose that  $q = 1$ . Using Lemma 2 and convexity of  $|f'|^q$ , we find that

$$\begin{aligned}
& |[\lambda^{\alpha+1}(1-\lambda)^\alpha + \lambda^\alpha(1-\lambda)^{\alpha+1}] f(\lambda a + (1-\lambda)b) \\
& - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ \lambda^{\alpha+1} J_{(\lambda a+(1-\lambda)b)^-}^\alpha f(a) + (1-\lambda)^{\alpha+1} J_{(\lambda a+(1-\lambda)b)^+}^\alpha f(b) \right] \Big| \\
\leq & \lambda^{\alpha+1}(1-\lambda)^{\alpha+1}(b-a) \left\{ \int_0^1 t^\alpha |f'[t(\lambda a + (1-\lambda)b) + (1-t)a]| dt \right. \\
& \left. - \int_0^1 (1-t)^\alpha |f'[tb + (1-t)(\lambda a + (1-\lambda)b)]| dt \right\} \\
\leq & \lambda^{\alpha+1}(1-\lambda)^{\alpha+1}(b-a) \left\{ \int_0^1 t^\alpha [t|f'(\lambda a + (1-\lambda)b)| + (1-t)|f'(a)|] dt \right. \\
& \left. + \int_0^1 (1-t)^\alpha [t|f'(b)| + (1-t)|f'(\lambda a + (1-\lambda)b)|] dt \right\} \\
= & \frac{2\lambda^{\alpha+1}(1-\lambda)^{\alpha+1}(b-a)}{(\alpha+2)} \left( |f'(\lambda a + (1-\lambda)b)| + \frac{|f'(a)| + |f'(b)|}{2(\alpha+1)} \right).
\end{aligned}$$

Secondly, we suppose that  $q > 1$ . Using Lemma 2 and power mean inequality, we obtain

$$\begin{aligned}
& \left\{ \int_0^1 t^\alpha |f'[t(\lambda a + (1-\lambda)b) + (1-t)a]| dt - \int_0^1 (1-t)^\alpha |f'[tb + (1-t)(\lambda a + (1-\lambda)b)]| dt \right\} \\
& \quad (2.6) \\
& \leq \left( \int_0^1 t^\alpha dt \right)^{1-\frac{1}{q}} \left( \int_0^1 t^\alpha |f'[t(\lambda a + (1-\lambda)b) + (1-t)a]|^q dt \right)^{\frac{1}{q}} \\
& \quad + \left( \int_0^1 (1-t)^\alpha dt \right)^{1-\frac{1}{q}} \left( \int_0^1 (1-t)^\alpha |f'[tb + (1-t)(\lambda a + (1-\lambda)b)]|^q dt \right)^{\frac{1}{q}}.
\end{aligned}$$

Hence, using convexity of  $|f'|^q$  and (2.6) we obtain

$$\begin{aligned}
& \left| [\lambda^{\alpha+1}(1-\lambda)^\alpha + \lambda^\alpha(1-\lambda)^{\alpha+1}] f(\lambda a + (1-\lambda)b) \right. \\
& \quad \left. - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ \lambda^{\alpha+1} J_{(\lambda a + (1-\lambda)b)^-}^\alpha f(a) + (1-\lambda)^{\alpha+1} J_{(\lambda a + (1-\lambda)b)^+}^\alpha f(b) \right] \right| \\
& \leq \frac{\lambda^{\alpha+1}(1-\lambda)^{\alpha+1}(b-a)}{(\alpha+1)^{1-\frac{1}{q}}} \left\{ \left( \int_0^1 t^\alpha [t|f'(\lambda a + (1-\lambda)b)|^q + (1-t)|f'(a)|^q] dt \right)^{\frac{1}{q}} \right. \\
& \quad \left. \left( \int_0^1 (1-t)^\alpha [t|f'(b)|^q + (1-t)|f'(\lambda a + (1-\lambda)b)|^q] dt \right)^{\frac{1}{q}} \right\} \\
& \leq \frac{\lambda^{\alpha+1}(1-\lambda)^{\alpha+1}(b-a)}{(\alpha+1)^{1-\frac{1}{q}}} \left\{ \left( \frac{(\alpha+1)|f'(\lambda a + (1-\lambda)b)|^q + |f'(a)|^q}{(\alpha+1)(\alpha+2)} \right)^{\frac{1}{q}} \right. \\
& \quad \left. + \left( \frac{(\alpha+1)|f'(\lambda a + (1-\lambda)b)|^q + |f'(b)|^q}{(\alpha+1)(\alpha+2)} \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

This completes the proof.  $\square$

**Corollary 1.** Under assumption Theorem 4 with  $\lambda = \frac{1}{2}$ , we have

$$\begin{aligned}
& \left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^\alpha} \left[ J_{(\frac{a+b}{2})^-}^\alpha f(a) + J_{(\frac{a+b}{2})^+}^\alpha f(b) \right] \right| \\
& \leq \frac{(b-a)}{4(\alpha+1)} \left\{ \left( \frac{(\alpha+1)|f'(\frac{a+b}{2})|^q + |f'(a)|^q}{(\alpha+2)} \right)^{\frac{1}{q}} \right. \\
& \quad \left. + \left( \frac{(\alpha+1)|f'(\frac{a+b}{2})|^q + |f'(b)|^q}{(\alpha+2)} \right)^{\frac{1}{q}} \right\} \\
& \quad (2.7)
\end{aligned}$$

where  $q \geq 1$ .

**Remark 2.** If we take  $\alpha = 1$  in Corollary 1, we have

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{8} \left\{ \left( \frac{2|f'(\frac{a+b}{2})|^q + |f'(a)|^q}{3} \right)^{\frac{1}{q}} + \left( \frac{2|f'(\frac{a+b}{2})|^q + |f'(b)|^q}{3} \right)^{\frac{1}{q}} \right\} \end{aligned}$$

where  $q \geq 1$ . Choosing  $q = 1$  in last inequality, it follows that

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{12} \left( 2 \left| f'\left(\frac{a+b}{2}\right) \right| + \frac{|f'(a)| + |f'(b)|}{2} \right).$$

In the last inequality, if we take  $2|f'(\frac{a+b}{2})| = \frac{|f'(a)| + |f'(b)|}{2}$ , then we reach a better the following result:

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{12} (|f'(a)| + |f'(b)|).$$

**Theorem 5.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  with  $0 \leq a < b$ . If  $|f'|^q$  convex on  $[a, b]$  for some fixed  $q > 1$ , then the following inequality for fractional integrals holds:

$$\begin{aligned} & \left| [\lambda^{\alpha+1}(1-\lambda)^\alpha + \lambda^\alpha(1-\lambda)^{\alpha+1}] f(\lambda a + (1-\lambda)b) \right. \\ & \left. - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ \lambda^{\alpha+1} J_{(\lambda a + (1-\lambda)b)^-}^\alpha f(a) + (1-\lambda)^{\alpha+1} J_{(\lambda a + (1-\lambda)b)^+}^\alpha f(b) \right] \right| \\ & \leq \frac{\lambda^{\alpha+1}(1-\lambda)^{\alpha+1}(b-a)}{(\alpha p + 1)^{\frac{1}{p}}} \left\{ \left( \frac{|f'(\lambda a + (1-\lambda)b)|^q + |f'(a)|^q}{2} \right)^{\frac{1}{q}} \right. \\ & \left. + \left( \frac{|f'(\lambda a + (1-\lambda)b)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

where  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $\alpha > 0$  and  $\lambda \in (0, 1)$ .

*Proof.* Using Lemma 2, convexity of  $|f'|^q$  and well-known Hölder's inequality, we obtain

$$\begin{aligned}
& \left| [\lambda^{\alpha+1}(1-\lambda)^\alpha + \lambda^\alpha(1-\lambda)^{\alpha+1}] f(\lambda a + (1-\lambda)b) \right. \\
& \quad \left. - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ \lambda^{\alpha+1} J_{(\lambda a+(1-\lambda)b)^-}^\alpha f(a) + (1-\lambda)^{\alpha+1} J_{(\lambda a+(1-\lambda)b)^+}^\alpha f(b) \right] \right| \\
& \leq \lambda^{\alpha+1}(1-\lambda)^{\alpha+1} (b-a) \left\{ \left( \int_0^1 t^{\alpha p} dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(t(\lambda a + (1-\lambda)b) + (1-t)a)|^q dt \right)^{\frac{1}{q}} \right. \\
& \quad \left. + \left( \int_0^1 (1-t)^{\alpha p} dt \right)^{\frac{1}{p}} \left( \int_0^1 |f'(tb + (1-t)(\lambda a + (1-\lambda)b))|^q dt \right)^{\frac{1}{q}} \right\} \\
& \leq \frac{\lambda^{\alpha+1}(1-\lambda)^{\alpha+1} (b-a)}{(\alpha p + 1)^{\frac{1}{p}}} \left\{ \left( \int_0^1 [t |f'(\lambda a + (1-\lambda)b)|^q + (1-t) |f'(a)|^q] dt \right)^{\frac{1}{q}} \right. \\
& \quad \left. + \left( \int_0^1 [t |f'(b)|^q + (1-t) |f'(\lambda a + (1-\lambda)b)|^q] dt \right)^{\frac{1}{q}} \right\} \\
& \leq \frac{\lambda^{\alpha+1}(1-\lambda)^{\alpha+1} (b-a)}{(\alpha p + 1)^{\frac{1}{p}}} \left\{ \left( \frac{|f'(\lambda a + (1-\lambda)b)|^q + |f'(a)|^q}{2} \right)^{\frac{1}{q}} \right. \\
& \quad \left. + \left( \frac{|f'(\lambda a + (1-\lambda)b)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

This completes the proof.  $\square$

**Corollary 2.** Under assumption Theorem 5 with  $\lambda = \frac{1}{2}$ , we have

$$\begin{aligned}
& \left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^\alpha} \left[ J_{\left(\frac{a+b}{2}\right)^-}^\alpha f(a) + J_{\left(\frac{a+b}{2}\right)^+}^\alpha f(b) \right] \right| \\
& \leq \frac{(b-a)}{4(\alpha p + 1)^{\frac{1}{p}}} \left\{ \left( \frac{|f'\left(\frac{a+b}{2}\right)|^q + |f'(a)|^q}{2} \right)^{\frac{1}{q}} + \left( \frac{|f'\left(\frac{a+b}{2}\right)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

**Remark 3.** If we take  $\alpha = 1$  in Corollary 2, we have

$$\begin{aligned}
& \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
& \leq \frac{(b-a)}{4(p+1)^{\frac{1}{p}}} \left\{ \left( \frac{|f'\left(\frac{a+b}{2}\right)|^q + |f'(a)|^q}{2} \right)^{\frac{1}{q}} + \left( \frac{|f'\left(\frac{a+b}{2}\right)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$



**Theorem 6.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be a differentiable mapping on  $(a, b)$  with  $0 \leq a < b$ . If  $|f'|^q$  convex on  $[a, b]$  for some fixed  $q \geq 1$ , then the following inequality for fractional integrals holds:

$$\begin{aligned} & |[\lambda^{\alpha+1}(1-\lambda)^\alpha + \lambda^\alpha(1-\lambda)^{\alpha+1}] f(\lambda a + (1-\lambda)b) \\ & - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ \lambda^{\alpha+1} J_{(\lambda a + (1-\lambda)b)^-}^\alpha f(a) + (1-\lambda)^{\alpha+1} J_{(\lambda a + (1-\lambda)b)^+}^\alpha f(b) \right] \\ \leq & \lambda^{\alpha+1}(1-\lambda)^{\alpha+1} (b-a) \left\{ \left( \frac{(\alpha q + 1) |f'(\lambda a + (1-\lambda)b)|^q + |f'(a)|^q}{(\alpha q + 1)(\alpha q + 2)} \right)^{\frac{1}{q}} \right. \\ & \left. + \left( \frac{(\alpha q + 1) |f'(\lambda a + (1-\lambda)b)|^q + |f'(b)|^q}{(\alpha q + 1)(\alpha q + 2)} \right)^{\frac{1}{q}} \right\} \end{aligned}$$

where  $\lambda \in (0, 1)$  and  $\alpha > 0$ .

*Proof.* Using Lemma 2, convexity of  $|f'|^q$ , and well-known Hölder's inequality, we have

$$\begin{aligned} & |[\lambda^{\alpha+1}(1-\lambda)^\alpha + \lambda^\alpha(1-\lambda)^{\alpha+1}] f(\lambda a + (1-\lambda)b) \\ & - \frac{\Gamma(\alpha+1)}{(b-a)^\alpha} \left[ \lambda^{\alpha+1} J_{(\lambda a + (1-\lambda)b)^-}^\alpha f(a) + (1-\lambda)^{\alpha+1} J_{(\lambda a + (1-\lambda)b)^+}^\alpha f(b) \right] \\ \leq & \lambda^{\alpha+1}(1-\lambda)^{\alpha+1} (b-a) \left\{ \left( \int_0^1 1^p dt \right)^{\frac{1}{p}} \left( \int_0^1 t^{\alpha q} |f'(t(\lambda a + (1-\lambda)b) + (1-t)a)|^q dt \right)^{\frac{1}{q}} \right. \\ & \left. + \left( \int_0^1 1^p dt \right)^{\frac{1}{p}} \left( \int_0^1 (1-t)^{\alpha q} |f'(tb + (1-t)(\lambda a + (1-\lambda)b))|^q dt \right)^{\frac{1}{q}} \right\} \\ \leq & \lambda^{\alpha+1}(1-\lambda)^{\alpha+1} (b-a) \left\{ \left( \int_0^1 t^{\alpha q} [t |f'(\lambda a + (1-\lambda)b)|^q + (1-t) |f'(a)|^q] dt \right)^{\frac{1}{q}} \right. \\ & \left. + \left( \int_0^1 (1-t)^{\alpha q} [t |f'(b)|^q + (1-t) |f'(\lambda a + (1-\lambda)b)|^q] dt \right)^{\frac{1}{q}} \right\} \\ = & \lambda^{\alpha+1}(1-\lambda)^{\alpha+1} (b-a) \left\{ \left( \frac{(\alpha q + 1) |f'(\lambda a + (1-\lambda)b)|^q + |f'(a)|^q}{(\alpha q + 1)(\alpha q + 2)} \right)^{\frac{1}{q}} \right. \\ & \left. + \left( \frac{(\alpha q + 1) |f'(\lambda a + (1-\lambda)b)|^q + |f'(b)|^q}{(\alpha q + 1)(\alpha q + 2)} \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

This completes the proof.  $\square$

**Corollary 3.** Under assumption Theorem 6 with  $\lambda = \frac{1}{2}$ , we have

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^\alpha} \left[ J_{\left(\frac{a+b}{2}\right)^-}^\alpha f(a) + J_{\left(\frac{a+b}{2}\right)^+}^\alpha f(b) \right] \right| \\ & \leq \frac{(b-a)}{4} \left\{ \left( \frac{(q\alpha+1)|f'(\frac{a+b}{2})|^q + |f'(a)|^q}{(q\alpha+1)(q\alpha+2)} \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left( \frac{(q\alpha+1)|f'(\frac{a+b}{2})|^q + |f'(b)|^q}{(q\alpha+1)(q\alpha+2)} \right)^{\frac{1}{q}} \right\} \end{aligned}$$

**Remark 4.** If we take  $\alpha = 1$  in Corollary 3, we have

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{4} \left\{ \left( \frac{(q+1)|f'(\frac{a+b}{2})|^q + |f'(a)|^q}{(q+1)(q+2)} \right)^{\frac{1}{q}} + \left( \frac{(q+1)|f'(\frac{a+b}{2})|^q + |f'(b)|^q}{(q+1)(q+2)} \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND ARTS, DÜZCE UNIVERSITY, DÜZCE-TURKEY

*E-mail address:* sarikayamz@gmail.com

*E-mail address:* yaldizhatice@gmail.com