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Approximation by Fuzzy Perturbed Neural Network Operators

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Abstract

This article deals with the determination of the rate of convergence to the unit of each of three newly introduced here fuzzy perturbed normalized neural network operators of one hidden layer. These are given through the fuzzy modulus of continuity of the involved fuzzy number valued function or its high order fuzzy derivative and that appears in the right-hand side of the associated fuzzy Jackson type inequalities. The activation function is very general, especially it can derive from any sigmoid or bell-shaped function. The right hand sides of our fuzzy convergence inequalities do not depend on the activation function. The sample fuzzy functionals are of Stancu, Kantorovich and Quadrature types. We give applications for the first fuzzy derivative of the involved function.

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1 Introduction

The Cardaliaguet-Euvrard real neural network operators were studied extensively in [14], where the authors among many other things proved that these operators converge uniformly on compacta, to the unit over continuous and bounded functions. Our fuzzy "perturbed normalized neural network operators" are motivated and inspired by the "bell" and "squashing functions" of [14]. The work in [14] is qualitative where the used bell-shaped function is general. However, our work, though greatly motivated by [14], is quantitative and the used activation functions are of compact support. Here we extend to fuzzy level our initial real work, see [12]. We derive a series of fuzzy Jackson type inequalities giving close upper bounds to the errors in approximating the unit operator by the above fuzzy perturbed neural network induced operators. All involved constants there are well determined. These are pointwise and uniform estimates involving the first fuzzy modulus of continuity of the engaged fuzzy function or the fuzzy derivative of the function under approximation. We give all necessary background of the fuzzy calculus needed.

Initial work of the subject was done in [10] and [11]. These works motivated the current work.

2 Fuzzy Mathematical Analysis Background

We need the following basic background

Definition 1 *(see [20]) Let $\mu : \mathbb{R} \to [0,1]$ with the following properties:*

(i) is normal, i.e., $\exists\ x_0 \in \mathbb{R} : \mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda) y) \geq \min\{\mu(x), \mu(y)\}$, $\forall\ x, y \in \mathbb{R}$, $\forall\ \lambda \in [0,1]$ (μ is called a convex fuzzy subset).

(iii) μ is upper semicontinuous on $\mathbb{R}$, i.e., $\forall\ x_0 \in \mathbb{R}$ and $\forall\ \varepsilon > 0$, $\exists$ neighborhood $V(x_0) : \mu(x) \leq \mu(x_0) + \varepsilon$, $\forall\ x \in V(x_0)$.

(iv) the set $\overline{supp(\mu)}$ is compact in $\mathbb{R}$ (where $supp(\mu) := \{x \in \mathbb{R} : \mu(x) > 0\}$).

We call μ a fuzzy real number. Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$.

E.g., $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$, for any $x_0 \in \mathbb{R}$, where $\chi_{\{x_0\}}$ is the characteristic function at x_0.

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define $[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\}$ and $[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) \geq 0\}}$.

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$ ([16]).

For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \ \ \forall\ r \in [0,1],$$

where

$[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and

$\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g., [20]).

Notice $1 \odot u = u$ and it holds

$$u \oplus v = v \oplus u, \quad \lambda \odot u = u \odot \lambda.$$

If $0 \leq r_1 \leq r_2 \leq 1$ then $[u]^{r_2} \subseteq [u]^{r_1}$. Actually $[u]^r = \left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} \leq u_+^{(r)}$, $u_-^{(r)}, u_+^{(r)} \in \mathbb{R}$, $\forall\ r \in [0,1]$.

For $\lambda > 0$ one has $\lambda u_{\pm}^{(r)} = (\lambda \odot u)_{\pm}^{(r)}$, respectively.

Define

$$D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \rightarrow \mathbb{R}_{+}$$

by

$$D(u, v) := \sup_{r \in [0,1]} \max \left\{ \left| u_{-}^{(r)} - v_{-}^{(r)} \right|, \left| u_{+}^{(r)} - v_{+}^{(r)} \right| \right\},$$

where

$$[v]^{r} = \left[v_{-}^{(r)}, v_{+}^{(r)} \right]; \quad u, v \in \mathbb{R}_{\mathcal{F}}.$$

We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$.

Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [21], [20].

Here $\overset{*}{\sum}$ stands for fuzzy summation and $\widetilde{0} := \chi_{\{0\}} \in \mathbb{R}_{\mathcal{F}}$ is the neural element with respect to $\oplus$, i.e.,

$$u \oplus \widetilde{0} = \widetilde{0} \oplus u = u, \ \ \forall \ u \in \mathbb{R}_{\mathcal{F}}.$$

Denote

$$D^{*}(f, g) := \sup_{x \in X \subseteq \mathbb{R}} D(f, g),$$

where $f, g : X \rightarrow \mathbb{R}_{\mathcal{F}}$.

We mention

Definition 2 *Let $f : X \subseteq \mathbb{R} \rightarrow \mathbb{R}_{\mathcal{F}}$, X interval, we define the (first) fuzzy modulus of continuity of f by*

$$\omega_{1}^{(\mathcal{F})}(f, \delta)_{X} = \sup_{x,y \in X, \ |x-y| \leq \delta} D(f(x), f(y)), \quad \delta > 0.$$

We define by $C_{\mathcal{F}}^{U}(\mathbb{R})$ the space of fuzzy uniformly continuous functions from $\mathbb{R} \rightarrow \mathbb{R}_{\mathcal{F}}$, also $C_{\mathcal{F}}(\mathbb{R})$ is the space of fuzzy continuous functions on $\mathbb{R}$, and $C_{B}(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ is the fuzzy continuous and bounded functions.

We mention

Proposition 3 *([6]) Let $f \in C_{\mathcal{F}}^{U}(X)$. Then $\omega_{1}^{(\mathcal{F})}(f, \delta)_{X} < \infty$, for any $\delta > 0$.*

Proposition 4 *([6]) It holds*

$$\lim_{\delta \rightarrow 0} \omega_{1}^{(\mathcal{F})}(f, \delta)_{X} = \omega_{1}^{(\mathcal{F})}(f, 0)_{X} = 0,$$

iff $f \in C_{\mathcal{F}}^{U}(X)$.

Proposition 5 *([6]) Here $[f]^{r} = \left[f_{-}^{(r)}, f_{+}^{(r)} \right]$, $r \in [0,1]$. Let $f \in C_{\mathcal{F}}(\mathbb{R})$. Then $f_{\pm}^{(r)}$ are equicontinuous with respect to $r \in [0,1]$ over $\mathbb{R}$, respectively in $\pm$.*

Note 6 *It is clear by Propositions 4, 5, that if* $f \in C_{\mathcal{F}}^{U}(\mathbb{R})$, *then* $f_{\pm}^{(r)} \in C_U(\mathbb{R})$ *(uniformly continuous on* $\mathbb{R}$*). Clearly also if* $f \in C_B(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$, *then* $f_{\pm}^{(r)} \in C_B(\mathbb{R})$ *(continuous and bounded functions on* $\mathbb{R}$*).*

Proposition 7 *Let* $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$. *Assume that* $\omega_1^{\mathcal{F}}(f, \delta)_X$, $\omega_1\left(f_{-}^{(r)}, \delta\right)_X$, $\omega_1\left(f_{+}^{(r)}, \delta\right)_X$ *are finite for any* $\delta > 0$, $r \in [0,1]$, *where X any interval of* $\mathbb{R}$.

Then

$$\omega_1^{(\mathcal{F})}(f, \delta)_X = \sup_{r \in [0,1]} \max\left\{\omega_1\left(f_{-}^{(r)}, \delta\right)_X, \omega_1\left(f_{+}^{(r)}, \delta\right)_X\right\}.$$

Proof. Similar to Proposition 14.15, p. 246 of [7]. ■

We need

Remark 8 *([4]). Here* $r \in [0,1]$, $x_i^{(r)}, y_i^{(r)} \in \mathbb{R}$, $i = 1, ..., m \in \mathbb{N}$. *Suppose that*

$$\sup_{r \in [0,1]} \max\left(x_i^{(r)}, y_i^{(r)}\right) \in \mathbb{R}, \text{ for } i = 1, ..., m.$$

Then one sees easily that

$$\sup_{r \in [0,1]} \max\left(\sum_{i=1}^{m} x_i^{(r)}, \sum_{i=1}^{m} y_i^{(r)}\right) \le \sum_{i=1}^{m} \sup_{r \in [0,1]} \max\left(x_i^{(r)}, y_i^{(r)}\right). \tag{1}$$

We need

Definition 9 *Let* $x, y \in \mathbb{R}_{\mathcal{F}}$. *If there exists* $z \in \mathbb{R}_{\mathcal{F}} : x = y \oplus z$, *then we call z the H-difference on x and y, denoted* $x - y$.

Definition 10 *([19]) Let* $T := [x_0, x_0 + \beta] \subset \mathbb{R}$, *with* $\beta > 0$. *A function* $f : T \to \mathbb{R}_{\mathcal{F}}$ *is H-difference at* $x \in T$ *if there exists an* $f'(x) \in \mathbb{R}_{\mathcal{F}}$ *such that the limits (with respect to D)*

$$\lim_{h \to 0+} \frac{f(x+h) - f(x)}{h}, \quad \lim_{h \to 0+} \frac{f(x) - f(x-h)}{h} \tag{2}$$

exist and are equal to $f'(x)$.

We call f' *the H-derivative or fuzzy derivative of* f *at* x.

Above is assumed that the H-differences $f(x+h) - f(x)$, $f(x) - f(x-h)$ exist in $\mathbb{R}_{\mathcal{F}}$ in a neighborhood of x.

Higher order H-fuzzy derivatives are defined the obvious way, like in the real case.

We denote by $C_{\mathcal{F}}^{N}(\mathbb{R})$, $N \ge 1$, the space of all N-times continuously H-fuzzy differentiable functions from $\mathbb{R}$ into $\mathbb{R}_{\mathcal{F}}$.

We mention

Theorem 11 *([17]) Let $f:\mathbb{R}\to\mathbb{R}_{\mathcal{F}}$ be H-fuzzy differentiable. Let $t\in\mathbb{R}$, $0\leq r\leq 1$. Clearly*

$$[f(t)]^r=\left[f(t)_-^{(r)},f(t)_+^{(r)}\right]\subseteq\mathbb{R}.$$

Then $(f(t))_\pm^{(r)}$ are differentiable and

$$[f'(t)]^r=\left[\left(f(t)_-^{(r)}\right)',\left(f(t)_+^{(r)}\right)'\right].$$

I.e.

$$(f')_\pm^{(r)}=\left(f_\pm^{(r)}\right)',\quad\forall\, r\in[0,1].$$

Remark 12 *([5]) Let $f\in C_{\mathcal{F}}^N(\mathbb{R})$, $N\geq 1$. Then by Theorem 11 we obtain*

$$\left[f^{(i)}(t)\right]^r=\left[\left(f(t)_-^{(r)}\right)^{(i)},\left(f(t)_+^{(r)}\right)^{(i)}\right],$$

for $i=0,1,2,...,N$, and in particular we have that

$$\left(f^{(i)}\right)_\pm^{(r)}=\left(f_\pm^{(r)}\right)^{(i)},$$

for any $r\in[0,1]$, all $i=0,1,2,...,N$.

Note 13 *([5]) Let $f\in C_{\mathcal{F}}^N(\mathbb{R})$, $N\geq 1$. Then by Theorem 11 we have $f_\pm^{(r)}\in C^N(\mathbb{R})$, for any $r\in[0,1]$.*

We need also a particular case of the Fuzzy Henstock integral ($\delta(x)=\frac{\delta}{2}$), see [20].

Definition 14 *([15], p. 644) Let $f:[a,b]\to\mathbb{R}_{\mathcal{F}}$. We say that f is Fuzzy-Riemann integrable to $I\in\mathbb{R}_{\mathcal{F}}$ if for any $\varepsilon>0$, there exists $\delta>0$ such that for any division $P=\{[u,v];\xi\}$ of $[a,b]$ with the norms $\Delta(P)<\delta$, we have*

$$D\left(\sum_P^{*}(v-u)\odot f(\xi),I\right)<\varepsilon.$$

We write

$$I:=(FR)\int_a^b f(x)\,dx. \tag{3}$$

We mention

Theorem 15 *([16]) Let $f:[a,b]\to\mathbb{R}_{\mathcal{F}}$ be fuzzy continuous. Then*

$$(FR)\int_a^b f(x)\,dx$$

exists and belongs to $\mathbb{R}_{\mathcal{F}}$, furthermore it holds

$$\left[(FR)\int_a^b f(x)\,dx\right]^r=\left[\int_a^b(f)_-^{(r)}(x)\,dx,\int_a^b(f)_+^{(r)}(x)\,dx\right], \tag{4}$$

$\forall\, r\in[0,1]$.

3 Real Neural Networks Approximation Basics (see [12])

Here the activation function $b : \mathbb{R} \to \mathbb{R}_+$ is of compact support $[-T, T]$, $T > 0$. That is $b(x) > 0$ for any $x \in [-T, T]$, and clearly b may have jump discontinuities. Also the shape of the graph of b could be anything. Typically in neural networks approximation we take b as a sigmoidal function or bell-shaped function, of course here of compact support $[-T, T]$, $T > 0$.

Example 16 *(i) b can be the characteristic function on $[-1, 1]$,*

(ii) b can be that hat function over $[-1, 1]$, i.e.,

$$b(x) = \begin{cases} 1 + x, & -1 \leq x \leq 0, \\ 1 - x, & 0 < x \leq 1, \\ 0, & \textit{elsewhere}, \end{cases}$$

(iii) the truncated sigmoidals

$$b(x) = \begin{cases} \frac{1}{1+e^{-x}} \textit{ or } \tanh x \textit{ or } \operatorname{erf}(x), & \textit{for } x \in [-T, T], \textit{ with large } T > 0, \\ 0, & x \in \mathbb{R} - [-T, T], \end{cases}$$

(iv) the truncated Gompertz function

$$b(x) = \begin{cases} e^{-\gamma e^{-\beta x}}, & x \in [-T, T]; \ \gamma, \beta > 0; \ \textit{large } T > 0, \\ 0, & x \in \mathbb{R} - [-T, T], \end{cases}$$

The Gompertz functions are also sigmoidal functions, with wide applications to many applied fields, e.g. demography and tumor growth modeling, etc.

So the general function b we will be using here covers all kinds of activation functions in neural network approximations.

In the real case we consider functions $f : \mathbb{R} \to \mathbb{R}$ that are either continuous and bounded, or uniformly continuous.

Let here the parameters $\mu, \nu \geq 0$; $\mu_i, \nu_i \geq 0$, $i = 1, ..., \overline{r} \in \mathbb{N}$; $w_i \geq 0$: $\sum_{i=1}^{\overline{r}} w_i = 1$; $0 < \alpha < 1$, $x \in \mathbb{R}$, $n \in \mathbb{N}$.

We use here the first modulus of continuity

$$\omega_1(f, \delta) := \sup_{\substack{x, y \in \mathbb{R} \\ |x - y| \leq \delta}} |f(x) - f(y)|,$$

and given that f is uniformly continuous we get $\lim_{\delta \to 0} \omega_1(f, \delta) = 0$.

Here we mention from [12] about the pointwise convergence with rates over $\mathbb{R}$, to the unit operator, of the following one hidden layer normalized neural network perturbed operators,

(i) the Stancu type ([18])

$$\left(H_n\left(f\right)\right)\left(x\right)=\frac{\sum_{k=-n^2}^{n^2}\left(\sum_{i=1}^{\overline{r}}w_i f\left(\frac{k+\mu_i}{n+\nu_i}\right)\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}, \tag{5}$$

(ii) the Kantorovich type

$$\left(K_n\left(f\right)\right)\left(x\right)=\frac{\sum_{k=-n^2}^{n^2}\left(\sum_{i=1}^{\overline{r}}w_i\left(n+\nu_i\right)\int_{\frac{k+\mu_i}{n+\nu_i}}^{\frac{k+\mu_i+1}{n+\nu_i}}f\left(t\right)dt\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}, \tag{6}$$

and

(iii) the quadrature type

$$\left(M_n\left(f\right)\right)\left(x\right)=\frac{\sum_{k=-n^2}^{n^2}\left(\sum_{i=1}^{\overline{r}}w_i f\left(\frac{k}{n}+\frac{i}{n\overline{r}}\right)\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}. \tag{7}$$

Similar operators defined for bell-shaped functions and sample coefficients $f\left(\frac{k}{n}\right)$ were studied initially in [14], [1], [2], [3], [8], [9], etc.

Here we care about the generalized perturbed cases of these operators (see [12], [13]).

Operator K_n in the corresponding Signal Processing context, represents the natural called "time-jitter" error, where the sample information is calculated in a perturbed neighborhood of $\frac{k+\mu}{n+\nu}$ rather than exactly at the node $\frac{k}{n}$.

The perturbed sample coefficients $f\left(\frac{k+\mu}{n+\nu}\right)$ with $0\leq\mu\leq\nu$, were first used by D. Stancu [18], in a totally different context, generalizing Bernstein operators approximation on $C\left(\left[0,1\right]\right)$.

The terms in the ratio of sums (5), (6), (7) are nonzero, iff

$$\left|n^{1-\alpha}\left(x-\frac{k}{n}\right)\right|\leq T,\ \text{ i.e. }\left|x-\frac{k}{n}\right|\leq\frac{T}{n^{1-\alpha}} \tag{8}$$

iff

$$nx-Tn^{\alpha}\leq k\leq nx+Tn^{\alpha}. \tag{9}$$

In order to have the desired order of the numbers

$$-n^2\leq nx-Tn^{\alpha}\leq nx+Tn^{\alpha}\leq n^2, \tag{10}$$

it is sufficiently enough to assume that

$$n\geq T+\left|x\right|. \tag{11}$$

When $x\in\left[-T,T\right]$ it is enough to assume $n\geq 2T$, which implies (10).

Proposition 17 *([1]) Let $a \leq b$, $a, b \in \mathbb{R}$. Let card (k) (≥ 0) be the maximum number of integers contained in $[a,b]$. Then*

$$\max(0,(b-a)-1) \leq card\,(k) \leq (b-a)+1. \tag{12}$$

Note 18 *We would like to establish a lower bound on card (k) over the interval $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$. From Proposition 17 we get that*

$$card\,(k) \geq \max\left(2Tn^{\alpha}-1,0\right). \tag{13}$$

We obtain card $(k) \geq 1$, if

$$2Tn^{\alpha}-1 \geq 1 \quad \text{iff } n \geq T^{-\frac{1}{\alpha}}. \tag{14}$$

So to have the desired order (10) and card $(k) \geq 1$ over $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$, we need to consider

$$n \geq \max\left(T+|x|, T^{-\frac{1}{\alpha}}\right). \tag{15}$$

Also notice that card $(k) \to +\infty$, as $n \to +\infty$.

Denote by $[\cdot]$ the integral part of a number and by $\lceil \cdot \rceil$ its ceiling.

So under assumption (15), the operators H_n, K_n, M_n, collapse to

(i)

$$\left(H_n(f)\right)(x) = \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\sum\limits_{i=1}^{\overline{r}} w_i f\left(\frac{k+\mu_i}{n+\nu_i}\right)\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}, \tag{16}$$

(ii)

$$\left(K_n(f)\right)(x) = \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\sum\limits_{i=1}^{\overline{r}} w_i (n+\nu_i) \int_{\frac{k+\mu_i}{n+\nu_i}}^{\frac{k+\mu_i+1}{n+\nu_i}} f(t)\,dt\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}, \tag{17}$$

and

(iii)

$$\left(M_n(f)\right)(x) = \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\sum\limits_{i=1}^{\overline{r}} w_i f\left(\frac{k}{n}+\frac{i}{n\overline{r}}\right)\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}. \tag{18}$$

From [12] we will mention and use the following results.

Theorem 19 *Let $x \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T+|x|, T^{-\frac{1}{\alpha}}\right)$. Then*

$$\left|\left(H_n(f)\right)(x) - f(x)\right| \leq \sum_{i=1}^{\overline{r}} w_i \omega_1\left(f, \left(\frac{\nu_i |x| + \mu_i}{n+\nu_i}\right) + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right) \tag{19}$$

$$\leq \max_{i\in\{1,\ldots,\overline{r}\}} \left\{ \omega_1 \left(f, \left(\frac{\nu_i |x| + \mu_i}{n+\nu_i} \right) + \left(1 + \frac{\nu_i}{n+\nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \right\}.$$

Corollary 20 *Let $x \in [-T^*, T^*]$, $T^* > 0$, $n \in \mathbb{N} : n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}} \right)$, $T > 0$. Then*

$$\|H_n(f) - f\|_{\infty,[-T^*,T^*]} \leq \sum_{i=1}^{\overline{r}} w_i \omega_1 \left(f, \left(\frac{\nu_i T^* + \mu_i}{n+\nu_i} \right) + \left(1 + \frac{\nu_i}{n+\nu_i} \right) \frac{T}{n^{1-\alpha}} \right)$$
$$\leq \max_{i\in\{1,\ldots,\overline{r}\}} \left\{ \omega_1 \left(f, \left(\frac{\nu_i T^* + \mu_i}{n+\nu_i} \right) + \left(1 + \frac{\nu_i}{n+\nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \right\}. \tag{20}$$

Theorem 21 *Let $x \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}} \right)$. Then*

$$|(K_n(f))(x) - f(x)| \leq \tag{21}$$
$$\max_{i\in\{1,\ldots,\overline{r}\}} \left\{ \omega_1 \left(f, \left(\frac{\nu_i |x| + \mu_i + 1}{n+\nu_i} \right) + \left(1 + \frac{\nu_i}{n+\nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \right\}.$$

Corollary 22 *Let $x \in [-T^*, T^*]$, $T^* > 0$, $n \in \mathbb{N} : n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}} \right)$, $T > 0$. Then*

$$\|K_n(f) - f\|_{\infty,[-T^*,T^*]} \leq \tag{22}$$
$$\max_{i\in\{1,\ldots,\overline{r}\}} \left\{ \omega_1 \left(f, \left(\frac{\nu_i T^* + \mu_i + 1}{n+\nu_i} \right) + \left(1 + \frac{\nu_i}{n+\nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \right\}.$$

Theorem 23 *Let $x \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}} \right)$. Then*

$$|M_n(f)(x) - f(x)| \leq \omega_1 \left(f, \frac{1}{n} + \frac{T}{n^{1-\alpha}} \right). \tag{23}$$

Corollary 24 *Let $x \in [-T^*, T^*]$, $T^* > 0$, $n \in \mathbb{N} : n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}} \right)$, $T > 0$. Then*

$$\|M_n(f) - f\|_{\infty,[-T^*,T^*]} \leq \omega_1 \left(f, \frac{1}{n} + \frac{T}{n^{1-\alpha}} \right). \tag{24}$$

In (19)-(24) all converges are at the speed $\frac{1}{n^{1-\alpha}}$.

Again from [12], taking into account the differentiation order of f we obtain:

Theorem 25 *Let $x \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}} \right)$, $0 < \alpha < 1$. Let $f \in C^N(\mathbb{R})$, $N \in \mathbb{N}$, such that $f^{(N)}$ is uniformly continuous or is continuous and bounded. Then*

$$|(H_n(f))(x) - f(x)| \leq$$

$$\sum_{j=1}^{N}\frac{\left|f^{(j)}(x)\right|}{j!}\left\{\sum_{i=1}^{\overline{r}}w_i\left[\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right]^j\right\}+ \tag{25}$$

$$\sum_{i=1}^{\overline{r}}w_i\omega_1\left(f^{(N)},\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)\cdot$$

$$\frac{\left(\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!}.$$

Inequality (25) implies the pointwise convergence with rates on $(H_n(f))(x)$to $f(x)$, as $n\to\infty$, at speed $\frac{1}{n^{1-\alpha}}$.

Corollary 26 *All as in Theorem 25, plus $f^{(j)}(x)=0$, $j=1,...,N$. Then*

$$|(H_n(f))(x)-f(x)|\leq$$

$$\sum_{i=1}^{\overline{r}}w_i\omega_1\left(f^{(N)},\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)\cdot$$

$$\frac{\left(\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!}, \tag{26}$$

a convergence at the high speed of $\frac{1}{n^{(1-\alpha)(N+1)}}$.

Corollary 27 *Let $x\in[-T^*,T^*]$, $T^*>0$; $T>0$ and $n\in\mathbb{N}$ such that $n\geq\max\left(T+T^*,T^{-\frac{1}{\alpha}}\right)$. Let $f\in C^N(\mathbb{R})$, $N\in\mathbb{N}$, such that $f^{(N)}$ is uniformly continuous or is continuous and bounded. Then*

$$\|H_n(f)-f\|_{\infty,[-T^*,T^*]}\leq \tag{27}$$

$$\sum_{j=1}^{N}\frac{\left\|f^{(j)}\right\|_{\infty,[-T^*,T^*]}}{j!}\left\{\sum_{i=1}^{\overline{r}}w_i\left[\left(\frac{\nu_iT^*+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right]^j\right\}+$$

$$\sum_{i=1}^{\overline{r}}w_i\omega_1\left(f^{(N)},\left(\frac{\nu_iT^*+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)\cdot$$

$$\frac{\left(\left(\frac{\nu_iT^*+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!}.$$

Corollary 28 *All as in Theorem 25, case of $N=1$. It holds*

$$|(H_n(f))(x)-f(x)|\leq$$

$$|f'(x)|\left\{\sum_{i=1}^{\overline{r}}w_i\left[\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right]\right\}+ \tag{28}$$

$$\sum_{i=1}^{\bar{r}} w_i \omega_1 \left(f', \left(\frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \cdot$$

$$\left(\left(\frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right).$$

Theorem 29 *Same assumptions as in Theorem 25, with* $0 < \alpha < 1$. *Then*

$$|(K_n(f))(x) - f(x)| \leq \sum_{j=1}^{N} \frac{\left| f^{(j)}(x) \right|}{j!} \cdot \tag{29}$$

$$\left(\sum_{i=1}^{\bar{r}} w_i \left(\left(\frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right)^j \right) +$$

$$\sum_{i=1}^{\bar{r}} w_i \left\{ \omega_1 \left(f^{(N)}, \left(\frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \cdot \right.$$

$$\left. \frac{\left(\left(\frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right)^N}{N!} \right\}.$$

Inequality (29) implies the pointwise convergence with rates of $(K_n(f))(x)$ *to* $f(x)$, *as* $n \to \infty$, *at the speed* $\frac{1}{n^{1-\alpha}}$.

Corollary 30 *All as in Theorem 25, plus* $f^{(j)}(x) = 0$, $j = 1, ..., N$; $0 < \alpha < 1$. *Then*

$$|(K_n(f))(x) - f(x)| \leq$$

$$\sum_{i=1}^{\bar{r}} w_i \left\{ \omega_1 \left(f^{(N)}, \left(\frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \cdot \right. \tag{30}$$

$$\left. \frac{\left(\left(\frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right)^N}{N!} \right\},$$

a convergence at speed $\frac{1}{n^{(1-\alpha)(N+1)}}$.

Corollary 31 *Let* $x \in [-T^*, T^*]$, $T^* > 0$; $T > 0$ *and* $n \in \mathbb{N}$ *such that* $n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}} \right)$, $0 < \alpha < 1$. *Let* $f \in C^N(\mathbb{R})$, $N \in \mathbb{N}$, *such that* $f^{(N)}$ *is uniformly continuous or is continuous and bounded. Then*

$$\|K_n(f) - f\|_{\infty, [-T^*, T^*]} \leq \sum_{j=1}^{N} \frac{\left\| f^{(j)} \right\|_{\infty, [-T^*, T^*]}}{j!} \cdot \tag{31}$$

$$\left(\sum_{i=1}^{\overline{r}} w_i \left[\left(\frac{\nu_i T^* + \mu_i + 1}{n+\nu_i}\right) + \left(1 + \frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right]^j\right) +$$

$$\sum_{i=1}^{\overline{r}} w_i \left\{\omega_1\left(f^{(N)}, \left(\frac{\nu_i T^* + \mu_i + 1}{n+\nu_i}\right) + \left(1 + \frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right) \cdot \right.$$

$$\left. \frac{\left(\left(\frac{\nu_i T^* + \mu_i + 1}{n+\nu_i}\right) + \left(1 + \frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!}\right\}.$$

Corollary 32 *All as in Theorem 25, case of $N = 1$. It holds*

$$|(K_n(f))(x) - f(x)| \leq$$

$$|f'(x)| \left(\sum_{i=1}^{\overline{r}} w_i \left(\left(\frac{\nu_i |x| + \mu_i + 1}{n+\nu_i}\right) + \left(1 + \frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)\right) + \tag{32}$$

$$\sum_{i=1}^{\overline{r}} w_i \left\{\omega_1\left(f', \left(\frac{\nu_i |x| + \mu_i + 1}{n+\nu_i}\right) + \left(1 + \frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right) \cdot \right.$$

$$\left. \left(\left(\frac{\nu_i |x| + \mu_i + 1}{n+\nu_i}\right) + \left(1 + \frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)\right\}.$$

Theorem 33 *Let all as in Theorem 25. Then*

$$|(M_n(f))(x) - f(x)| \leq \sum_{j=1}^{N} \frac{\left|f^{(j)}(x)\right|}{j!}\left[\frac{T}{n^{1-\alpha}} + \frac{1}{n}\right]^j + \tag{33}$$

$$\omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}} + \frac{1}{n}\right)\frac{\left(\frac{T}{n^{1-\alpha}} + \frac{1}{n}\right)^N}{N!}.$$

Inequality (33) implies the pointwise convergence with rates of $(M_n(f))(x)$ to $f(x)$, as $n \to \infty$, at the speed $\frac{1}{n^{1-\alpha}}$.

Corollary 34 *All as in Theorem 25, plus $f^{(j)}(x) = 0$, $j = 1, ..., N$. Then*

$$|(M_n(f))(x) - f(x)| \leq \omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}} + \frac{1}{n}\right)\frac{\left(\frac{T}{n^{1-\alpha}} + \frac{1}{n}\right)^N}{N!}, \tag{34}$$

a convergence at speed $\frac{1}{n^{(1-\alpha)(N+1)}}$.

Corollary 35 *All here as in Corollary 27. Then*

$$\|M_n(f) - f\|_{\infty,[-T^*,T^*]} \leq \sum_{j=1}^{N} \frac{\left\|f^{(j)}\right\|_{\infty,[-T^*,T^*]}}{j!}\left(\frac{T}{n^{1-\alpha}} + \frac{1}{n}\right)^j \tag{35}$$

$$+\omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}} + \frac{1}{n}\right)\frac{\left(\frac{T}{n^{1-\alpha}} + \frac{1}{n}\right)^N}{N!}.$$

Finally we mention also from [12]

Corollary 36 *All as in Theorem 25, $N = 1$ case. It holds*

$$|(M_n(f))(x) - f(x)| \le \left[|f'(x)| + \omega_1 \left(f', \frac{T}{n^{1-\alpha}} + \frac{1}{n} \right) \right] \left(\frac{T}{n^{1-\alpha}} + \frac{1}{n} \right). \quad (36)$$

Remark 37 *1) By change of variable method, the operator K_n could be written conveniently as follows:*

(ii)'

$$(K_n(f))(x) =$$

$$\frac{\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{\overline{r}} w_i (n+\nu_i) \int_0^{\frac{1}{n+\nu_i}} f\left(t + \frac{k+\mu_i}{n+\nu_i} \right) dt \right) b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}{\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}. \quad (37)$$

2) Next we apply the principle of iterated suprema. Let $W \subseteq \mathbb{R}$ and $f : W \to \mathbb{R}_{\mathcal{F}}$ with $D^*(f, \widetilde{o}) < \infty$, that is f is a fuzzy bounded function. Then easily we derive that

$$D^*(f, \widetilde{o}) = \sup_{r \in [0,1]} \max \left\{ \left\| f_-^{(r)} \right\|_\infty, \left\| f_+^{(r)} \right\|_\infty \right\}, \quad (38)$$

where $\|\cdot\|_\infty$ is the supremum norm of the function over W, and

$$D^*(f, \widetilde{o}) = \sup_{x \in W} D(f(x), \widetilde{o}).$$

4 Fuzzy Neural Network Approximations

Here we consider $f \in C_{\mathcal{F}}^U(\mathbb{R})$ or $f \in C_B(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$, b is as in section 3, $0 < \alpha < 1$, also the rest of parameters, are as in section 3. For $x \in \mathbb{R}$, we take always that $n \ge \max\left(T + |x|, T^{-\frac{1}{\alpha}} \right)$. The fuzzy analog of operators H_n, K_n, M_n, see (16), (17), (18) and (37) follows, $n \in \mathbb{N}$.

We define the corresponding fuzzy operators next

(i)

$$\left(H_n^{\mathcal{F}}(f) \right)(x) = \frac{\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]*} \left(\sum_{i=1}^{\overline{r}*} w_i \odot f\left(\frac{k+\mu_i}{n+\nu_i} \right) \right) \odot b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}{\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}, \quad (39)$$

(ii)

$$\left(K_n^{\mathcal{F}}(f) \right)(x) =$$

$$\frac{\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]*} \left(\sum_{i=1}^{\overline{r}*} w_i (n+\nu_i) \odot (FR) \int_0^{\frac{1}{n+\nu_i}} f\left(t + \frac{k+\mu_i}{n+\nu_i} \right) dt \right) \odot b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}{\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}, \quad (40)$$

and

(iii)

$$\left(M_n^{\mathcal{F}}(f)\right)(x)=\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]^*}\left(\sum_{i=1}^{\overline{r}*}w_i\odot f\left(\frac{k}{n}+\frac{i}{n\overline{r}}\right)\right)\odot b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}. \tag{41}$$

Call

$$V(x)=\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right). \tag{42}$$

We notice that ($r\in[0,1]$)

$$\left[\left(H_n^{\mathcal{F}}(f)\right)(x)\right]^r=\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{\overline{r}}w_i\left[f\left(\frac{k+\mu_i}{n+\nu_i}\right)\right]^r\right)\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}$$

$$=\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{\overline{r}}w_i\left[f_-^{(r)}\left(\frac{k+\mu_i}{n+\nu_i}\right),f_+^{(r)}\left(\frac{k+\mu_i}{n+\nu_i}\right)\right]\right)\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}$$

$$=\left[\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{\overline{r}}w_i f_-^{(r)}\left(\frac{k+\mu_i}{n+\nu_i}\right)\right)\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)},\right. \tag{43}$$

$$\left.\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{\overline{r}}w_i f_+^{(r)}\left(\frac{k+\mu_i}{n+\nu_i}\right)\right)\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\right]$$

$$=\left[\left(H_n\left(f_-^{(r)}\right)\right)(x),\left(H_n\left(f_+^{(r)}\right)\right)(x)\right].$$

We have proved that

$$\left(H_n^{\mathcal{F}}(f)\right)_{\pm}^{(r)}=H_n\left(f_{\pm}^{(r)}\right),\quad \forall\, r\in[0,1], \tag{44}$$

respectively.

For convinience also we call

$$A(x)=\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}. \tag{45}$$

We observe that

$$\left[\left(K_n^{\mathcal{F}}(f)\right)(x)\right]^r=$$

$$\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{\overline{r}}w_i(n+\nu_i)\left[\int_0^{\frac{1}{n+\nu_i}}f\left(t+\frac{k+\mu_i}{n+\nu_i}\right)dt\right]^r\right)A(x)\overset{(4)}{=}$$

$$\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{\overline{r}}w_{i}\left(n+\nu_{i}\right)\left[\int_{0}^{\frac{1}{n+\nu_{i}}}f_{-}^{(r)}\left(t+\frac{k+\mu_{i}}{n+\nu_{i}}\right)dt,\right.\right.$$

$$\left.\left.\int_{0}^{\frac{1}{n+\nu_{i}}}f_{+}^{(r)}\left(t+\frac{k+\mu_{i}}{n+\nu_{i}}\right)dt\right]\right)A\left(x\right)= \tag{46}$$

$$\left[\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{\overline{r}}w_{i}\left(n+\nu_{i}\right)\int_{0}^{\frac{1}{n+\nu_{i}}}f_{-}^{(r)}\left(t+\frac{k+\mu_{i}}{n+\nu_{i}}\right)dt\right)A\left(x\right),\right.$$

$$\left.\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{\overline{r}}w_{i}\left(n+\nu_{i}\right)\int_{0}^{\frac{1}{n+\nu_{i}}}f_{+}^{(r)}\left(t+\frac{k+\mu_{i}}{n+\nu_{i}}\right)dt\right)A\left(x\right)\right]$$

$$=\left[\left(K_{n}\left(f_{-}^{(r)}\right)\right)(x),\left(K_{n}\left(f_{+}^{(r)}\right)\right)(x)\right].$$

We have proved that

$$\left(K_{n}^{\mathcal{F}}\left(f\right)\right)_{\pm}^{(r)}=K_{n}\left(f_{\pm}^{(r)}\right),\quad\forall\ r\in[0,1], \tag{47}$$

respectively.

By linear change of variable of fuzzy Riemann integrals, see [7], pp. 242-243, we easily obtain that

$$\left(K_{n}^{\mathcal{F}}\left(f\right)\right)(x)=$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]^{*}}\left(\sum_{i=1}^{\overline{r}*}w_{i}\left(n+\nu_{i}\right)\odot\left(FR\right)\int_{\frac{k+\mu_{i}}{n+\nu_{i}}}^{\frac{k+\mu_{i}+1}{n+\nu_{i}}}f\left(t\right)dt\right)\odot b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}, \tag{48}$$

which is the fuzzy analog of $\left(K_{n}\left(f\right)\right)(x)$ as in (17).

Next we observe that

$$\left[\left(M_{n}^{\mathcal{F}}\left(f\right)\right)(x)\right]^{r}=\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{\overline{r}}w_{i}\left[f\left(\frac{k}{n}+\frac{i}{n\overline{r}}\right)\right]^{r}\right)A\left(x\right)=$$

$$\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{\overline{r}}w_{i}\left[f_{-}^{(r)}\left(\frac{k}{n}+\frac{i}{n\overline{r}}\right),f_{+}^{(r)}\left(\frac{k}{n}+\frac{i}{n\overline{r}}\right)\right]\right)A\left(x\right)= \tag{49}$$

$$\left[\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{\overline{r}}w_{i}f_{-}^{(r)}\left(\frac{k}{n}+\frac{i}{n\overline{r}}\right)\right)A\left(x\right),\right.$$

$$\left.\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{\overline{r}}w_{i}f_{+}^{(r)}\left(\frac{k}{n}+\frac{i}{n\overline{r}}\right)\right)A\left(x\right)\right]$$

$$= \left[\left(M_n \left(f_-^{(r)} \right) \right) (x) , \left(M_n \left(f_+^{(r)} \right) \right) (x) \right].$$

That is proving

$$\left(M_n^{\mathcal{F}} (f) \right)_{\pm}^{(r)} = M_n \left(f_{\pm}^{(r)} \right), \quad \forall\, r \in [0,1], \tag{50}$$

respectively.

We present our first main result

Theorem 38 *It holds*

$$D \left(\left(H_n^{\mathcal{F}} (f) \right) (x) , f (x) \right) \leq$$

$$\omega_1^{(\mathcal{F})} \left(f, \max_{i \in \{1,\ldots,\overline{r}\}} \left[\left(\frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right] \right), \tag{51}$$

where $\omega_1^{(\mathcal{F})} (f, \cdot) = \omega_1^{(\mathcal{F})} (f, \cdot)_{\mathbb{R}}$.

Notice that (51) implies $H_n^{\mathcal{F}} (f) \xrightarrow{D} f$*, pointwise, as* $n \to \infty$ *when* $f \in C_{\mathcal{F}}^U (\mathbb{R})$*, at speed* $\frac{1}{n^{1-\alpha}}$*.*

Proof. We observe that

$$D \left(\left(H_n^{\mathcal{F}} (f) \right) (x) , f (x) \right) =$$

$$\sup_{r \in [0,1]} \max \left\{ \left| \left(H_n^{\mathcal{F}} (f) \right)_-^{(r)} (x) - f_-^{(r)} (x) \right| , \left| \left(H_n^{\mathcal{F}} (f) \right)_+^{(r)} (x) - f_+^{(r)} (x) \right| \right\} \overset{(44)}{=} \tag{52}$$

$$\sup_{r \in [0,1]} \max \left\{ \left| \left(H_n \left(f_-^{(r)} \right) \right) (x) - f_-^{(r)} (x) \right| , \left| \left(H_n \left(f_+^{(r)} \right) \right) (x) - f_+^{(r)} (x) \right| \right\} \overset{(19)}{\leq}$$

$$\sup_{r \in [0,1]} \max \left\{ \max_{i \in \{1,\ldots,\overline{r}\}} \left\{ \omega_1 \left(f_-^{(r)}, \left(\frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \right\}, \right.$$

$$\left. \max_{i \in \{1,\ldots,\overline{r}\}} \left\{ \omega_1 \left(f_+^{(r)}, \left(\frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \right\} \right\} =$$

$$\sup_{r \in [0,1]} \max \left\{ \omega_1 \left(f_-^{(r)}, \max_{i \in \{1,\ldots,\overline{r}\}} \left[\left(\frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right] \right), \right.$$

$$\left. \omega_1 \left(f_+^{(r)}, \max_{i \in \{1,\ldots,\overline{r}\}} \left[\left(\frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right] \right) \right\} =$$

(by Proposition 7)

$$\omega_1^{(\mathcal{F})} \left(f, \max_{i \in \{1,\ldots,\overline{r}\}} \left[\left(\frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right] \right), \tag{53}$$

proving the claim. ■

Corollary 39 *Let* $x \in [-T^*, T^*]$, $T^* > 0$, $n \in \mathbb{N} : n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}}\right)$, $T > 0$. *Then*

$$D^*\left(H_n^{\mathcal{F}}(f), f\right)_{[-T^*,T^*]} \leq$$

$$\omega_1^{(\mathcal{F})}\left(f, \max_{i \in \{1,\dots,\overline{r}\}}\left[\left(\frac{\nu_i T^* + \mu_i}{n + \nu_i}\right) + \left(1 + \frac{\nu_i}{n + \nu_i}\right)\frac{T}{n^{1-\alpha}}\right]\right). \tag{54}$$

Notice that (54) implies $H_n^{\mathcal{F}}(f) \overset{D^*}{\to} f$, *uniformly, as* $n \to \infty$ *when* $f \in C_{\mathcal{F}}^U(\mathbb{R})$, *at speed* $\frac{1}{n^{1-\alpha}}$.

Proof. By (51). ■

We continue with

Theorem 40 *It holds*

$$D\left(\left(K_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq$$

$$\omega_1^{(\mathcal{F})}\left(f, \max_{i \in \{1,\dots,\overline{r}\}}\left[\left(\frac{\nu_i |x| + \mu_i + 1}{n + \nu_i}\right) + \left(1 + \frac{\nu_i}{n + \nu_i}\right)\frac{T}{n^{1-\alpha}}\right]\right). \tag{55}$$

Notice that (55) implies $K_n^{\mathcal{F}}(f) \overset{D}{\to} f$, *pointwise, as* $n \to \infty$ *when* $f \in C_{\mathcal{F}}^U(\mathbb{R})$, *at speed* $\frac{1}{n^{1-\alpha}}$.

Proof. Similar to Theorem 38, we use (47) and (21) into $D\left(\left(K_n^{\mathcal{F}}(f)\right)(x), f(x)\right)$. Finally we use Proposition 7. ■

Corollary 41 *All as in Corollary 39. Then*

$$D^*\left(\left(K_n^{\mathcal{F}}(f), f\right)\right)_{[-T^*,T^*]} \leq \tag{56}$$

$$\omega_1^{(\mathcal{F})}\left(f, \max_{i \in \{1,\dots,\overline{r}\}}\left[\left(\frac{\nu_i T^* + \mu_i + 1}{n + \nu_i}\right) + \left(1 + \frac{\nu_i}{n + \nu_i}\right)\frac{T}{n^{1-\alpha}}\right]\right).$$

Notice that (56) implies $K_n^{\mathcal{F}}(f) \overset{D^*}{\to} f$, *uniformly, as* $n \to \infty$ *when* $f \in C_{\mathcal{F}}^U(\mathbb{R})$, *at speed* $\frac{1}{n^{1-\alpha}}$.

Proof. By (55). ■

We also have

Theorem 42 *It holds*

$$D\left(\left(M_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq \omega_1^{(\mathcal{F})}\left(f, \frac{1}{n} + \frac{T}{n^{1-\alpha}}\right). \tag{57}$$

Notice that (57) implies $M_n^{\mathcal{F}}(f) \overset{D}{\to} f$, *pointwise, as* $n \to \infty$ *when* $f \in C_{\mathcal{F}}^U(\mathbb{R})$, *at speed* $\frac{1}{n^{1-\alpha}}$.

Proof. Similar to Theorem 38, we use (50) and (23) into $D\left(\left(M_n^{\mathcal{F}}(f)\right)(x), f(x)\right)$. Finally we use Proposition 7. ■

Corollary 43 *All as in Corollary 39. Then*

$$D^*\left(\left(M_n^{\mathcal{F}}(f), f\right)\right)_{[-T^*,T^*]} \leq \omega_1^{(\mathcal{F})}\left(f, \frac{1}{n} + \frac{T}{n^{1-\alpha}}\right). \tag{58}$$

Notice that (58) implies $M_n^{\mathcal{F}}(f) \overset{D^*}{\to} f$, uniformly, as $n \to \infty$ when $f \in C_{\mathcal{F}}^U(\mathbb{R})$, at speed $\frac{1}{n^{1-\alpha}}$.

Proof. By (57). ■

We proceed to the following results where we use the smoothness of a fuzzy derivative of f.

Theorem 44 *Let $f \in C_{\mathcal{F}}^N(\mathbb{R})$, $N \in \mathbb{N}$, with H-fuzzy $f^{(N)}$ either fuzzy continuous and bounded or fuzzy uniformly continuous.*

Here $x \in \mathbb{R}$, $T > 0$, $n \in \mathbb{N} : n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$, $0 < \alpha < 1$, the rest of parameters as in Section 3. Then

$$D\left(\left(H_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq$$

$$\sum_{j=1}^{N} \frac{D\left(f^{(j)}(x), \widetilde{o}\right)}{j!}\left\{\sum_{i=1}^{\overline{r}} w_i\left[\left(\frac{\nu_i|x| + \mu_i}{n + \nu_i}\right) + \left(1 + \frac{\nu_i}{n + \nu_i}\right)\frac{T}{n^{1-\alpha}}\right]^j\right\} +$$

$$\sum_{i=1}^{\overline{r}} w_i \frac{\left(\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right) + \left(1 + \frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!} \cdot \tag{59}$$

$$\omega_1^{(\mathcal{F})}\left(f^{(N)}, \left(\frac{\nu_i|x| + \mu_i}{n + \nu_i}\right) + \left(1 + \frac{\nu_i}{n + \nu_i}\right)\frac{T}{n^{1-\alpha}}\right).$$

Inequality (59) implies the pointwise convergence with rates on $\left(H_n^{\mathcal{F}}(f)\right)(x) \overset{D}{\to} f(x)$, as $n \to \infty$, at speed $\frac{1}{n^{1-\alpha}}$.

Proof. We observe that

$$D\left(\left(H_n^{\mathcal{F}}(f)\right)(x), f(x)\right) =$$

$$\sup_{r \in [0,1]} \max\left\{\left|\left(H_n^{\mathcal{F}}(f)\right)_-^{(r)}(x) - f_-^{(r)}(x)\right|, \left|\left(H_n^{\mathcal{F}}(f)\right)_+^{(r)}(x) - f_+^{(r)}(x)\right|\right\} \overset{(44)}{=}$$

$$\sup_{r \in [0,1]} \max\left\{\left|\left(H_n\left(f_-^{(r)}\right)\right)(x) - f_-^{(r)}(x)\right|, \left|\left(H_n\left(f_+^{(r)}\right)\right)(x) - f_+^{(r)}(x)\right|\right\} \overset{(25)}{\leq} \tag{60}$$

$$\sup_{r\in[0,1]} \max\left\{\sum_{j=1}^{N} \frac{\left|\left(f_{-}^{(r)}\right)^{(j)}(x)\right|}{j!}\left\{\sum_{i=1}^{\overline{r}} w_i\left[\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right]^j\right\}\right.$$

$$+\sum_{i=1}^{\overline{r}} w_i\omega_1\left(\left(f_{-}^{(r)}\right)^{(N)}, \left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)\cdot$$

$$\frac{\left(\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!},$$

$$\sum_{j=1}^{N} \frac{\left|\left(f_{+}^{(r)}\right)^{(j)}(x)\right|}{j!}\left\{\sum_{i=1}^{\overline{r}} w_i\left[\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right]^j\right\}+$$

$$\sum_{i=1}^{\overline{r}} w_i\omega_1\left(\left(f_{+}^{(r)}\right)^{(N)}, \left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)\cdot$$

$$\left.\frac{\left(\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!}\right\} = \tag{61}$$

(by Remark 12)

$$\sup_{r\in[0,1]} \max\left\{\sum_{j=1}^{N} \frac{\left|\left(f^{(j)}\right)_{-}^{(r)}(x)\right|}{j!}\left\{\sum_{i=1}^{\overline{r}} w_i\left[\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right]^j\right\}\right.$$

$$+\sum_{i=1}^{\overline{r}} w_i\omega_1\left(\left(f^{(N)}\right)_{-}^{(r)}, \left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)\cdot$$

$$\frac{\left(\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!},$$

$$\sum_{j=1}^{N} \frac{\left|\left(f^{(j)}\right)_{+}^{(r)}(x)\right|}{j!}\left\{\sum_{i=1}^{\overline{r}} w_i\left[\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right]^j\right\}+ \tag{62}$$

$$\sum_{i=1}^{\overline{r}} w_i\omega_1\left(\left(f^{(N)}\right)_{+}^{(r)}, \left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)\cdot$$

$$\left.\frac{\left(\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!}\right\} \overset{\text{(by (1))}}{\leq}$$

$$\sum_{j=1}^{N} \frac{1}{j!} \left\{ \sum_{i=1}^{\overline{r}} w_i \left[\left(\frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right]^j \right\} \cdot$$

$$\sup_{r \in [0,1]} \max \left\{ \left| \left(f^{(j)} \right)_{-}^{(r)} (x) \right|, \left| \left(f^{(j)} \right)_{+}^{(r)} (x) \right| \right\} + \tag{63}$$

$$\sum_{i=1}^{\overline{r}} w_i \frac{\left(\left(\frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right)^N}{N!}$$

$$\sup_{r \in [0,1]} \max \left\{ \omega_1 \left(\left(f^{(N)} \right)_{-}^{(r)}, \left(\frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right), \right.$$

$$\left. \omega_1 \left(\left(f^{(N)} \right)_{+}^{(r)}, \left(\frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \right\} =$$

(by definition of D and Proposition 7)

$$\sum_{j=1}^{N} \frac{1}{j!} \left\{ \sum_{i=1}^{\overline{r}} w_i \left[\left(\frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right]^j \right\} D \left(f^{(j)} (x), \widetilde{o} \right) +$$

$$\sum_{i=1}^{\overline{r}} w_i \frac{\left(\left(\frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right)^N}{N!} \cdot$$

$$\omega_1^{(F)} \left(f^{(N)}, \left(\frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right), \tag{64}$$

proving the claim. ■

Corollary 45 *(to Theorem 44) Additionally assume that* $D \left(f^{(j)} (x), \widetilde{o} \right) = 0$, $j = 1, ..., N$. *Then*

$$D \left(\left(H_n^{\mathcal{F}} (f) \right) (x), f (x) \right) \leq$$

$$\sum_{i=1}^{\overline{r}} w_i \frac{\left(\left(\frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right)^N}{N!} \cdot$$

$$\omega_1^{(\mathcal{F})} \left(f^{(N)}, \left(\frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right). \tag{65}$$

Inequality (65) implies the pointwise convergence with rates of $\left(H_n^{\mathcal{F}} (f) \right) (x) \xrightarrow{D} f (x)$, *as* $n \to \infty$, *at high speed of* $\frac{1}{n^{(1-\alpha)(N+1)}}$.

Proof. By (59). ■

Corollary 46 *(to Theorem 44) Here we take* $x \in [-T^*, T^*]$, $T^* > 0$; $T > 0$ *and* $n \in \mathbb{N} : n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}}\right)$. *Let* $f \in C^N_{\mathcal{F}}(\mathbb{R})$, $N \in \mathbb{N}$, $f^{(N)}$ *is either fuzzy continuous and bounded or fuzzy uniformly continuous;* $0 < \alpha < 1$. *Then*

$$D^*\left(H_n^{\mathcal{F}}(f), f\right)_{[-T^*,T^*]} \leq$$

$$\sum_{j=1}^{N} \frac{D^*\left(f^{(j)}, \widetilde{o}\right)_{[-T^*,T^*]}}{j!} \left\{\sum_{i=1}^{\overline{r}} w_i \left[\left(\frac{\nu_i T^* + \mu_i}{n + \nu_i}\right) + \left(1 + \frac{\nu_i}{n + \nu_i}\right)\frac{T}{n^{1-\alpha}}\right]^j\right\} +$$

$$\sum_{i=1}^{\overline{r}} w_i \frac{\left(\left(\frac{\nu_i T^* + \mu_i}{n+\nu_i}\right) + \left(1 + \frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!} \cdot \tag{66}$$

$$\omega_1^{(\mathcal{F})}\left(f^{(N)}, \left(\frac{\nu_i T^* + \mu_i}{n + \nu_i}\right) + \left(1 + \frac{\nu_i}{n + \nu_i}\right)\frac{T}{n^{1-\alpha}}\right).$$

Inequality (66) implies the uniform convergence with rates of $H_n^{\mathcal{F}}(f) \overset{D^*}{\to} f$, *as* $n \to \infty$, *at speed* $\frac{1}{n^{1-\alpha}}$.

Proof. By (59). ■

Corollary 47 *(to Theorem 44) Case* $N = 1$. *It holds*

$$D\left(\left(H_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq$$

$$\left[D\left(f'(x), \widetilde{o}\right) + \omega_1^{(\mathcal{F})}\left(f', \left(\frac{\nu_i |x| + \mu_i}{n + \nu_i}\right) + \left(1 + \frac{\nu_i}{n + \nu_i}\right)\frac{T}{n^{1-\alpha}}\right)\right] \cdot$$

$$\left\{\sum_{i=1}^{\overline{r}} w_i \left[\left(\frac{\nu_i |x| + \mu_i}{n + \nu_i}\right) + \left(1 + \frac{\nu_i}{n + \nu_i}\right)\frac{T}{n^{1-\alpha}}\right]\right\}. \tag{67}$$

Inequality (67) implies the pointwise convergence with rates of $\left(H_n^{\mathcal{F}}(f)\right)(x) \overset{D}{\to} f(x)$, *as* $n \to \infty$, *at speed* $\frac{1}{n^{1-\alpha}}$.

Proof. By (59). ■

We continue with

Theorem 48 *Let* $f \in C^N_{\mathcal{F}}(\mathbb{R})$, $N \in \mathbb{N}$, *with* H*-fuzzy* $f^{(N)}$ *either fuzzy continuous and bounded or fuzzy uniformly continuous.*

Here $x \in \mathbb{R}$, $T > 0$, $n \in \mathbb{N} : n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$, $0 < \alpha < 1$, *the rest of parameters as in Section 3. Then*

$$D\left(\left(K_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq$$

$$\sum_{j=1}^{N} \frac{D\left(f^{(j)}(x), \widetilde{o}\right)}{j!} \left\{\sum_{i=1}^{\overline{r}} w_i \left[\left(\frac{\nu_i |x| + \mu_i + 1}{n + \nu_i}\right) + \left(1 + \frac{\nu_i}{n + \nu_i}\right)\frac{T}{n^{1-\alpha}}\right]^j\right\} +$$

$$\sum_{i=1}^{\overline{r}} w_i \frac{\left(\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!} \cdot \tag{68}$$

$$\omega_1^{(\mathcal{F})}\left(f^{(N)}, \left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right).$$

Inequality (68) implies the pointwise convergence with rates on $\left(K_n^{\mathcal{F}}(f)\right)(x) \xrightarrow{D} f(x)$, *as* $n \to \infty$, *at speed* $\frac{1}{n^{1-\alpha}}$.

Proof. Similar to Theorem 44. Here we use (47), Theorem 29, Remark 12 along with (1), the definition of D and Proposition 7. ■

Corollary 49 *(to Theorem 48) Additionally assume that* $D\left(f^{(j)}(x), \widetilde{o}\right) = 0$, $j = 1, ..., N$. *Then*

$$D\left(\left(K_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq$$

$$\sum_{i=1}^{\overline{r}} w_i \frac{\left(\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!} \cdot \tag{69}$$

$$\omega_1^{(\mathcal{F})}\left(f^{(N)}, \left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right).$$

Inequality (69) implies the pointwise convergence with rates of $\left(K_n^{\mathcal{F}}(f)\right)(x) \xrightarrow{D} f(x)$, *as* $n \to \infty$, *at high speed of* $\frac{1}{n^{(1-\alpha)(N+1)}}$.

Proof. By (68). ■

Corollary 50 *(to Theorem 48) Here we take* $x \in [-T^*, T^*]$, $T^* > 0$; $T > 0$ *and* $n \in \mathbb{N} : n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}}\right)$. *Let* $f \in C_{\mathcal{F}}^N(\mathbb{R})$, $N \in \mathbb{N}$, $f^{(N)}$ *is either fuzzy continuous and bounded or fuzzy uniformly continuous;* $0 < \alpha < 1$. *Then*

$$D^*\left(K_n^{\mathcal{F}}(f), f\right)_{[-T^*,T^*]} \leq$$

$$\sum_{j=1}^{N} \frac{D^*\left(f^{(j)}, \widetilde{o}\right)_{[-T^*,T^*]}}{j!}\left\{\sum_{i=1}^{\overline{r}} w_i\left[\left(\frac{\nu_i T^*+\mu_i+1}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right]^j\right\}+$$

$$\sum_{i=1}^{\overline{r}} w_i \frac{\left(\left(\frac{\nu_i T^*+\mu_i+1}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!} \cdot \tag{70}$$

$$\omega_1^{(\mathcal{F})}\left(f^{(N)}, \left(\frac{\nu_i T^*+\mu_i+1}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right).$$

Inequality (70) implies the uniform convergence with rates of $K_n^{\mathcal{F}}(f) \xrightarrow{D^*} f$, *as* $n \to \infty$, *at speed* $\frac{1}{n^{1-\alpha}}$.

Proof. By (68). ■

Corollary 51 *(to Theorem 48) Case $N=1$. It holds*

$$D\left(\left(K_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq$$

$$\left[D\left(f'(x), \widetilde{o}\right) + \omega_1^{(\mathcal{F})}\left(f', \left(\frac{\nu_i |x| + \mu_i + 1}{n + \nu_i}\right) + \left(1 + \frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)\right] \cdot$$

$$\left\{\sum_{i=1}^{\overline{r}} w_i \left[\left(\frac{\nu_i |x| + \mu_i + 1}{n + \nu_i}\right) + \left(1 + \frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right]\right\}. \quad (71)$$

Inequality (71) implies the pointwise convergence with rates of $\left(K_n^{\mathcal{F}}(f)\right)(x) \overset{D}{\to} f(x)$, *as* $n \to \infty$, *at speed* $\frac{1}{n^{1-\alpha}}$.

Proof. By (68). ■

We also have

Theorem 52 *All here as in Theorem 44. Then*

$$D\left(\left(M_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq \sum_{j=1}^{N} \frac{D\left(f^{(j)}(x), \widetilde{o}\right)}{j!}\left[\frac{T}{n^{1-\alpha}} + \frac{1}{n}\right]^j +$$

$$\omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{T}{n^{1-\alpha}} + \frac{1}{n}\right)\frac{\left(\frac{T}{n^{1-\alpha}} + \frac{1}{n}\right)^N}{N!}. \quad (72)$$

Inequality (72) implies the pointwise convergence with rates on $\left(M_n^{\mathcal{F}}(f)\right)(x) \overset{D}{\to} f(x)$, *as* $n \to \infty$, *at the speed* $\frac{1}{n^{1-\alpha}}$.

Proof. As in Theorem 44. We use here (50), Theorem 33, Remark 12, (1), definition of D and Proposition 7. ■

Corollary 53 *(to Theorem 52) Additionally assume that* $D\left(f^{(j)}(x), \widetilde{o}\right) = 0$, $j = 1, ..., N$. *Then*

$$D\left(\left(M_n^{\mathcal{F}}(f)\right)(x), f(x)\right) \leq$$

$$\omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{T}{n^{1-\alpha}} + \frac{1}{n}\right)\frac{\left(\frac{T}{n^{1-\alpha}} + \frac{1}{n}\right)^N}{N!}. \quad (73)$$

Inequality (73) implies the pointwise convergence with rates of $\left(M_n^{\mathcal{F}}(f)\right)(x) \overset{D}{\to} f(x)$, *as* $n \to \infty$, *at high speed of* $\frac{1}{n^{(1-\alpha)(N+1)}}$.

Proof. By (72). ■

Corollary 54 *(to Theorem 52) Here all as in Corollary 46. Then*

$$D^{*}\left(M_{n}^{\mathcal{F}}\left(f\right),f\right)_{\left[-T^{*},T^{*}\right]}\leq$$

$$\sum_{j=1}^{N}\frac{D^{*}\left(f^{(j)},\widetilde{o}\right)_{\left[-T^{*},T^{*}\right]}}{j!}\left[\frac{T}{n^{1-\alpha}}+\frac{1}{n}\right]^{j}+\omega_{1}^{(\mathcal{F})}\left(f^{(N)},\frac{T}{n^{1-\alpha}}+\frac{1}{n}\right)\frac{\left(\frac{T}{n^{1-\alpha}}+\frac{1}{n}\right)^{N}}{N!}. \tag{74}$$

Inequality (74) implies the uniform convergence with rates of $M_{n}^{\mathcal{F}}\left(f\right)\overset{D^{*}}{\rightarrow}f$, *as* $n\rightarrow\infty$, *at speed* $\frac{1}{n^{1-\alpha}}$.

Proof. By (72). ■

Corollary 55 *(to Theorem 52) Case* $N=1$. *It holds*

$$D\left(\left(M_{n}^{\mathcal{F}}\left(f\right)\right)\left(x\right),f\left(x\right)\right)\leq$$

$$\left[D\left(f'\left(x\right),\widetilde{o}\right)+\omega_{1}^{(\mathcal{F})}\left(f',\frac{T}{n^{1-\alpha}}+\frac{1}{n}\right)\right]\left(\frac{T}{n^{1-\alpha}}+\frac{1}{n}\right). \tag{75}$$

Inequality (75) implies the pointwise convergence with rates of $\left(M_{n}^{\mathcal{F}}\left(f\right)\right)\left(x\right)\overset{D}{\rightarrow}f\left(x\right)$, *as* $n\rightarrow\infty$, *at the speed* $\frac{1}{n^{1-\alpha}}$.

Proof. By (72). ■

Remark 56 *All real neural network approximation results listed here were transferred to the fuzzy setting.*

References

[1] G.A. Anastassiou, *Rate of convergence of some neural network operators to the unit-univariate case*, J. Math. Anal. Appli. 212 (1997), 237-262.

[2] G.A. Anastassiou, *Rate of convergence of some multivariate neural network operators to the unit*, J. Comp. Math. with Appl., 40 (2000), 1-19.

[3] G.A. Anastassiou, *Quantitative Approximations*, Chapman&Hall/CRC, Boca Raton, New York, 2001.

[4] G.A. Anastassiou, *Fuzzy Approximation by Fuzzy Convolution type Operators*, Computers and Mathematics, 48(2004), 1369-1386.

[5] G.A. Anastassiou, *Higher order Fuzzy Korovkin Theory via inequalities*, Communications in Applied Analysis, 10(2006), No. 2, 359-392.

[6] G.A. Anastassiou, *Fuzzy Korovkin Theorems and Inequalities*, Journal of Fuzzy Mathematics, 15(2007), No. 1, 169-205.

[7] G.A. Anastassiou, *Fuzzy Mathematics: Approximation Theory*, Springer, Heidelberg, New York, 2010.

[8] G.A. Anastassiou, *Rate of convergence of some neural network operators to the unit-univariate case, revisited,* Vesnik 65, 4 (2013), 511-518.

[9] G.A. Anastassiou, *Rate of convergence of some multivariate neural network operators to the unit, revisited*, J. of Computational Analysis and Applications, Vol. 15, No. 7 (2013), 1300-1309.

[10] G.A. Anastassiou, *Fuzzy fractional approximations by fuzzy normalized bell and squashing type neural network operators*, J. of Fuzzy Mathematics, Vol. 22, No. 1, 2014, pp. 139-156.

[11] G.A. Anastassiou, *Higher Order Multivariate Fuzzy Approximation by basic Neural Network Operators*, Cubo, accepted, 2013.

[12] G.A. Anastassiou, *Approximation by Perturbed Neural Network Operators*, submitted, 2014.

[13] G.A. Anastassiou, *Approximations by Multivariate Perturbed Neural Network Operators*, submitted, 2014.

[14] P. Cardaliaguet, G. Euvrard, *Approximation of a function and its derivative with a neural network*, Neural Networks 5(1992), 207-220.

[15] S. Gal, *Approximation Theory in Fuzzy Setting*, Chapter 13 in Handbook of Analytic-Computational Methods in Applied Mathematics, 617-666, edited by G. Anastassiou, Chapman & Hall/CRC, Boca Raton, New York, 2000.

[16] R. Goetschel Jr., W. Voxman, *Elementary fuzzy calculus*, Fuzzy Sets and Systems, 18(1986), 31-43.

[17] O. Kaleva, *Fuzzy differential equations*, Fuzzy Sets and Systems, 24(1987), 301-317.

[18] D.D. Stancu, *On a generalization of the Bernstein polynomials*, Studia Universitatis Babeş-Bolyai, Series Mathematica-Physica, 14 (1969), 31-45.

[19] Wu Congxin, Gong Zengtai, *On Henstock integrals of interval-valued functions and fuzzy-valued functions*, Fuzzy Sets and Systems, Vol. 115, No. 3, 2000, 377-391.

[20] C. Wu, Z. Gong, *On Henstock integral of fuzzy number valued functions (I)*, Fuzzy Sets and Systems, 120(2001), No. 3, 523-532.

[21] C. Wu, M. Ma, *On embedding problem of fuzzy number spaces: Part 1*, Fuzzy Sets and Systems, 44 (1991), 33-38.