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Voronovskaya type asymptotic expansions for perturbed neural network operators

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here are studied further perturbed normalized neural network operators of Cardaliaguet-Euvrard type. We derive univariate and multivariate Voronovskaya type asymptotic expansions for the error of approximation of these operators to the unit operator.

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1 Introduction

P. Cardaliaguet and G. Euvrard were the first, see [11], to describe precisely and study neural network approximation operators to the unit operator. Namely they proved: be given $f : \mathbb{R} \to \mathbb{R}$ a continuous bounded function and b a centered bell-shaped function, then the functions

$$F_n(x) = \sum_{k=-n^2}^{n^2} \frac{f\left(\frac{k}{n}\right)}{In^{\alpha}} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right),$$

where $I := \int_{-\infty}^{\infty} b(t)\,dt$, $0 < \alpha < 1$, converge uniformly on compacta to f.

You see above that the weights $\frac{f\left(\frac{k}{n}\right)}{In^{\alpha}}$ are explicitly known, for the first time shown in [11].

Furthermore the authors in [11] proved that: let $f : \mathbb{R}^p \to \mathbb{R}$, $p \in \mathbb{N}$, be a continuous bounded function and b a p-dimensional bell-shaped function. Then the functions

$$G_n(x) =$$

$$\sum_{k_1=-n^2}^{n^2} \cdots \sum_{k_p=-n^2}^{n^2} \frac{f\left(\frac{k_1}{n},\ldots,\frac{k_p}{n}\right)}{In^{\alpha p}} b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),\ldots,n^{1-\alpha}\left(x_p-\frac{k_p}{n}\right)\right),$$

where I is the integral of b on $\mathbb{R}^p$ and $0<\alpha<1$, converge uniformly on compacta to f.

Still the work [11] is qualitative and not quantitative.

The author in [1], [2] and [3], see chapters 2-5, was the first to establish neural network approximations to continuous functions with rates, that is quantitative works, by very specifically defined neural network operators of Cardaliagnet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative or partial derivatives, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" function are assumed to be of compact support. Also in [3] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see chapters 4-5 there.

Though the work in [1], [2], [3], was quantitative, the rate of convergence was not precisely determined.

Finally the author in [6], [8], by normalizing his operators he achieved to determine the exact rates of convergence.

Recently the author in [9], [10] studied the convergence of perturbed cases of the above neural network operators. These perturbations actually occur in what we perceive in our computations from neural network operations. We continue here this last study by giving Voronovskaya type asymptotic expansions for the pointwise approximation of these perturbed operators to the unit operator, see also the related [5], [7].

For more about neural networks in general we refer to [12], [13], [14], [15], [16], [17].

The article is presented in two parts, the univariate and multivariate.

Part I
Univariate Theory

2 Univariate Basics

Here the univariate activation function $b:\mathbb{R}\to\mathbb{R}_+$ is of compact support $[-T,T]$, $T>0$. That is $b(x)>0$ for any $x\in[-T,T]$, and clearly b may have jump discontinuities. Also the shape of the graph of b could be anything. Typically in neural networks approximation we take b as a sigmoidal function or bell-shaped function, of course here of compact support $[-T,T]$, $T>0$.

Example 1 *(i) b can be the characteristic function on* $[-1,1]$,

(ii) b can be that hat function over $[-1,1]$, *i.e.,*

$$b(x)=\begin{cases}1+x, & -1\leq x\leq 0,\\ 1-x, & 0<x\leq 1,\\ 0, & \textit{elsewhere},\end{cases}$$

(iii) the truncated sigmoidals

$$b(x)=\begin{cases}\frac{1}{1+e^{-x}} \textit{ or } \tanh x \textit{ or } \operatorname{erf}(x), & \textit{for } x\in[-T,T], \textit{ with large } T>0,\\ 0, & x\in\mathbb{R}-[-T,T],\end{cases}$$

(iv) the truncated Gompertz function

$$b(x)=\begin{cases}e^{-\alpha e^{-\beta x}}, & x\in[-T,T];\ \alpha,\beta>0;\ \textit{large } T>0,\\ 0, & x\in\mathbb{R}-[-T,T],\end{cases}$$

So the general function b we will be using here covers all kinds of activation functions in univariate neural network approximations.

Typically we consider functions $f:\mathbb{R}\to\mathbb{R}$ that are either continuous and bounded, or uniformly continuous.

Let here the parameters $\mu,\nu\geq 0$; $\mu_i,\nu_i\geq 0$, $i=1,...,r\in\mathbb{N}$; $w_i\geq 0$: $\sum_{i=1}^{r} w_i=1$; $0<\alpha<1$, $x\in\mathbb{R}$, $n\in\mathbb{N}$.

In this first part we study the asymptotic expansions of Voronovskaya type of the following one hidden layer univariate normalized neural network perturbed operators,

(i) the Stancu type (see [18])

$$\left(H_n(f)\right)(x)=\frac{\sum_{k=-n^2}^{n^2}\left(\sum_{i=1}^{r} w_i f\left(\frac{k+\mu_i}{n+\nu_i}\right)\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}, \tag{1}$$

(ii) the Kantorovich type

$$\left(K_n(f)\right)(x)=\frac{\sum_{k=-n^2}^{n^2}\left(\sum_{i=1}^{r} w_i\left(n+\nu_i\right)\int_{\frac{k+\mu_i}{n+\nu_i}}^{\frac{k+\mu_i+1}{n+\nu_i}} f(t)\,dt\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}, \tag{2}$$

and

(iii) the quadrature type

$$\left(M_n(f)\right)(x)=\frac{\sum_{k=-n^2}^{n^2}\left(\sum_{i=1}^{r} w_i f\left(\frac{k}{n}+\frac{i}{nr}\right)\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}. \tag{3}$$

Similar operators defined for bell-shaped functions and sample coefficients $f\left(\frac{k}{n}\right)$ were studied initially in [11], [1], [2], [3], [6], [8], [5], [7].

Operator K_n in the corresponding Signal Processing context, represents the natural called "time-jitter" error, where the sample information is calculated in a perturbed neighborhood of $\frac{k+\mu}{n+\nu}$ rather than exactly at the node $\frac{k}{n}$.

The perturbed sample coefficients $f\left(\frac{k+\mu}{n+\nu}\right)$ with $0 \leq \mu \leq \nu$, were first used by D. Stancu [18], in a totally different context, generalizing Bernstein operators approximation on $C([0,1])$. For related approximation properties of these perturbed operators see [9], [10].

The terms in the ratio of sums (1), (2), (3) are nonzero, iff

$$\left|n^{1-\alpha}\left(x-\frac{k}{n}\right)\right| \leq T, \text{ i.e. } \left|x-\frac{k}{n}\right| \leq \frac{T}{n^{1-\alpha}} \tag{4}$$

iff

$$nx - Tn^{\alpha} \leq k \leq nx + Tn^{\alpha}. \tag{5}$$

In order to have the desired order of the numbers

$$-n^2 \leq nx - Tn^{\alpha} \leq nx + Tn^{\alpha} \leq n^2, \tag{6}$$

it is sufficiently enough to assume that

$$n \geq T + |x|. \tag{7}$$

When $x \in [-T, T]$ it is enough to assume $n \geq 2T$, which implies (6).

Proposition 2 *([1]) Let $a \leq b$, $a, b \in \mathbb{R}$. Let $card(k)$ (≥ 0) be the maximum number of integers contained in $[a, b]$. Then*

$$\max(0, (b-a)-1) \leq card(k) \leq (b-a)+1. \tag{8}$$

Note 3 *We would like to establish a lower bound on $card(k)$ over the interval $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$. From Proposition 2 we get that*

$$card(k) \geq \max(2Tn^{\alpha} - 1, 0). \tag{9}$$

We obtain $card(k) \geq 1$, if

$$2Tn^{\alpha} - 1 \geq 1 \quad \textit{iff } n \geq T^{-\frac{1}{\alpha}}. \tag{10}$$

So to have the desired order (6) and $card(k) \geq 1$ over $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$, we need to consider

$$n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right). \tag{11}$$

Also notice that $card(k) \to +\infty$, as $n \to +\infty$.

Denote by $[\cdot]$ the integral part of a number and by $\lceil\cdot\rceil$ its ceiling.
So under assumption (11), the operators H_n, K_n, M_n, collapse to
(i)

$$\left(H_n\left(f\right)\right)\left(x\right)=\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{r}w_i f\left(\frac{k+\mu_i}{n+\nu_i}\right)\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}, \tag{12}$$

(ii)

$$\left(K_n\left(f\right)\right)\left(x\right)=\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{r}w_i\left(n+\nu_i\right)\int_{\frac{k+\mu_i}{n+\nu_i}}^{\frac{k+\mu_i+1}{n+\nu_i}}f\left(t\right)dt\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}, \tag{13}$$

and
(iii)

$$\left(M_n\left(f\right)\right)\left(x\right)=\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{r}w_i f\left(\frac{k}{n}+\frac{i}{nr}\right)\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}. \tag{14}$$

We make

Remark 4 *Let k as in (5). We observe that*

$$\left|\frac{k+\mu}{n+\nu}-x\right|\leq\left|\frac{k}{n+\nu}-x\right|+\frac{\mu}{n+\nu}. \tag{15}$$

Next we see

$$\left|\frac{k}{n+\nu}-x\right|\leq\left|\frac{k}{n+\nu}-\frac{k}{n}\right|+\left|\frac{k}{n}-x\right|\overset{(4)}{\leq}\frac{\nu\left|k\right|}{n\left(n+\nu\right)}+\frac{T}{n^{1-\alpha}}$$

(by $\left|k\right|\leq\max\left(\left|nx-Tn^{\alpha}\right|,\left|nx+Tn^{\alpha}\right|\right)\leq n\left|x\right|+Tn^{\alpha}$)

$$\leq\left(\frac{\nu}{n+\nu}\right)\left(\left|x\right|+\frac{T}{n^{1-\alpha}}\right)+\frac{T}{n^{1-\alpha}}. \tag{16}$$

Consequently it holds the useful inequality

$$\left|\frac{k+\mu}{n+\nu}-x\right|\leq\left(\frac{\nu}{n+\nu}\right)\left(\left|x\right|+\frac{T}{n^{1-\alpha}}\right)+\frac{T}{n^{1-\alpha}}+\frac{\mu}{n+\nu}$$

$$=\left(\frac{\nu\left|x\right|+\mu}{n+\nu}\right)+\left(1+\frac{\nu}{n+\nu}\right)\frac{T}{n^{1-\alpha}}, \tag{17}$$

where $\mu,\nu\geq0$, $0<\alpha<1$.

Also, by change of variable method, the operator K_n *could be written conveniently as follows:*

(ii)'

$$\left(K_n\left(f\right)\right)(x) =$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{r} w_i\left(n+\nu_i\right)\int_0^{\frac{1}{n+\nu_i}} f\left(t+\frac{k+\mu_i}{n+\nu_i}\right)dt\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}. \quad (18)$$

Let $N \in \mathbb{N}$, *we denote by* $AC^N(\mathbb{R})$ *the space of functions* f, *such that* $f^{(N-1)}$ *is absolutely continuous function on compacta.*

3 Univariate Results

We give our first univariate main result

Theorem 5 *Let* $f \in AC^N(\mathbb{R})$, $N \in \mathbb{N}$, *with* $\left\|f^{(N)}\right\|_{\infty} := \left\|f^{(N)}\right\|_{\infty,\mathbb{R}} < \infty$, *also* $x \in \mathbb{R}$. *Here* $n \geq \max\left(T+|x|, T^{-\frac{1}{\alpha}}\right)$, $0<\alpha<1$, $n \in \mathbb{N}$, $T>0$. *Then*

$$\left(H_n\left(f\right)\right)(x) - f(x) = \sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}H_n\left((\cdot-x)^j, x\right) + o\left(\frac{1}{n^{(N-\varepsilon)(1-\alpha)}}\right), \quad (19)$$

where $0<\varepsilon\leq N$.

If $N=1$, *the sum in (19) collapses.*

The last (19) implies that

$$n^{(N-\varepsilon)(1-\alpha)}\left[\left(H_n\left(f\right)\right)(x) - f(x) - \sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}H_n\left((\cdot-x)^j, x\right)\right] \to 0, \quad (20)$$

as $n\to\infty$, $0<\varepsilon\leq N$.

When $N=1$, *or* $f^{(j)}(x)=0$, *for all* $j=1,...,N-1$, *then we derive*

$$n^{(N-\varepsilon)(1-\alpha)}\left[\left(H_n\left(f\right)\right)(x) - f(x)\right] \to 0, \quad (21)$$

as $n\to\infty$, $0<\varepsilon\leq N$.

Proof. Let k as in (5). We observe that

$$w_i f\left(\frac{k+\mu_i}{n+\nu_i}\right) = \sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}w_i\left(\frac{k+\mu_i}{n+\nu_i}-x\right)^j + \quad (22)$$

$$w_i\int_x^{\frac{k+\mu_i}{n+\nu_i}} f^{(N)}(t)\frac{\left(\frac{k+\mu_i}{n+\nu_i}-t\right)^{N-1}}{(N-1)!}dt, \quad i=1,...,r.$$

Call

$$V(x) = \sum_{k=\lceil nx - Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right). \tag{23}$$

Hence

$$\frac{\left(\sum_{i=1}^{r} w_i f\left(\frac{k+\mu_i}{n+\nu_i}\right)\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} =$$

$$\sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!}\left(\sum_{i=1}^{r} w_i \left(\frac{k+\mu_i}{n+\nu_i} - x\right)^{j}\right) \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} + \tag{24}$$

$$\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \left(\sum_{i=1}^{r} w_i \int_{x}^{\frac{k+\mu_i}{n+\nu_i}} f^{(N)}(t) \frac{\left(\frac{k+\mu_i}{n+\nu_i} - t\right)^{N-1}}{(N-1)!} dt\right).$$

Therefore it holds (see (12))

$$(H_n(f))(x) - f(x) - \tag{25}$$

$$\sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!}\left(\sum_{k=\lceil nx - Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left\{\left(\sum_{i=1}^{r} w_i \left(\frac{k+\mu_i}{n+\nu_i} - x\right)^{j}\right) \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\right\}\right)$$

$$= R(x),$$

where

$$R(x) = \frac{\sum_{k=\lceil nx - Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \cdot$$

$$\left(\sum_{i=1}^{r} w_i \int_{x}^{\frac{k+\mu_i}{n+\nu_i}} f^{(N)}(t) \frac{\left(\frac{k+\mu_i}{n+\nu_i} - t\right)^{N-1}}{(N-1)!} dt\right). \tag{26}$$

Next we upper bound $R(x)$.

We notice the following:

1) Case of $\frac{k+\mu_i}{n+\nu_i} \geq x$. Then

$$\left|\int_{x}^{\frac{k+\mu_i}{n+\nu_i}} f^{(N)}(t) \frac{\left(\frac{k+\mu_i}{n+\nu_i} - t\right)^{N-1}}{(N-1)!} dt\right| \leq \int_{x}^{\frac{k+\mu_i}{n+\nu_i}} \left|f^{(N)}(t)\right| \frac{\left(\frac{k+\mu_i}{n+\nu_i} - t\right)^{N-1}}{(N-1)!} dt \leq \tag{27}$$

$$\left\|f^{(N)}\right\|_{\infty} \frac{\left(\frac{k+\mu_i}{n+\nu_i} - x\right)^{N}}{N!} \overset{(17)}{\leq}$$

$$\left(\frac{\left\|f^{(N)}\right\|_{\infty}}{N!}\right)\left(\left(\frac{\nu_i |x| + \mu_i}{n+\nu_i}\right) + \left(1 + \frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^{N}. \tag{28}$$

2) Case of $\frac{k+\mu_i}{n+\nu_i} \leq x$. Then

$$\left| \int_x^{\frac{k+\mu_i}{n+\nu_i}} f^{(N)}(t) \frac{\left(\frac{k+\mu_i}{n+\nu_i} - t\right)^{N-1}}{(N-1)!} dt \right| = \left| \int_{\frac{k+\mu_i}{n+\nu_i}}^{x} f^{(N)}(t) \frac{\left(t - \frac{k+\mu_i}{n+\nu_i}\right)^{N-1}}{(N-1)!} dt \right| \tag{29}$$

$$\leq \int_{\frac{k+\mu_i}{n+\nu_i}}^{x} \left| f^{(N)}(t) \right| \frac{\left(t - \frac{k+\mu_i}{n+\nu_i}\right)^{N-1}}{(N-1)!} dt \leq \left\| f^{(N)} \right\|_\infty \frac{\left(x - \frac{k+\mu_i}{n+\nu_i}\right)^{N}}{N!} \overset{(17)}{\leq}$$
$$\left(\frac{\left\| f^{(N)} \right\|_\infty}{N!} \right) \left(\left(\frac{\nu_i |x| + \mu_i}{n+\nu_i} \right) + \left(1 + \frac{\nu_i}{n+\nu_i} \right) \frac{T}{n^{1-\alpha}} \right)^N . \tag{30}$$

So in either case we get

$$\left| \int_x^{\frac{k+\mu_i}{n+\nu_i}} f^{(N)}(t) \frac{\left(\frac{k+\mu_i}{n+\nu_i} - t\right)^{N-1}}{(N-1)!} dt \right| \leq$$
$$\left(\frac{\left\| f^{(N)} \right\|_\infty}{N!} \right) \left(\left(\frac{\nu_i |x| + \mu_i}{n+\nu_i} \right) + \left(1 + \frac{\nu_i}{n+\nu_i} \right) \frac{T}{n^{1-\alpha}} \right)^N . \tag{31}$$

Consequently we obtain

$$|R(x)| \leq \left(\frac{\left\| f^{(N)} \right\|_\infty}{N!} \right) \sum_{i=1}^{r} w_i \left(\left(\frac{\nu_i |x| + \mu_i}{n+\nu_i} \right) + \left(1 + \frac{\nu_i}{n+\nu_i} \right) \frac{T}{n^{1-\alpha}} \right)^N \leq \tag{32}$$
$$\left(\frac{\left\| f^{(N)} \right\|_\infty}{N!} \right) \max_{i \in \{1,\dots,r\}} \left[\left(\left(\frac{\nu_i |x| + \mu_i}{n+\nu_i} \right) + \left(1 + \frac{\nu_i}{n+\nu_i} \right) \frac{T}{n^{1-\alpha}} \right)^N \right] \leq$$
$$\left(\frac{\left\| f^{(N)} \right\|_\infty}{N!} \right) \max_{i \in \{1,\dots,r\}} \left\{ \left(\left(\frac{\nu_i |x| + \mu_i}{n^{1-\alpha} n^{\alpha}} \right) + \left(1 + \frac{\nu_i}{n} \right) \frac{T}{n^{1-\alpha}} \right)^N \right\} =$$
$$\left(\frac{\left\| f^{(N)} \right\|_\infty}{N!} \right) \max_{i \in \{1,\dots,r\}} \left\{ \left(\left(\frac{\nu_i |x| + \mu_i}{n^{\alpha}} \right) + \left(1 + \frac{\nu_i}{n} \right) T \right)^N \right\} \frac{1}{n^{(1-\alpha)N}} \leq \tag{33}$$
$$\left(\frac{\left\| f^{(N)} \right\|_\infty}{N!} \right) \max_{i \in \{1,\dots,r\}} \left\{ \left((\nu_i |x| + \mu_i) + (1 + \nu_i) T \right)^N \right\} \frac{1}{n^{(1-\alpha)N}} .$$

We have proved that

$$|R(x)| \leq$$
$$\left(\frac{\left\| f^{(N)} \right\|_\infty}{N!} \right) \max_{i \in \{1,\dots,r\}} \left\{ \left((\nu_i |x| + \mu_i) + (1 + \nu_i) T \right)^N \right\} \frac{1}{n^{(1-\alpha)N}} =: \frac{A}{n^{(1-\alpha)N}} . \tag{34}$$

That is we proved

$$|R(x)| = O\left(\frac{1}{n^{(1-\alpha)N}}\right), \tag{35}$$

and

$$|R(x)| = o(1). \tag{36}$$

And, letting $0 < \varepsilon \leq N$, we derive

$$\frac{|R(x)|}{\left(\frac{1}{n^{(N-\varepsilon)(1-\alpha)}}\right)} \leq \frac{A}{n^{(1-\alpha)N}} n^{(N-\varepsilon)(1-\alpha)} = \frac{A}{n^{\varepsilon(1-\alpha)}} \to 0, \tag{37}$$

as $n \to \infty$.

That is

$$|R(x)| = o\left(\frac{1}{n^{(N-\varepsilon)(1-\alpha)}}\right). \tag{38}$$

Clearly here we can rewrite (25), as

$$(H_n(f))(x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} H_n\left((\cdot - x)^j, x\right) = R(x). \tag{39}$$

Based on (38) and (39) we derive (19). ■

We continue with

Theorem 6 *Let $f \in AC^N(\mathbb{R})$, $N \in \mathbb{N}$, with $\left\|f^{(N)}\right\|_\infty := \left\|f^{(N)}\right\|_{\infty,\mathbb{R}} < \infty$, also $x \in \mathbb{R}$. Here $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$, $0 < \alpha < 1$, $n \in \mathbb{N}$, $T > 0$. Then*

$$(K_n(f))(x) - f(x) = \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} K_n\left((\cdot - x)^j, x\right) + o\left(\frac{1}{n^{(N-\varepsilon)(1-\alpha)}}\right), \tag{40}$$

where $0 < \varepsilon \leq N$.

If $N = 1$, the sum in (40) collapses.

The last (40) implies that

$$n^{(N-\varepsilon)(1-\alpha)} \left[(K_n(f))(x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} K_n\left((\cdot - x)^j, x\right)\right] \to 0, \tag{41}$$

as $n \to \infty$, $0 < \varepsilon \leq N$.

When $N = 1$, or $f^{(j)}(x) = 0$, for all $j = 1, ..., N-1$, then we derive

$$n^{(N-\varepsilon)(1-\alpha)} \left[(K_n(f))(x) - f(x)\right] \to 0, \tag{42}$$

as $n \to \infty$, $0 < \varepsilon \leq N$.

Proof. Let k as in (5). We observe that

$$\int_0^{\frac{1}{n+\nu_i}} f\left(t+\frac{k+\mu_i}{n+\nu_i}\right)dt=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\int_0^{\frac{1}{n+\nu_i}}\left(t+\frac{k+\mu_i}{n+\nu_i}-x\right)^j dt+$$

$$\int_0^{\frac{1}{n+\nu_i}}\left(\int_x^{t+\frac{k+\mu_i}{n+\nu_i}} f^{(N)}(z)\frac{\left(t+\frac{k+\mu_i}{n+\nu_i}-z\right)^{N-1}}{(N-1)!}dz\right)dt, \tag{43}$$

$i=1,...,r.$

Hence it holds

$$\left(\sum_{i=1}^{r} w_i(n+\nu_i)\int_0^{\frac{1}{n+\nu_i}} f\left(t+\frac{k+\mu_i}{n+\nu_i}\right)dt\right)=$$

$$\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\sum_{i=1}^{r} w_i(n+\nu_i)\int_0^{\frac{1}{n+\nu_i}}\left(t+\frac{k+\mu_i}{n+\nu_i}-x\right)^j dt\right)+ \tag{44}$$

$$\left(\sum_{i=1}^{r} w_i(n+\nu_i)\int_0^{\frac{1}{n+\nu_i}}\left(\int_x^{t+\frac{k+\mu_i}{n+\nu_i}} f^{(N)}(z)\frac{\left(t+\frac{k+\mu_i}{n+\nu_i}-z\right)^{N-1}}{(N-1)!}dz\right)dt\right).$$

Call

$$V(x)=\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right). \tag{45}$$

Therefore we obtain

$$(K_n(f))(x)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\cdot \tag{46}$$

$$\left(\sum_{i=1}^{r} w_i(n+\nu_i)\int_0^{\frac{1}{n+\nu_i}}\left(t+\frac{k+\mu_i}{n+\nu_i}-x\right)^j dt\right)+R(x),$$

where

$$R(x)=\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\cdot$$

$$\left(\sum_{i=1}^{r} w_i(n+\nu_i)\int_0^{\frac{1}{n+\nu_i}}\left(\int_x^{t+\frac{k+\mu_i}{n+\nu_i}} f^{(N)}(z)\frac{\left(t+\frac{k+\mu_i}{n+\nu_i}-z\right)^{N-1}}{(N-1)!}dz\right)dt\right). \tag{47}$$

So far we have

$$\left(K_n(f)\right)(x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \cdot \tag{48}$$

$$\left(\sum_{i=1}^{r} w_i (n+\nu_i) \int_0^{\frac{1}{n+\nu_i}} \left(t+\frac{k+\mu_i}{n+\nu_i}-x\right)^j dt\right) = R(x).$$

Next we upper bound $R(x)$.

As in the proof of Theorem 5 we find that

$$\left|\int_x^{t+\frac{k+\mu_i}{n+\nu_i}} f^{(N)}(z) \frac{\left(t+\frac{k+\mu_i}{n+\nu_i}-z\right)^{N-1}}{(N-1)!} dz\right| \leq$$

$$\left\|f^{(N)}\right\|_{\infty} \frac{\left|t+\frac{k+\mu_i}{n+\nu_i}-x\right|^N}{N!} \leq \left\|f^{(N)}\right\|_{\infty} \frac{\left(|t|+\left|\frac{k+\mu_i}{n+\nu_i}-x\right|\right)^N}{N!}. \tag{49}$$

Therefore we get

$$(n+\nu_i)\int_0^{\frac{1}{n+\nu_i}} \left|\int_x^{t+\frac{k+\mu_i}{n+\nu_i}} f^{(N)}(z) \frac{\left(t+\frac{k+\mu_i}{n+\nu_i}-z\right)^{N-1}}{(N-1)!} dz\right| dt \leq$$

$$\left(\frac{\left\|f^{(N)}\right\|_{\infty}}{N!}\right)(n+\nu_i)\int_0^{\frac{1}{n+\nu_i}} \left(|t|+\left|\frac{k+\mu_i}{n+\nu_i}-x\right|\right)^N dt \leq \tag{50}$$

$$\left(\frac{\left\|f^{(N)}\right\|_{\infty}}{N!}\right)\left(\frac{1}{n+\nu_i}+\left|\frac{k+\mu_i}{n+\nu_i}-x\right|\right)^N \overset{(17)}{\leq}$$

$$\left(\frac{\left\|f^{(N)}\right\|_{\infty}}{N!}\right)\left(\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N. \tag{51}$$

Consequently we infer that

$$|R(x)| \leq \left(\frac{\left\|f^{(N)}\right\|_{\infty}}{N!}\right)\sum_{i=1}^{r} w_i \left[\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right]^N \leq \tag{52}$$

$$\left(\frac{\left\|f^{(N)}\right\|_{\infty}}{N!}\right)\max_{i\in\{1,\dots,r\}}\left[\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right]^N \leq$$

$$\left(\frac{\left\|f^{(N)}\right\|_{\infty}}{N!}\right)\max_{i\in\{1,\dots,r\}}\left[\left(\frac{\nu_i|x|+\mu_i+1}{n^{1-\alpha}n^{\alpha}}\right)+\left(1+\frac{\nu_i}{n}\right)\frac{T}{n^{1-\alpha}}\right]^N =$$

$$\left(\frac{\left\|f^{(N)}\right\|_{\infty}}{N!}\right)\max_{i\in\{1,\ldots,r\}}\left[\left(\frac{\nu_i\left|x\right|+\mu_i+1}{n^{\alpha}}\right)+\left(1+\frac{\nu_i}{n}\right)T\right]^N\frac{1}{n^{(1-\alpha)N}}\leq \tag{53}$$

$$\left(\frac{\left\|f^{(N)}\right\|_{\infty}}{N!}\right)\max_{i\in\{1,\ldots,r\}}\left[\left(\nu_i\left|x\right|+\mu_i+1\right)+\left(1+\nu_i\right)T\right]^N\frac{1}{n^{(1-\alpha)N}}.$$

We have proved that

$$\left|R\left(x\right)\right|\leq$$

$$\left(\frac{\left\|f^{(N)}\right\|_{\infty}}{N!}\right)\max_{i\in\{1,\ldots,r\}}\left[\left(\nu_i\left|x\right|+\mu_i+1\right)+\left(1+\nu_i\right)T\right]^N\frac{1}{n^{(1-\alpha)N}}=:\frac{B}{n^{(1-\alpha)N}}. \tag{54}$$

That is we proved

$$\left|R\left(x\right)\right|=O\left(\frac{1}{n^{(1-\alpha)N}}\right), \tag{55}$$

and

$$\left|R\left(x\right)\right|=o\left(1\right). \tag{56}$$

And, letting $0<\varepsilon\leq N$, we derive

$$\frac{\left|R\left(x\right)\right|}{\left(\frac{1}{n^{(N-\varepsilon)(1-\alpha)}}\right)}\leq\frac{B}{n^{(1-\alpha)N}}n^{(N-\varepsilon)(1-\alpha)}=\frac{B}{n^{\varepsilon(1-\alpha)}}\to 0, \tag{57}$$

as $n\to\infty$.

That is

$$\left|R\left(x\right)\right|=o\left(\frac{1}{n^{(N-\varepsilon)(1-\alpha)}}\right). \tag{58}$$

Clearly here we can rewrite (48), as

$$\left(K_n\left(f\right)\right)\left(x\right)-f\left(x\right)-\sum_{j=1}^{N-1}\frac{f^{(j)}\left(x\right)}{j!}K_n\left(\left(\cdot-x\right)^j,x\right)=R\left(x\right). \tag{59}$$

Based on (58) and (59) we derive (40). ■

We also give

Theorem 7 *Let $f\in AC^N\left(\mathbb{R}\right)$, $N\in\mathbb{N}$, with $\left\|f^{(N)}\right\|_{\infty}:=\left\|f^{(N)}\right\|_{\infty,\mathbb{R}}<\infty$, also $x\in\mathbb{R}$. Here $n\geq\max\left(T+\left|x\right|,T^{-\frac{1}{\alpha}}\right)$, $0<\alpha<1$, $n\in\mathbb{N}$, $T>0$. Then*

$$\left(M_n\left(f\right)\right)\left(x\right)-f\left(x\right)=\sum_{j=1}^{N-1}\frac{f^{(j)}\left(x\right)}{j!}M_n\left(\left(\cdot-x\right)^j,x\right)+o\left(\frac{1}{n^{(N-\varepsilon)(1-\alpha)}}\right), \tag{60}$$

where $0<\varepsilon\leq N$.

If $N=1$, the sum in (60) collapses.

The last (60) implies that

$$n^{(N-\varepsilon)(1-\alpha)}\left[\left(M_n(f)\right)(x)-f(x)-\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}M_n\left((\cdot-x)^j,x\right)\right]\to 0, \quad (61)$$

as $n\to\infty$, $0<\varepsilon\leq N$.

When $N=1$, *or* $f^{(j)}(x)=0$, *for all* $j=1,...,N-1$, *then we derive*

$$n^{(N-\varepsilon)(1-\alpha)}\left[\left(M_n(f)\right)(x)-f(x)\right]\to 0, \quad (62)$$

as $n\to\infty$, $0<\varepsilon\leq N$.

Proof. Let k as in (5). Again by Taylor's formula we have that

$$\sum_{i=1}^{r}w_i f\left(\frac{k}{n}+\frac{i}{nr}\right)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{i=1}^{r}w_i\left(\frac{k}{n}+\frac{i}{nr}-x\right)^j+ \quad (63)$$

$$\sum_{i=1}^{r}w_i\int_x^{\frac{k}{n}+\frac{i}{nr}}f^{(N)}(t)\frac{\left(\frac{k}{n}+\frac{i}{nr}-t\right)^{N-1}}{(N-1)!}dt.$$

Call

$$V(x)=\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right). \quad (64)$$

Then

$$\left(M_n(f)\right)(x)=\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{r}w_i f\left(\frac{k}{n}+\frac{i}{nr}\right)\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}$$

$$=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\cdot$$

$$\sum_{i=1}^{r}w_i\left(\frac{k}{n}+\frac{i}{nr}-x\right)^j+\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\cdot \quad (65)$$

$$\left(\sum_{i=1}^{r}w_i\int_x^{\frac{k}{n}+\frac{i}{nr}}f^{(N)}(t)\frac{\left(\frac{k}{n}+\frac{i}{nr}-t\right)^{N-1}}{(N-1)!}dt\right).$$

Therefore we get

$$\left(M_n(f)\right)(x)-f(x)-\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\cdot$$

$$\left(\sum_{i=1}^{r} w_i \left(\frac{k}{n} + \frac{i}{nr} - x\right)^j\right) = R(x), \tag{66}$$

where

$$R(x) = \frac{\sum_{k=\lceil nx - Tn^{\alpha}\rceil}^{\lceil nx + Tn^{\alpha}\rceil} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)}.$$

$$\left(\sum_{i=1}^{r} w_i \int_x^{\frac{k}{n}+\frac{i}{nr}} f^{(N)}(t) \frac{\left(\frac{k}{n} + \frac{i}{nr} - t\right)^{N-1}}{(N-1)!} dt\right). \tag{67}$$

As in the proof of Theorem 5 we obtain

$$\left|\int_x^{\frac{k}{n}+\frac{i}{nr}} f^{(N)}(t) \frac{\left(\frac{k}{n} + \frac{i}{nr} - t\right)^{N-1}}{(N-1)!} dt\right| \leq$$

$$\left(\frac{\left\|f^{(N)}\right\|_{\infty}}{N!}\right) \left|\frac{k}{n} + \frac{i}{nr} - x\right|^N \leq \left(\frac{\left\|f^{(N)}\right\|_{\infty}}{N!}\right) \left(\frac{1}{n} + \left|\frac{k}{n} - x\right|\right)^N \leq \tag{68}$$

$$\left(\frac{\left\|f^{(N)}\right\|_{\infty}}{N!}\right) \left(\frac{1}{n} + \frac{T}{n^{1-\alpha}}\right)^N. \tag{69}$$

Clearly then it holds

$$|R(x)| \leq \left(\frac{\left\|f^{(N)}\right\|_{\infty}}{N!}\right) \left(\frac{1}{n} + \frac{T}{n^{1-\alpha}}\right)^N =$$

$$\left(\frac{\left\|f^{(N)}\right\|_{\infty}}{N!}\right) \left(\frac{1}{n^{1-\alpha} n^{\alpha}} + \frac{T}{n^{1-\alpha}}\right)^N = \left(\frac{\left\|f^{(N)}\right\|_{\infty}}{N!}\right) \left(\frac{1}{n^{\alpha}} + T\right)^N \frac{1}{n^{(1-\alpha)N}} \leq \tag{70}$$

$$\left(\frac{\left\|f^{(N)}\right\|_{\infty}}{N!}\right) (1+T)^N \frac{1}{n^{(1-\alpha)N}}.$$

That is

$$|R(x)| \leq \left[\left(\frac{\left\|f^{(N)}\right\|_{\infty}}{N!}\right) (1+T)^N\right] \frac{1}{n^{(1-\alpha)N}} \tag{71}$$

$$=: \frac{C}{n^{(1-\alpha)N}}. \tag{72}$$

That is we proved

$$|R(x)| = O\left(\frac{1}{n^{(1-\alpha)N}}\right), \tag{73}$$

and

$$|R(x)| = o(1). \tag{74}$$

And, letting $0 < \varepsilon \leq N$, we derive

$$\frac{|R(x)|}{\left(\frac{1}{n^{(N-\varepsilon)(1-\alpha)}}\right)} \leq \frac{C}{n^{(1-\alpha)N}} n^{(N-\varepsilon)(1-\alpha)} = \frac{C}{n^{\varepsilon(1-\alpha)}} \to 0, \quad (75)$$

as $n \to \infty$.

That is

$$|R(x)| = o\left(\frac{1}{n^{(N-\varepsilon)(1-\alpha)}}\right). \quad (76)$$

Clearly here we can rewrite (66), as

$$(M_n(f))(x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} M_n\left((\cdot - x)^j, x\right) = R(x). \quad (77)$$

Based on (76) and (77) we derive (60). ■

Part II

Multivariate Theory

4 Multivariate Basics

Here the activation function $b : \mathbb{R}^d \to \mathbb{R}_+$, $d \in \mathbb{N}$, is of compact support $B := \prod_{j=1}^{d} [-T_j, T_j]$, $T_j > 0$, $j = 1, ..., d$. That is $b(x) > 0$ for any $x \in B$, and clearly b may have jump discontinuities. Also the shape of the graph of b is immaterial.

Typically in neural networks approximation we take b to be a d-dimensional bell-shaped function (i.e. per coordinate is a centered bell-shaped function), or a product of univariate centered bell-shaped functions, or a product of sigmoid functions, in our case all them of compact support B.

Example 8 *Take $b(x) = \beta(x_1)\beta(x_2)...\beta(x_d)$, where β is any of the following functions, $j = 1, ..., d$:*

(i) $\beta(x_j)$ is the characteristic function on $[-1, 1]$,

(ii) $\beta(x_j)$ is the hat function over $[-1, 1]$, that is,

$$\beta(x_j) = \begin{cases} 1 + x_j, & -1 \leq x_j \leq 0, \\ 1 - x_j, & 0 < x_j \leq 1, \\ 0, & \text{elsewhere,} \end{cases}$$

(iii) the truncated sigmoids

$$\beta(x_j) = \begin{cases} \frac{1}{1+e^{-x_j}} \text{ or } \tanh x_j \text{ or } \operatorname{erf}(x_j), & \text{for } x_j \in [-T_j, T_j], \text{ with large } T_j > 0, \\ 0, \quad x_j \in \mathbb{R} - [-T_j, T_j], \end{cases}$$

(*iv*) *the truncated Gompertz function*

$$\beta(x_j) = \begin{cases} e^{-\alpha e^{-\beta x_j}}, & x_j \in [-T_j, T_j]; \ \alpha, \beta > 0; \ \textit{large } T_j > 0, \\ 0, & x_j \in \mathbb{R} - [-T_j, T_j], \end{cases}$$

The Gompertz functions are also sigmoid functions, with wide applications to many applied fields, e.g. demography and tumor growth modeling, etc.

Thus the general activation function b we will be using here includes all kinds of activation functions in neural network approximations.

Typically we consider functions $f : \mathbb{R}^d \to \mathbb{R}$ that either continuous and bounded, or uniformly continuous.

Let here the parameters: $0 < \alpha < 1$, $x = (x_1, ..., x_d) \in \mathbb{R}^d$, $n \in \mathbb{N}$; $r = (r_1, ..., r_d) \in \mathbb{N}^d$, $i = (i_1, ..., i_d) \in \mathbb{N}^d$, with $i_j = 1, 2, ..., r_j$, $j = 1, ..., d$; also let $w_i = w_{i_1,...,i_d} \geq 0$, such that $\sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} ... \sum_{i_d=1}^{r_d} w_{i_1,...,i_d} = 1$, in brief written as $\sum_{i=1}^{r} w_i = 1$. We further consider the parameters $k = (k_1, ..., k_d) \in \mathbb{Z}^d$; $\mu_i = (\mu_{i_1}, ..., \mu_{i_d}) \in \mathbb{R}_+^d$, $\nu_i = (\nu_{i_1}, ..., \nu_{i_d}) \in \mathbb{R}_+^d$; and $\lambda_i = \lambda_{i_1,...,i_d}$, $\rho_i = \rho_{i_1,...,i_d} \geq 0$. Call $\nu_i^{\min} = \min\{\nu_{i_1}, ..., \nu_{i_d}\}$ and $\|x\|_\infty = \max(|x_1|, ..., |x_d|)$.

In this second part we study the asymptotic expansions of Voronovskaya type of the following one hidden layer multivariate normalized neural network perturbed operators,

(i) the Stancu type (see [18])

$$(H_n^*(f))(x) = (H_n^*(f))(x_1, ..., x_d) = \tag{78}$$

$$\frac{\sum_{k=-n^2}^{n^2} \left(\sum_{i=1}^{r} w_i f\left(\frac{k+\mu_i}{n+\nu_i}\right) \right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)} =$$

$$\frac{\sum_{k_1=-n^2}^{n^2} ... \sum_{k_d=-n^2}^{n^2} \left(\sum_{i_1=1}^{r} ... \sum_{i_d=1}^{r} w_{i_1,...,i_d} f\left(\frac{k_1+\mu_{i_1}}{n+\nu_{i_1}}, ..., \frac{k_d+\mu_{i_d}}{n+\nu_{i_d}}\right) \right) \cdot}{\sum_{k_1=-n^2}^{n^2} ... \sum_{k_d=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}$$

$$b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right),$$

(ii) the Kantorovich type

$$(K_n^*(f))(x) = \tag{79}$$

$$\frac{\sum_{k=-n^2}^{n^2} \left(\sum_{i=1}^{r} w_i (n+\rho_i)^d \int_0^{\frac{1}{n+\rho_i}} f\left(t + \frac{k+\lambda_i}{n+\rho_i}\right) dt \right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)} =$$

$$\sum_{k_1=-n^2}^{n^2} \ldots \sum_{k_d=-n^2}^{n^2} \left(\sum_{i_1=1}^{r} \ldots \sum_{i_d=1}^{r} w_{i_1,\ldots,i_d} \left(n+\rho_{i_1,\ldots,i_d}\right)^d \cdot \right.$$

$$\frac{\left. \int \ldots \int \ldots \int_0^{\frac{1}{n+\rho_{i_1,\ldots,i_d}}} f\left(t_1 + \frac{k_1+\lambda_{i_1,\ldots,i_d}}{n+\rho_{i_1,\ldots,i_d}}, \ldots, t_d + \frac{k_d+\lambda_{i_1,\ldots,i_d}}{n+\rho_{i_1,\ldots,i_d}}\right) dt_1 \ldots dt_d \right) \cdot}{\sum_{k_1=-n^2}^{n^2} \ldots \sum_{k_d=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), \ldots, n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)} \tag{80}$$

$$b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), \ldots, n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right),$$

and

(iii) the quadrature type

$$\left(M_n^*(f)\right)(x) = \frac{\sum_{k=-n^2}^{n^2} \left(\sum_{i=1}^{r} w_i f\left(\frac{k}{n} + \frac{i}{nr}\right)\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)} = \tag{81}$$

$$\frac{\sum_{k_1=-n^2}^{n^2} \ldots \sum_{k_d=-n^2}^{n^2} \left(\sum_{i_1=1}^{r_1} \ldots \sum_{i_d=1}^{r_d} w_{i_1,\ldots,i_d} f\left(\frac{k_1}{n} + \frac{i_1}{nr_1}, \ldots, \frac{k_d}{n} + \frac{i_d}{nr_d}\right)\right) \cdot}{\sum_{k_1=-n^2}^{n^2} \ldots \sum_{k_d=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), \ldots, n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}$$

$$b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), \ldots, n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right).$$

Similar operators defined for d-dimensional bell-shaped activation functions and sample coefficients $f\left(\frac{k}{n}\right) = f\left(\frac{k_1}{n}, \ldots, \frac{k_d}{n}\right)$ were studied initially in [11], [1], [2], [3], [6], [8], [5], [7], etc. Also see the newer research in [9], [10].

The terms in the ratio of sums (78)-(81) can be nonzero, iff simultaneously

$$\left| n^{1-\alpha}\left(x_j - \frac{k_j}{n}\right)\right| \leq T_j, \text{ all } j = 1, \ldots, d, \tag{82}$$

i.e. $\left|x_j - \frac{k_j}{n}\right| \leq \frac{T_j}{n^{1-\alpha}}$, all $j = 1, \ldots, d$, iff

$$nx_j - T_j n^{\alpha} \leq k_j \leq nx_j + T_j n^{\alpha}, \quad \text{all } j = 1, \ldots, d. \tag{83}$$

To have the order

$$-n^2 \leq nx_j - T_j n^{\alpha} \leq k_j \leq nx_j + T_j n^{\alpha} \leq n^2, \tag{84}$$

we need $n \geq T_j + |x_j|$, all $j = 1, \ldots, d$. So (84) is true when we take

$$n \geq \max_{j \in \{1,\ldots,d\}} \left(T_j + |x_j|\right). \tag{85}$$

When $x \in B$ in order to have (84) it is enough to assume that $n \geq 2T^*$, where $T^* := \max\{T_1, \ldots, T_d\} > 0$. Consider

$$\widetilde{I}_j := \left[nx_j - T_j n^{\alpha}, nx_j + T_j n^{\alpha}\right], \;\; j = 1, \ldots, d, \;\; n \in \mathbb{N}.$$

The length of $\widetilde{I}_j$ is $2T_j n^{\alpha}$. By Proposition 1 of [1], we get that the cardinality of $k_j \in \mathbb{Z}$ that belong to $\widetilde{I}_j := card(k_j) \geq \max(2T_j n^{\alpha} - 1, 0)$, any $j \in \{1, ..., d\}$. In order to have $card(k_j) \geq 1$, we need $2T_j n^{\alpha} - 1 \geq 1$ iff $n \geq T_j^{-\frac{1}{\alpha}}$, any $j \in \{1, ..., d\}$.

Therefore, a sufficient condition in order to obtain the order (84) along with the interval $\widetilde{I}_j$ to contain at least one integer for all $j = 1, ..., d$ is that

$$n \geq \max_{j \in \{1,...,d\}} \left\{ T_j + |x_j|, T_j^{-\frac{1}{\alpha}} \right\}. \tag{86}$$

Clearly as $n \to +\infty$ we get that $card(k_j) \to +\infty$, all $j = 1, ..., d$. Also notice that $card(k_j)$ equals to the cardinality of integers in $[\lceil nx_j - T_j n^{\alpha} \rceil, [nx_j + T_j n^{\alpha}]]$ for all $j = 1, ..., d$.

From now on, in this part two we assume (86).

We denote by $T = (T_1, ..., T_d)$, $[nx + Tn^{\alpha}] = ([nx_1 + T_1 n^{\alpha}], ..., [nx_d + T_d n^{\alpha}])$, and $\lceil nx - Tn^{\alpha} \rceil = (\lceil nx_1 - T_1 n^{\alpha} \rceil, ..., \lceil nx_d - T_d n^{\alpha} \rceil)$. Furthermore it holds

(i)

$$(H_n^*(f))(x) = (H_n^*(f))(x_1, ..., x_d) = \tag{87}$$

$$\frac{\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{r} w_i f\left(\frac{k+\mu_i}{n+\nu_i} \right) \right) b\left(n^{1-\alpha}\left(x - \frac{k}{n} \right) \right)}{\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x - \frac{k}{n} \right) \right)} =$$

$$\frac{\sum_{k_1=\lceil nx_1 - T_1 n^{\alpha} \rceil}^{[nx_1+T_1 n^{\alpha}]} \cdots \sum_{k_d=\lceil nx_d - T_d n^{\alpha} \rceil}^{[nx_d+T_d n^{\alpha}]} \left(\sum_{i_1=1}^{r_1} \cdots \sum_{i_d=1}^{r_d} w_{i_1,...,i_d} f\left(\frac{k_1+\mu_{i_1}}{n+\nu_{i_1}}, ..., \frac{k_d+\mu_{i_d}}{n+\nu_{i_d}} \right) \right) \cdot}{\sum_{k_1=\lceil nx_1 - T_1 n^{\alpha} \rceil}^{[nx_1+T_1 n^{\alpha}]} \cdots \sum_{k_d=\lceil nx_d - T_d n^{\alpha} \rceil}^{[nx_d+T_d n^{\alpha}]} b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n} \right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n} \right) \right)}$$

$$b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n} \right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n} \right) \right),$$

(ii)

$$(K_n^*(f))(x) = \tag{88}$$

$$\frac{\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{r} w_i (n+\rho_i) \int_0^{\frac{1}{n+\rho_i}} f\left(t + \frac{k+\lambda_i}{n+\rho_i} \right) dt \right) b\left(n^{1-\alpha}\left(x - \frac{k}{n} \right) \right)}{\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x - \frac{k}{n} \right) \right)} =$$

$$\frac{\begin{array}{c} \sum\limits_{k_1=\lceil nx_1 - T_1 n^{\alpha} \rceil}^{[nx_1+T_1 n^{\alpha}]} \cdots \sum\limits_{k_d=\lceil nx_d - T_d n^{\alpha} \rceil}^{[nx_d+T_d n^{\alpha}]} \left(\sum\limits_{i_1=1}^{r_1} \cdots \sum\limits_{i_d=1}^{r_d} w_{i_1,...,i_d} \left(n + \rho_{i_1,...,i_d} \right)^d \cdot \right. \\ \left. \int \cdots \int \cdots \int_0^{\frac{1}{n+\rho_{i_1,...,i_d}}} f\left(t_1 + \frac{k_1+\lambda_{i_1,...,i_d}}{n+\rho_{i_1,...,i_d}}, ..., t_d + \frac{k_d+\lambda_{i_1,...,i_d}}{n+\rho_{i_1,...,i_d}} \right) dt_1 ... dt_d \right) \cdot \end{array}}{\sum_{k_1=\lceil nx_1 - T_1 n^{\alpha} \rceil}^{[nx_1+T_1 n^{\alpha}]} \cdots \sum_{k_d=\lceil nx_d - T_d n^{\alpha} \rceil}^{[nx_d+T_d n^{\alpha}]} b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n} \right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n} \right) \right)} \tag{89}$$

$$b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),...,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right),$$

and

(iii)

$$\left(M_n^*(f)\right)(x)=\frac{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}\left(\sum_{i=1}^{r}w_i f\left(\frac{k}{n}+\frac{i}{nr}\right)\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}= \tag{90}$$

$$\frac{\sum_{k_1=\lceil nx_1-T_1n^\alpha\rceil}^{[nx_1+T_1n^\alpha]}...\sum_{k_d=\lceil nx_d-T_dn^\alpha\rceil}^{[nx_d+T_dn^\alpha]}\left(\sum_{i_1=1}^{r_1}...\sum_{i_d=1}^{r_d}w_{i_1,...,i_d}f\left(\frac{k_1}{n}+\frac{i_1}{nr_1},...,\frac{k_d}{n}+\frac{i_d}{nr_d}\right)\right)\cdot}{\sum_{k_1=\lceil nx_1-T_1n^\alpha\rceil}^{[nx_1+T_1n^\alpha]}...\sum_{k_d=\lceil nx_d-T_dn^\alpha\rceil}^{[nx_d+T_dn^\alpha]}b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),...,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)}$$

$$b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),...,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right).$$

So if $\left|n^{1-\alpha}\left(x_j-\frac{k_j}{n}\right)\right|\leq T_j$, all $j=1,...,d$, we get that

$$\left\|x-\frac{k}{n}\right\|_\infty\leq\frac{T^*}{n^{1-\alpha}}. \tag{91}$$

For convinience we call

$$V(x)=\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)=$$

$$\sum_{k_1=\lceil nx_1-T_1n^\alpha\rceil}^{[nx_1+T_1n^\alpha]}...\sum_{k_d=\lceil nx_d-T_dn^\alpha\rceil}^{[nx_d+T_dn^\alpha]}b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right),...,n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right). \tag{92}$$

We make

Remark 9 *Here always k is as in (84).*

I) We observe that

$$\left\|\frac{k+\mu_i}{n+\nu_i}-x\right\|_\infty\leq\left\|\frac{k}{n+\nu_i}-x\right\|_\infty+\left\|\frac{\mu_i}{n+\nu_i}\right\|_\infty \tag{93}$$

$$\leq\left\|\frac{k}{n+\nu_i}-x\right\|_\infty+\frac{\|\mu_i\|_\infty}{n+\nu_i^{\min}}.$$

Next we see

$$\left\|\frac{k}{n+\nu_i}-x\right\|_\infty\leq\left\|\frac{k}{n+\nu_i}-\frac{k}{n}\right\|_\infty+\left\|\frac{k}{n}-x\right\|_\infty\overset{(91)}{\leq}\left\|\frac{\nu_i k}{n(n+\nu_i)}\right\|_\infty+\frac{T^*}{n^{1-\alpha}} \tag{94}$$

$$\leq \|k\|_\infty \frac{\|\nu_i\|_\infty}{n\left(n+\nu_i^{\min}\right)} + \frac{T^*}{n^{1-\alpha}} =: (*)\,.$$

We notice for $j = 1, ..., d$ *we get that*

$$|k_j| \leq n\,|x_j| + T_j n^\alpha.$$

Therefore

$$\|k\|_\infty \leq \|n\,|x| + Tn^\alpha\|_\infty \leq n\,\|x\|_\infty + T^* n^\alpha, \tag{95}$$

where $|x| = (|x_1|, ..., |x|_d)\,.$

Thus

$$(*) \leq (n\,\|x\|_\infty + T^* n^\alpha)\, \frac{\|\nu_i\|_\infty}{n\left(n+\nu_i^{\min}\right)} + \frac{T^*}{n^{1-\alpha}} = \tag{96}$$

$$\left(\|x\|_\infty + T^* n^{\alpha-1}\right) \frac{\|\nu_i\|_\infty}{\left(n+\nu_i^{\min}\right)} + \frac{T^*}{n^{1-\alpha}}.$$

So we get

$$\left\|\frac{k}{n+\nu_i} - x\right\|_\infty \leq \left(\|x\|_\infty + \frac{T^*}{n^{1-\alpha}}\right) \frac{\|\nu_i\|_\infty}{\left(n+\nu_i^{\min}\right)} + \frac{T^*}{n^{1-\alpha}}. \tag{97}$$

Consequently we obtain the useful

$$\left\|\frac{k+\mu_i}{n+\nu_i} - x\right\|_\infty \leq \left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n+\nu_i^{\min}}\right) + \left(1 + \frac{\|\nu_i\|_\infty}{\left(n+\nu_i^{\min}\right)}\right) \frac{T^*}{n^{1-\alpha}}. \tag{98}$$

II) We also have for

$$0 \leq t_j \leq \frac{1}{n+\rho_{i_1,\ldots,i_d}}, \quad j = 1, ..., d, \tag{99}$$

that

$$\left\|t + \frac{k+\lambda_{i_1,\ldots,i_d}}{n+\rho_{i_1,\ldots,i_d}} - x\right\|_\infty \leq \frac{1}{n+\rho_{i_1,\ldots,i_d}} + \left\|\frac{k+\lambda_{i_1,\ldots,i_d}}{n+\rho_{i_1,\ldots,i_d}} - x\right\|_\infty \leq \tag{100}$$

$$\frac{1+\lambda_{i_1,\ldots,i_d}}{n+\rho_{i_1,\ldots,i_d}} + \left\|\frac{k}{n+\rho_{i_1,\ldots,i_d}} - x\right\|_\infty \leq$$

$$\frac{1+\lambda_{i_1,\ldots,i_d}}{n+\rho_{i_1,\ldots,i_d}} + \left\|\frac{k}{n+\rho_{i_1,\ldots,i_d}} - \frac{k}{n}\right\|_\infty + \left\|\frac{k}{n} - x\right\|_\infty \leq$$

$$\frac{1+\lambda_{i_1,\ldots,i_d}}{n+\rho_{i_1,\ldots,i_d}} + \frac{T^*}{n^{1-\alpha}} + \frac{\rho_{i_1,\ldots,i_d}\,\|k\|_\infty}{\left(n+\rho_{i_1,\ldots,i_d}\right) n} \leq$$

$$\frac{1+\lambda_{i_1,\ldots,i_d}}{n+\rho_{i_1,\ldots,i_d}} + \frac{T^*}{n^{1-\alpha}} + \frac{\rho_{i_1,\ldots,i_d}}{n\left(n+\rho_{i_1,\ldots,i_d}\right)} \left(n\,\|x\|_\infty + T^* n^\alpha\right) = \tag{101}$$

$$\frac{1+\lambda_{i_1,...,i_d}}{n+\rho_{i_1,...,i_d}}+\frac{T^*}{n^{1-\alpha}}+\left(\frac{\rho_{i_1,...,i_d}}{n+\rho_{i_1,...,i_d}}\right)\left(\|x\|_\infty+\frac{T^*}{n^{1-\alpha}}\right). \qquad (102)$$

We have found the useful

$$\left\|t+\frac{k+\lambda_{i_1,...,i_d}}{n+\rho_{i_1,...,i_d}}-x\right\|_\infty \leq \qquad (103)$$

$$\left(\frac{\rho_{i_1,...,i_d}\|x\|_\infty+\lambda_{i_1,...,i_d}+1}{n+\rho_{i_1,...,i_d}}\right)+\left(1+\frac{\rho_{i_1,...,i_d}}{n+\rho_{i_1,...,i_d}}\right)\frac{T^*}{n^{1-\alpha}}.$$

III) We also observe that it holds

$$\left\|\frac{k}{n}+\frac{i}{nr}-x\right\|_\infty \leq \left\|\frac{k}{n}-x\right\|_\infty+\frac{1}{n}\left\|\frac{i}{r}\right\|_\infty \leq \frac{T^*}{n^{1-\alpha}}+\frac{1}{n}. \qquad (104)$$

Next we follow [4], pp. 284-286.

About Multivariate Taylor formula and estimates

Let $\mathbb{R}^d$; $d\geq 2$; $z:=(z_1,...,z_d)$, $x_0:=(x_{01},...,x_{0d})\in\mathbb{R}^d$. We consider the space of functions $AC^N(\mathbb{R}^d)$ with $f:\mathbb{R}^d\to\mathbb{R}$ be such that all partial derivatives of order $(N-1)$ are coordinatewise absolutely continuous functions on convex compacta, $N\in\mathbb{N}$. Also $f\in C^{N-1}(\mathbb{R}^d)$. Each N^{th} order partial derivative is denoted by $f_{\widetilde{\alpha}}:=\frac{\partial^{\widetilde{\alpha}}f}{\partial x^{\widetilde{\alpha}}}$, where $\widetilde{\alpha}:=(\alpha_1,...,\alpha_N)$, $\alpha_j\in\mathbb{Z}^+$, $j=1,...,d$ and $|\widetilde{\alpha}|:=\sum_{j=1}^d\alpha_j=N$. Consider $g_z(t):=f(x_0+t(z-x_0))$, $t\geq 0$. Then

$$g_z^{(l)}(t)=\left[\left(\sum_{j=1}^d(z_j-x_{0j})\frac{\partial}{\partial x_j}\right)^l f\right](x_{01}+t(z_1-x_{01}),...,x_{0N}+t(z_N-x_{0N})), \qquad (105)$$

for all $l=0,1,2,...,N$.

Example 10 *Let $d=N=2$. Then*

$$g_z(t)=f(x_{01}+t(z_1-x_{01}),x_{02}+t(z_2-x_{02})),\quad t\in\mathbb{R},$$

and

$$g_z'(t)=(z_1-x_{01})\frac{\partial f}{\partial x_1}(x_0+t(z-x_0))+(z_2-x_{02})\frac{\partial f}{\partial x_2}(x_0+t(z-x_0)). \qquad (106)$$

Setting

$$(*)=(x_{01}+t(z_1-x_{01}),x_{02}+t(z_2-x_{02}))=(x_0+t(z-x_0)),$$

we get

$$g_z''(t)=(z_1-x_{01})^2\frac{\partial f^2}{\partial x_1^2}(*)+(z_1-x_{01})(z_2-x_{02})\frac{\partial f^2}{\partial x_2\partial x_1}(*)+$$

$$(z_1 - x_{01})(z_2 - x_{02}) \frac{\partial f^2}{\partial x_1 \partial x_2}(*) + (z_2 - x_{02})^2 \frac{\partial f^2}{\partial x_2^2}(*). \tag{107}$$

Similarly, we have the general case of $d, N \in \mathbb{N}$ *for* $g_z^{(N)}(t)$.

We mention the following multivariate Taylor theorem.

Theorem 11 *Under the above assumptions we have*

$$f(z_1, ..., z_N) = g_z(1) = \sum_{l=0}^{N-1} \frac{g_z^{(l)}(0)}{l!} + R_N(z, 0), \tag{108}$$

where

$$R_N(z, 0) := \int_0^1 \left(\int_0^{t_1} ... \left(\int_0^{t_{N-1}} g_z^{(N)}(t_N) \, dt_N \right) ... \right) dt_1, \tag{109}$$

or

$$R_N(z, 0) = \frac{1}{(N-1)!} \int_0^1 (1-\theta)^{N-1} g_z^{(N)}(\theta) \, d\theta. \tag{110}$$

Notice that $g_z(0) = f(x_0)$.

We make

Remark 12 *Assume here that*

$$\|f_\alpha\|_{\infty, \mathbb{R}^d, N}^{\max} := \max_{|\alpha|=N} \|f_\alpha\|_{\infty, \mathbb{R}^d} < \infty. \tag{111}$$

Then

$$\left\| g_z^{(m)} \right\|_{\infty, [0,1]} = \left\| \left[\left(\sum_{j=1}^d (z_j - x_{0j}) \frac{\partial}{\partial x_j} \right)^N f \right] (x_0 + t(z - x_0)) \right\|_{\infty, [0,1]} \leq$$
$$\left(\sum_{j=1}^d |z_j - x_{0j}| \right)^N \|f_\alpha\|_{\infty, \mathbb{R}^d, N}^{\max}, \tag{112}$$

that is

$$\left\| g_z^{(N)} \right\|_{\infty, [0,1]} \leq \left(\|z - x_0\|_{l_1} \right)^N \|f_\alpha\|_{\infty, \mathbb{R}^d, N}^{\max} < \infty. \tag{113}$$

Hence we get by (110) that

$$|R_N(z, 0)| \leq \frac{\left\| g_z^{(N)} \right\|_{\infty, [0,1]}}{N!} < \infty. \tag{114}$$

And it holds

$$|R_N(z, 0)| \leq \frac{\left(\|z - x_0\|_{l_1} \right)^N}{N!} \|f_\alpha\|_{\infty, \mathbb{R}^d, N}^{\max}, \tag{115}$$

$\forall \ z, x_0 \in \mathbb{R}^d$.

Inequality (115) will be an important tool in proving our main multivariate results.

5 Multivariate Results

We present our first multivariate main result

Theorem 13 *Let* $f \in AC^N(\mathbb{R}^d) \cap C^{N-1}(\mathbb{R}^d)$, $d \in \mathbb{N} - \{1\}$, $N \in \mathbb{N}$, *with* $\|f_{\widetilde{\alpha}}\|_{\infty,\mathbb{R}^d,N}^{\max} < \infty$. *Here* $n \geq \max_{j \in \{1,...,d\}} \left(T_j + |x_j|, T_j^{-\frac{1}{\alpha}}\right)$, *where* $x \in \mathbb{R}^d$, $0 < \alpha < 1$, $n \in \mathbb{N}$, $T_j > 0$. *Then*

$$(H_n^*(f))(x) - f(x) =$$

$$\sum_{l=1}^{N-1} \left(\sum_{|\widetilde{\alpha}|=l} \left(\frac{f_{\widetilde{\alpha}(x)}}{\prod_{j=1}^{d} \alpha_j!} \right) H_n^* \left(\prod_{j=1}^{d} (\cdot - x_j)^{\alpha_j}, x \right) \right) + o\left(\frac{1}{n^{(N-\varepsilon)(1-\alpha)}} \right), \quad (116)$$

where $0 < \varepsilon \leq N$.

If $N = 1$, *the sum in (116) collapses.*

The last (116) implies that

$$n^{(N-\varepsilon)(1-\alpha)} \cdot$$

$$\left[(H_n^*(f))(x) - f(x) - \sum_{l=1}^{N-1} \left(\sum_{|\widetilde{\alpha}|=l} \left(\frac{f_{\widetilde{\alpha}(x)}}{\prod_{j=1}^{d} \alpha_j!} \right) H_n^* \left(\prod_{j=1}^{d} (\cdot - x_j)^{\alpha_j}, x \right) \right) \right] \to 0, \quad (117)$$

as $n \to \infty$, $0 < \varepsilon \leq N$.

When $N = 1$, *or* $f_{\widetilde{\alpha}}(x) = 0$, *for all* $\widetilde{\alpha} : |\widetilde{\alpha}| = l = 1, ..., N-1$, *then we derive*

$$n^{(N-\varepsilon)(1-\alpha)} \left[(H_n^*(f))(x) - f(x) \right] \to 0, \quad (118)$$

as $n \to \infty$, $0 < \varepsilon \leq N$.

Proof. Set

$$g_{\frac{k+\mu_i}{n+\nu_i}}(t) = f\left(x + t\left(\frac{k+\mu_i}{n+\nu_i} - x \right) \right), \quad 0 \leq t \leq 1. \quad (119)$$

Then we have

$$g_{\frac{k+\mu_i}{n+\nu_i}}^{(l)}(t) = \left[\left(\sum_{j=1}^{d} \left(\frac{k_j + \mu_{i_j}}{n + \nu_{i_j}} - x_j \right) \frac{\partial}{\partial x_j} \right)^l f \right] \quad (120)$$

$$\left(x_1 + t\left(\frac{k_1 + \mu_{i_1}}{n + \nu_{i_1}} - x_1 \right), ..., x_d + t\left(\frac{k_d + \mu_{i_d}}{n + \nu_{i_d}} - x_d \right) \right), \; l = 0, ..., N,$$

and

$$g_{\frac{k+\mu_i}{n+\nu_i}}(0) = f(x). \tag{121}$$

By Taylor's formula, we get

$$f\left(\frac{k_1+\mu_{i_1}}{n+\nu_{i_1}}, ..., \frac{k_d+\mu_{i_d}}{n+\nu_{i_d}}\right) = g_{\frac{k+\mu_i}{n+\nu_i}}(1) = \tag{122}$$

$$\sum_{l=0}^{N-1} \frac{g^{(l)}_{\frac{k+\mu_i}{n+\nu_i}}(0)}{l!} + R_N\left(\frac{k+\mu_i}{n+\nu_i}, 0\right),$$

where

$$R_N\left(\frac{k+\mu_i}{n+\nu_i}, 0\right) = \frac{1}{(N-1)!}\int_0^1 (1-\theta)^{N-1} g^{(N)}_{\frac{k+\mu_i}{n+\nu_i}}(\theta)\, d\theta. \tag{123}$$

Here we denote by

$$f_{\widetilde{\alpha}} := \frac{\partial^{\widetilde{\alpha}} f}{\partial x^{\widetilde{\alpha}}}, \quad \widetilde{\alpha} := (\alpha_1, ..., \alpha_d), \ \ \alpha_j \in \mathbb{Z}^+, \ j = 1, ..., d, \tag{124}$$

such that $|\widetilde{\alpha}| = \sum_{j=1}^{d} \alpha_j = l$, $0 \leq l \leq N$.

More precisely we can rewrite

$$f\left(\frac{k+\mu_i}{n+\nu_i}\right) - f(x) =$$

$$\sum_{l=1}^{N-1} \sum_{\left(\substack{\widetilde{\alpha}:=(\alpha_1,...,\alpha_d),\ \alpha_j \in \mathbb{Z}^+,\\ j=1,...,d,\ |\widetilde{\alpha}|=\sum_{j=1}^{d}\alpha_j = l}\right)} \left(\frac{1}{\prod_{j=1}^{d} \alpha_j!}\right) \left(\prod_{j=1}^{d}\left(\frac{k_j+\mu_{i_j}}{n+\nu_{i_j}} - x_j\right)^{\alpha_j}\right) f_{\widetilde{\alpha}}(x)$$

$$+R_N\left(\frac{k+\mu_i}{n+\nu_i}, 0\right), \tag{125}$$

where

$$R_N\left(\frac{k+\mu_i}{n+\nu_i}, 0\right) = N\int_0^1 (1-\theta)^{N-1} \sum_{\left(\substack{\widetilde{\alpha}:=(\alpha_1,...,\alpha_d),\ \alpha_j \in \mathbb{Z}^+,\\ j=1,...,d,\ |\widetilde{\alpha}|=\sum_{j=1}^{d}\alpha_j = N}\right)} \left(\frac{1}{\prod_{j=1}^{d} \alpha_j!}\right).$$

$$\left(\prod_{j=1}^{d}\left(\frac{k_j+\mu_{i_j}}{n+\nu_{i_j}}-x_j\right)^{\alpha_j}\right) f_{\widetilde{\alpha}}\left(x+\theta\left(\frac{k+\mu_i}{n+\nu_i}-x\right)\right) d\theta. \tag{126}$$

Thus

$$\sum_{i=1}^{r} w_i f\left(\frac{k+\mu_i}{n+\nu_i}\right) = \sum_{l=0}^{N-1}\sum_{i=1}^{r} w_i \frac{g^{(l)}_{\frac{k+\mu_i}{n+\nu_i}}(0)}{l!} + \sum_{i=1}^{r} w_i R_N\left(\frac{k+\mu_i}{n+\nu_i},0\right), \tag{127}$$

and

$$\left(H_n^*(f)\right)(x) = \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{r} w_i f\left(\frac{k+\mu_i}{n+\nu_i}\right)\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} =$$

$$\sum_{l=0}^{N-1}\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{r} w_i \frac{g^{(l)}_{\frac{k+\mu_i}{n+\nu_i}}(0)}{l!}\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} + \tag{128}$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\left(\sum_{i=1}^{r} w_i R_N\left(\frac{k+\mu_i}{n+\nu_i},0\right)\right).$$

Therefore it holds

$$\left(H_n^*(f)\right)(x) - f(x) -$$

$$\sum_{l=1}^{N-1}\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{r} w_i \frac{g^{(l)}_{\frac{k+\mu_i}{n+\nu_i}}(0)}{l!}\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} = R^*(x), \tag{129}$$

where

$$R^*(x) = \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\left(\sum_{i=1}^{r} w_i R_N\left(\frac{k+\mu_i}{n+\nu_i},0\right)\right). \tag{130}$$

By (115) we get that

$$\left\| g^{(N)}_{\frac{k+\mu_i}{n+\nu_i}} \right\|_{\infty,[0,1]} \le \left(\left\|\frac{k+\mu_i}{n+\nu_i}-x\right\|_{l_1}\right)^N \|f_{\widetilde{\alpha}}\|_{\infty,\mathbb{R}^d,N}^{\max} < \infty. \tag{131}$$

And furthermore it holds

$$\left| R_N\left(\frac{k+\mu_i}{n+\nu_i},0\right)\right| \le \frac{\left(\left\|\frac{k+\mu_i}{n+\nu_i}-x\right\|_{l_1}\right)^N}{N!} \|f_{\widetilde{\alpha}}\|_{\infty,\mathbb{R}^d,N}^{\max}. \tag{132}$$

By (98) now we obtain

$$\left\| \frac{k+\mu_i}{n+\nu_i} - x \right\|_{l_1} \leq \tag{133}$$

$$d\left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n+\nu_i^{\min}}\right) + \left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right],$$

and thus we have

$$\left| R_N\left(\frac{k+\mu_i}{n+\nu_i}, 0\right)\right| \leq$$

$$\frac{d^N}{N!}\left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n+\nu_i^{\min}}\right) + \left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]^N \|f_{\widetilde{\alpha}}\|_{\infty,\mathbb{R}^d,N}^{\max}. \tag{134}$$

Clearly now we can deduce

$$|R^*(x)| \leq \left(\frac{d^N}{N!}\|f_{\widetilde{\alpha}}\|_{\infty,\mathbb{R}^d,N}\right)\cdot$$

$$\sum_{i=1}^{r} w_i \left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n+\nu_i^{\min}}\right) + \left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]^N \leq \tag{135}$$

$$\left(\frac{d^N}{N!}\|f_{\widetilde{\alpha}}\|_{\infty,\mathbb{R}^d,N}\right)\cdot$$

$$\max_{\left(\begin{array}{c}\text{all } i=(i_1,...,i_d):\\ i_j=1,...,r_j;\ j=1,...,d\end{array}\right)} \left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n+\nu_i^{\min}}\right) + \left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]^N$$

$$\leq \left(\frac{d^N}{N!}\|f_{\widetilde{\alpha}}\|_{\infty,\mathbb{R}^d,N}\right)\cdot$$

$$\max_i \left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n^{1-\alpha}n^\alpha}\right) + \left(1+\frac{\|\nu_i\|_\infty}{n}\right)\frac{T^*}{n^{1-\alpha}}\right]^N = \tag{136}$$

$$\left(\frac{d^N}{N!}\|f_{\widetilde{\alpha}}\|_{\infty,\mathbb{R}^d,N}\right)\cdot$$

$$\max_i \left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n^\alpha}\right) + \left(1+\frac{\|\nu_i\|_\infty}{n}\right)T^*\right]^N \frac{1}{n^{(1-\alpha)N}} \leq$$

$$\left(\frac{d^N}{N!}\|f_{\widetilde{\alpha}}\|_{\infty,\mathbb{R}^d,N}\right)\cdot$$

$$\max_i \left[\left(\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty\right) + \left(1+\|\nu_i\|_\infty\right)T^*\right]^N \frac{1}{n^{(1-\alpha)N}} =: \frac{\Lambda}{n^{(1-\alpha)N}}. \tag{137}$$

That is we proved

$$|R^*(x)| = O\left(\frac{1}{n^{(1-\alpha)N}}\right), \tag{138}$$

and

$$|R^*(x)| = o(1). \tag{139}$$

And, letting $0 < \varepsilon \leq N$, we derive

$$\frac{|R^*(x)|}{\left(\frac{1}{n^{(N-\varepsilon)(1-\alpha)}}\right)} \leq \frac{\Lambda}{n^{(1-\alpha)N}} n^{(N-\varepsilon)(1-\alpha)} = \frac{\Lambda}{n^{\varepsilon(1-\alpha)}} \to 0, \tag{140}$$

as $n \to \infty$.

I.e.

$$|R^*(x)| = o\left(\frac{1}{n^{(N-\varepsilon)(1-\alpha)}}\right). \tag{141}$$

Clearly here we can rewrite (129), as

$$(H_n^*(f))(x) - f(x) - \sum_{l=1}^{N-1}\left(\sum_{|\widetilde{\alpha}|=l}\left(\frac{f_{\widetilde{\alpha}}(x)}{\prod_{j=1}^{d}\alpha_j!}\right) H_n^*\left(\prod_{j=1}^{d}(\cdot - x_j)^{\alpha_j}, x\right)\right) = R^*(x). \tag{142}$$

Based on (141) and (142) we derive (116). ■

We continue with

Theorem 14 *Let* $f \in AC^N(\mathbb{R}^d) \cap C^{N-1}(\mathbb{R}^d)$, $d \in \mathbb{N} - \{1\}$, $N \in \mathbb{N}$, *with* $\|f_{\widetilde{\alpha}}\|_{\infty,\mathbb{R}^d,N}^{\max} < \infty$. *Here* $n \geq \max_{j \in \{1,\ldots,d\}}\left(T_j + |x_j|, T_j^{-\frac{1}{\alpha}}\right)$, *where* $x \in \mathbb{R}^d$, $0 < \alpha < 1$, $n \in \mathbb{N}$, $T_j > 0$. *Then*

$$(K_n^*(f))(x) - f(x) =$$

$$\sum_{l=1}^{N-1}\left(\sum_{|\widetilde{\alpha}|=l}\left(\frac{f_{\widetilde{\alpha}}(x)}{\prod_{j=1}^{d}\alpha_j!}\right) K_n^*\left(\prod_{j=1}^{d}(\cdot - x_j)^{\alpha_j}, x\right)\right) + o\left(\frac{1}{n^{(N-\varepsilon)(1-\alpha)}}\right), \tag{143}$$

where $0 < \varepsilon \leq N$.

If $N = 1$, *the sum in (143) collapses.*

The last (143) implies that

$$n^{(N-\varepsilon)(1-\alpha)} \cdot$$

$$\left[(K_n^*(f))(x) - f(x) - \sum_{l=1}^{N-1}\left(\sum_{|\widetilde{\alpha}|=l}\left(\frac{f_{\widetilde{\alpha}}(x)}{\prod_{j=1}^{d}\alpha_j!}\right) K_n^*\left(\prod_{j=1}^{d}(\cdot - x_j)^{\alpha_j}, x\right)\right)\right] \to 0, \tag{144}$$

as $n \to \infty$, $0 < \varepsilon \leq N$.

When $N=1$, *or* $f_{\widetilde{\alpha}}(x)=0$, *for all* $\widetilde{\alpha}:|\widetilde{\alpha}|=l=1,...,N-1$, *then we derive*

$$n^{(N-\varepsilon)(1-\alpha)}\left[\left(K_n^*(f)\right)(x)-f(x)\right]\to 0, \tag{145}$$

as $n\to\infty$, $0<\varepsilon\leq N$.

Proof. Set

$$g_{t+\frac{k+\lambda_i}{n+\rho_i}}(\lambda^*)=f\left(x+\lambda^*\left(t+\frac{k+\lambda_i}{n+\rho_i}-x\right)\right),\quad 0\leq\lambda^*\leq 1. \tag{146}$$

Then we have

$$g^{(l)}_{t+\frac{k+\lambda_i}{n+\rho_i}}(\lambda^*)=$$

$$\left[\left(\sum_{j=1}^{d}\left(t_j+\frac{k_j+\lambda_i}{n+\rho_i}-x_j\right)\frac{\partial}{\partial x_j}\right)^l f\right]\left(x+\lambda^*\left(t+\frac{k+\lambda_i}{n+\rho_i}-x\right)\right), \tag{147}$$

$l=0,...,N$, and

$$g_{t+\frac{k+\lambda_i}{n+\rho_i}}(0)=f(x). \tag{148}$$

By Taylor's formula, we get

$$f\left(t+\frac{k+\lambda_i}{n+\rho_i}\right)=g_{t+\frac{k+\lambda_i}{n+\rho_i}}(1)=$$

$$\sum_{l=0}^{N-1}\frac{g^{(l)}_{t+\frac{k+\lambda_i}{n+\rho_i}}(0)}{l!}+R_N\left(t+\frac{k+\lambda_i}{n+\rho_i},0\right), \tag{149}$$

where

$$R_N\left(t+\frac{k+\lambda_i}{n+\rho_i},0\right)=\frac{1}{(N-1)!}\int_0^1(1-\theta)^{N-1}g^{(N)}_{t+\frac{k+\lambda_i}{n+\rho_i}}(\theta)\,d\theta. \tag{150}$$

Here we denote by

$$f_{\widetilde{\alpha}}:=\frac{\partial^{\widetilde{\alpha}}f}{\partial x^{\widetilde{\alpha}}},\quad \widetilde{\alpha}:=(\alpha_1,...,\alpha_d),\ \ \alpha_j\in\mathbb{Z}^+,\ j=1,...,d, \tag{151}$$

such that $|\widetilde{\alpha}|=\sum_{j=1}^{d}\alpha_j=l$, $0\leq l\leq N$.

More precisely we can rewrite

$$f\left(t+\frac{k+\lambda_i}{n+\rho_i}\right)-f(x)=$$

$$\sum_{l=1}^{N-1} \sum_{\left(\substack{\widetilde{\alpha}:=(\alpha_1,\dots,\alpha_d),\ \alpha_j\in\mathbb{Z}^+,\\ j=1,\dots,d,\ |\widetilde{\alpha}|=\sum_{j=1}^{d}\alpha_j=l}\right)} \left(\frac{1}{\prod_{j=1}^{d}\alpha_j!}\right)\left(\prod_{j=1}^{d}\left(\left(t_j+\frac{k_j+\lambda_i}{n+\rho_i}\right)-x_j\right)^{\alpha_j}\right) f_{\widetilde{\alpha}}(x)$$

$$+R_N\left(t+\frac{k+\lambda_i}{n+\rho_i},0\right), \tag{152}$$

where

$$R_N\left(t+\frac{k+\lambda_i}{n+\rho_i},0\right)=N\int_0^1(1-\theta)^{N-1}\cdot$$

$$\sum_{\left(\substack{\widetilde{\alpha}:=(\alpha_1,\dots,\alpha_d),\ \alpha_j\in\mathbb{Z}^+,\\ j=1,\dots,d,\ |\widetilde{\alpha}|=\sum_{j=1}^{d}\alpha_j=N}\right)} \left(\frac{1}{\prod_{j=1}^{d}\alpha_j!}\right)\left(\prod_{j=1}^{d}\left(\left(t_j+\frac{k_j+\lambda_i}{n+\rho_i}\right)-x_j\right)^{\alpha_j}\right)\cdot \tag{153}$$

$$f_{\widetilde{\alpha}}\left(x+\theta\left(\left(t+\frac{k+\lambda_i}{n+\rho_i}\right)-x\right)\right)d\theta.$$

Thus it holds

$$\sum_{i=1}^{r}w_i(n+\rho_i)^d\int_0^{\frac{1}{n+\rho_i}}f\left(t+\frac{k+\lambda_i}{n+\rho_i}\right)dt= \tag{154}$$

$$\sum_{l=0}^{N-1}\frac{\sum_{i=1}^{r}w_i(n+\rho_i)^d\int_0^{\frac{1}{n+\rho_i}}g^{(l)}_{t+\frac{k+\lambda_i}{n+\rho_i}}(0)\,dt}{l!}+$$

$$\sum_{i=1}^{r}w_i(n+\rho_i)^d\int_0^{\frac{1}{n+\rho_i}}R_N\left(t+\frac{k+\lambda_i}{n+\rho_i},0\right)dt.$$

Hence it holds

$$(K_n^*(f))(x)=$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{r}w_i(n+\rho_i)^d\int_0^{\frac{1}{n+\rho_i}}f\left(t+\frac{k+\lambda_i}{n+\rho_i}\right)dt\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}=$$

$$\sum_{l=0}^{N-1}\frac{1}{l!}\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\left(\sum_{i=1}^{r}w_i(n+\rho_i)^d\int_0^{\frac{1}{n+\rho_i}}g^{(l)}_{t+\frac{k+\lambda_i}{n+\rho_i}}(0)\,dt\right)+ \tag{155}$$

$$\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V\left(x\right)}\left(\sum_{i=1}^{r}w_i\left(n+\rho_i\right)^d\int_0^{\frac{1}{n+\rho_i}}R_N\left(t+\frac{k+\lambda_i}{n+\rho_i},0\right)dt\right).$$

So we see that

$$\left(K_n^*\left(f\right)\right)\left(x\right)-f\left(x\right)-$$

$$\sum_{l=1}^{N-1}\frac{1}{l!}\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V\left(x\right)}\left(\sum_{i=1}^{r}w_i\left(n+\rho_i\right)^d\int_0^{\frac{1}{n+\rho_i}}g_{t+\frac{k+\lambda_i}{n+\rho_i}}^{(l)}\left(0\right)dt\right) \tag{156}$$

$$=R^*\left(x\right),$$

where

$$R^*\left(x\right)=$$

$$\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V\left(x\right)}\left(\sum_{i=1}^{r}w_i\left(n+\rho_i\right)^d\int_0^{\frac{1}{n+\rho_i}}R_N\left(t+\frac{k+\lambda_i}{n+\rho_i},0\right)dt\right). \tag{157}$$

By (115) we get that

$$\left\|g_{t+\frac{k+\lambda_i}{n+\rho_i}}^{(N)}\right\|_{\infty,[0,1]}\leq\left(\left\|t+\frac{k+\lambda_i}{n+\rho_i}-x\right\|_{l_1}\right)^N\left\|f_{\widetilde{\alpha}}\right\|_{\infty,\mathbb{R}^d,N}^{\max}<\infty. \tag{158}$$

And furthermore it holds

$$\left|R_N\left(t+\frac{k+\lambda_i}{n+\rho_i},0\right)\right|\leq$$

$$\frac{\left(\left\|t+\frac{k+\lambda_i}{n+\rho_i}-x\right\|_{l_1}\right)^N}{N!}\left\|f_{\widetilde{\alpha}}\right\|_{\infty,\mathbb{R}^d,N}^{\max}. \tag{159}$$

By (103) now we obtain

$$\left\|t+\frac{k+\lambda_i}{n+\rho_i}-x\right\|_{l_1}\leq \tag{160}$$

$$d\left[\left(\frac{\rho_i\left\|x\right\|_{\infty}+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right],$$

and thus we have

$$\left|R_N\left(t+\frac{k+\lambda_i}{n+\rho_i},0\right)\right|\leq\left(\frac{d^N}{N!}\left\|f_{\widetilde{\alpha}}\right\|_{\infty,\mathbb{R}^d,N}^{\max}\right)\cdot$$

$$\left[\left(\frac{\rho_i\left\|x\right\|_{\infty}+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]^N. \tag{161}$$

Clearly now we can infer

$$|R^*(x)| \leq \left(\frac{d^N}{N!}\|f_{\widetilde{\alpha}}\|^{\max}_{\infty,\mathbb{R}^d,N}\right) \cdot$$

$$\sum_{i=1}^{r} w_i \left[\left(\frac{\rho_i\|x\|_\infty + \lambda_i + 1}{n+\rho_i}\right) + \left(1 + \frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]^N \leq \tag{162}$$

$$\left(\frac{d^N}{N!}\|f_{\widetilde{\alpha}}\|^{\max}_{\infty,\mathbb{R}^d,N}\right) \cdot$$

$$\left(\begin{array}{c}\max \\ \text{all } i = (i_1,...,i_d): \\ i_j = 1,...,r_j;\ j = 1,...,d\end{array}\right)_i \left[\left(\frac{\rho_i\|x\|_\infty + \lambda_i + 1}{n+\rho_i}\right) + \left(1 + \frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]^N \leq$$

$$\left(\frac{d^N}{N!}\|f_{\widetilde{\alpha}}\|^{\max}_{\infty,\mathbb{R}^d,N}\right) \cdot$$

$$\max_i \left[\left(\frac{\rho_i\|x\|_\infty + \lambda_i + 1}{n^{1-\alpha}n^\alpha}\right) + \left(1 + \frac{\rho_i}{n}\right)\frac{T^*}{n^{1-\alpha}}\right]^N = \tag{163}$$

$$\left(\frac{d^N}{N!}\|f_{\widetilde{\alpha}}\|^{\max}_{\infty,\mathbb{R}^d,N}\right) \cdot$$

$$\max_i \left[\left(\frac{\rho_i\|x\|_\infty + \lambda_i + 1}{n^\alpha}\right) + \left(1 + \frac{\rho_i}{n}\right)T^*\right]^N \frac{1}{n^{(1-\alpha)N}} \leq$$

$$\left(\frac{d^N}{N!}\|f_{\widetilde{\alpha}}\|^{\max}_{\infty,\mathbb{R}^d,N}\right) \cdot$$

$$\max_i \left[(\rho_i\|x\|_\infty + \lambda_i + 1) + (1+\rho_i)T^*\right]^N \frac{1}{n^{(1-\alpha)N}} =: \frac{D}{n^{(1-\alpha)N}}. \tag{164}$$

That is we proved

$$|R^*(x)| = O\left(\frac{1}{n^{(1-\alpha)N}}\right), \tag{165}$$

and

$$|R^*(x)| = o(1). \tag{166}$$

And, letting $0 < \varepsilon \leq N$, we derive

$$\frac{|R^*(x)|}{\left(\frac{1}{n^{(N-\varepsilon)(1-\alpha)}}\right)} \leq \frac{D}{n^{(1-\alpha)N}} n^{(N-\varepsilon)(1-\alpha)} = \frac{D}{n^{\varepsilon(1-\alpha)}} \to 0, \tag{167}$$

as $n \to \infty$.

I.e.

$$|R^*(x)| = o\left(\frac{1}{n^{(N-\varepsilon)(1-\alpha)}}\right). \tag{168}$$

Clearly here we can rewrite (156), as

$$\left(K_n^*(f)\right)(x)-f(x)-\sum_{l=1}^{N-1}\left(\sum_{|\widetilde{\alpha}|=l}\left(\frac{f_{\widetilde{\alpha}(x)}}{\prod_{j=1}^{d}\alpha_j!}\right)K_n^*\left(\prod_{j=1}^{d}(\cdot-x_j)^{\alpha_j},x\right)\right)=R^*(x). \tag{169}$$

Based on (168) and (169) we derive (143). ■

We finish with

Theorem 15 *Let* $f\in AC^N\left(\mathbb{R}^d\right)\cap C^{N-1}\left(\mathbb{R}^d\right)$, $d\in\mathbb{N}-\{1\}$, $N\in\mathbb{N}$, *with* $\|f_{\widetilde{\alpha}}\|_{\infty,\mathbb{R}^d,N}^{\max}<\infty$. *Here* $n\geq\max_{j\in\{1,\ldots,d\}}\left(T_j+|x_j|,T_j^{-\frac{1}{\alpha}}\right)$, *where* $x\in\mathbb{R}^d$, $0<\alpha<1$, $n\in\mathbb{N}$, $T_j>0$. *Then*

$$\left(M_n^*(f)\right)(x)-f(x)=$$

$$\sum_{l=1}^{N-1}\left(\sum_{|\widetilde{\alpha}|=l}\left(\frac{f_{\widetilde{\alpha}(x)}}{\prod_{j=1}^{d}\alpha_j!}\right)M_n^*\left(\prod_{j=1}^{d}(\cdot-x_j)^{\alpha_j},x\right)\right)+o\left(\frac{1}{n^{(N-\varepsilon)(1-\alpha)}}\right), \tag{170}$$

where $0<\varepsilon\leq N$.

If $N=1$, *the sum in (170) collapses.*

The last (170) implies that

$$n^{(N-\varepsilon)(1-\alpha)}\cdot$$

$$\left[\left(M_n^*(f)\right)(x)-f(x)-\sum_{l=1}^{N-1}\left(\sum_{|\widetilde{\alpha}|=l}\left(\frac{f_{\widetilde{\alpha}(x)}}{\prod_{j=1}^{d}\alpha_j!}\right)M_n^*\left(\prod_{j=1}^{d}(\cdot-x_j)^{\alpha_j},x\right)\right)\right]\to 0, \tag{171}$$

as $n\to\infty$, $0<\varepsilon\leq N$.

When $N=1$, *or* $f_{\widetilde{\alpha}}(x)=0$, *for all* $\widetilde{\alpha}:|\widetilde{\alpha}|=l=1,...,N-1$, *then we derive*

$$n^{(N-\varepsilon)(1-\alpha)}\left[\left(M_n^*(f)\right)(x)-f(x)\right]\to 0, \tag{172}$$

as $n\to\infty$, $0<\varepsilon\leq N$.

Proof. Set

$$g_{\frac{k}{n}+\frac{i}{nr}}(t)=f\left(x+t\left(\frac{k}{n}+\frac{i}{nr}-x\right)\right),\quad 0\leq t\leq 1. \tag{173}$$

Then we have

$$g_{\frac{k}{n}+\frac{i}{nr}}^{(l)}(t)=\left[\left(\sum_{j=1}^{d}\left(\frac{k_j}{n}+\frac{i_j}{nr_j}-x_j\right)\frac{\partial}{\partial x_j}\right)^l f\right]\left(x+t\left(\frac{k}{n}+\frac{i}{nr}-x\right)\right), \tag{174}$$

$l = 0, ..., N$, and

$$g_{\frac{k}{n}+\frac{i}{nr}}(0) = f(x). \tag{175}$$

By Taylor's formula, we get

$$f\left(\frac{k}{n}+\frac{i}{nr}\right) = g_{\frac{k}{n}+\frac{i}{nr}}(1) =$$

$$\sum_{l=0}^{N-1} \frac{g^{(l)}_{\frac{k}{n}+\frac{i}{nr}}(0)}{l!} + R_N\left(\frac{k}{n}+\frac{i}{nr}, 0\right), \tag{176}$$

where

$$R_N\left(\frac{k}{n}+\frac{i}{nr}, 0\right) = \frac{1}{(N-1)!}\int_0^1 (1-\theta)^N g^{(N)}_{\frac{k}{n}+\frac{i}{nr}}(\theta)\, d\theta. \tag{177}$$

Here we denote by

$$f_{\widetilde{\alpha}} := \frac{\partial^{\widetilde{\alpha}} f}{\partial x^{\widetilde{\alpha}}}, \quad \widetilde{\alpha} := (\alpha_1, ..., \alpha_d), \ \ \alpha_j \in \mathbb{Z}^+, \ j = 1, ..., d, \tag{178}$$

such that $|\widetilde{\alpha}| = \sum_{j=1}^{d} \alpha_j = l$, $0 \leq l \leq N$.

More precisely we can rewrite

$$f\left(\frac{k}{n}+\frac{i}{nr}\right) - f(x) =$$

$$\sum_{l=1}^{N-1} \sum_{\widetilde{\alpha}:|\widetilde{\alpha}|=l} \left(\frac{f_{\widetilde{\alpha}}(x)}{\prod_{j=1}^{d} \alpha_j!}\right) \left(\prod_{j=1}^{d} \left(\frac{k_j}{n}+\frac{i_j}{nr_j}-x_j\right)^{\alpha_j}\right)$$

$$+R_N\left(\frac{k}{n}+\frac{i}{nr}, 0\right), \tag{179}$$

where

$$R_N\left(\frac{k}{n}+\frac{i}{nr}, 0\right) = N\int_0^1 (1-\theta)^{N-1} \cdot$$

$$\sum_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \left(\frac{1}{\prod_{j=1}^{d} \alpha_j!}\right) \left(\prod_{j=1}^{d} \left(\frac{k_j}{n}+\frac{i_j}{nr_j}-x_j\right)^{\alpha_j}\right) \cdot \tag{180}$$

$$f_{\widetilde{\alpha}}\left(x+\theta\left(\frac{k}{n}+\frac{i}{nr}-x\right)\right) d\theta.$$

Thus

$$\left(M_n^*(f)\right)(x) = \frac{\sum_{k=\lceil nx-Tn^\alpha \rceil}^{[nx+Tn^\alpha]} \left(\sum_{i=1}^{r} w_i f\left(\frac{k}{n}+\frac{i}{nr}\right)\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} =$$

$$\frac{\sum_{l=0}^{N-1} \frac{1}{l!} \sum_{k=\lceil nx-Tn^\alpha \rceil}^{[nx+Tn^\alpha]} \left(\sum_{i=1}^{r} w_i g_{\frac{k}{n}+\frac{i}{nr}}^{(l)}(0)\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} + \tag{181}$$

$$\frac{\sum_{k=\lceil nx-Tn^\alpha \rceil}^{[nx+Tn^\alpha]} \left(\sum_{i=1}^{r} w_i R_N\left(\frac{k}{n}+\frac{i}{nr},0\right)\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}.$$

Therefore it holds

$$\left(M_n^*(f)\right)(x) - f(x) -$$

$$\sum_{l=1}^{N-1} \frac{1}{l!} \frac{\sum_{k=\lceil nx-Tn^\alpha \rceil}^{[nx+Tn^\alpha]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \left(\sum_{i=1}^{r} w_i g_{\frac{k}{n}+\frac{i}{nr}}^{(l)}(0)\right) = R^*(x), \tag{182}$$

where

$$R^*(x) = \sum_{k=\lceil nx-Tn^\alpha \rceil}^{[nx+Tn^\alpha]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \left(\sum_{i=1}^{r} w_i R_N\left(\frac{k}{n}+\frac{i}{nr},0\right)\right). \tag{183}$$

By (115) we get that

$$\left\| g_{\frac{k}{n}+\frac{i}{nr}}^{(N)} \right\|_{\infty,[0,1]} \leq \left(\left\| \frac{k}{n}+\frac{i}{nr}-x \right\|_{l_1} \right)^N \|f_{\widetilde{\alpha}}\|_{\infty,\mathbb{R}^d,N}^{\max} < \infty. \tag{184}$$

And furthermore it holds

$$\left| R_N\left(\frac{k}{n}+\frac{i}{nr},0\right) \right| \leq$$

$$\frac{\left(\left\|\frac{k}{n}+\frac{i}{nr}-x\right\|_{l_1}\right)^N}{N!} \|f_{\widetilde{\alpha}}\|_{\infty,\mathbb{R}^d,N}^{\max}. \tag{185}$$

By (104) we obtain

$$\left\| \frac{k}{n}+\frac{i}{nr}-x \right\|_{l_1} \leq d\left(\frac{T^*}{n^{1-\alpha}}+\frac{1}{n}\right), \tag{186}$$

and thus we have

$$\left| R_N\left(\frac{k}{n}+\frac{i}{nr},0\right) \right| \leq \left(\frac{d^N}{N!} \|f_{\widetilde{\alpha}}\|_{\infty,\mathbb{R}^d,N}^{\max}\right) \left(\frac{T^*}{n^{1-\alpha}}+\frac{1}{n}\right)^N. \tag{187}$$

Clearly we get also that

$$|R^*(x)| \leq \left(\frac{d^N}{N!}\left\|f_{\widetilde{\alpha}}\right\|_{\infty,\mathbb{R}^d,N}^{\max}\right)\left(\frac{T^*}{n^{1-\alpha}}+\frac{1}{n^{1-\alpha}n^{\alpha}}\right)^N = \tag{188}$$

$$\left(\frac{d^N}{N!}\left\|f_{\widetilde{\alpha}}\right\|_{\infty,\mathbb{R}^d,N}^{\max}\right)\left(T^*+\frac{1}{n^{\alpha}}\right)^N \frac{1}{n^{(1-\alpha)N}} \leq$$

$$\left[\left(\frac{d^N}{N!}\left\|f_{\widetilde{\alpha}}\right\|_{\infty,\mathbb{R}^d,N}^{\max}\right)(T^*+1)^N\right]\frac{1}{n^{(1-\alpha)N}}. \tag{189}$$

That is we find

$$|R^*(x)| \leq \left[\left(\frac{d^N}{N!}\left\|f_{\widetilde{\alpha}}\right\|_{\infty,\mathbb{R}^d,N}^{\max}\right)(1+T^*)^N\right]\frac{1}{n^{(1-\alpha)N}} =: \frac{E}{n^{(1-\alpha)N}}. \tag{190}$$

That is we proved

$$|R^*(x)| = O\left(\frac{1}{n^{(1-\alpha)N}}\right), \tag{191}$$

and

$$|R^*(x)| = o(1). \tag{192}$$

And, letting $0 < \varepsilon \leq N$, we derive

$$\frac{|R^*(x)|}{\left(\frac{1}{n^{(N-\varepsilon)(1-\alpha)}}\right)} \leq \frac{E}{n^{(1-\alpha)N}}n^{(N-\varepsilon)(1-\alpha)} = \frac{E}{n^{\varepsilon(1-\alpha)}} \to 0, \tag{193}$$

as $n \to \infty$.

I.e.

$$|R^*(x)| = o\left(\frac{1}{n^{(N-\varepsilon)(1-\alpha)}}\right). \tag{194}$$

Clearly here we can rewrite (182), as

$$(M_n^*(f))(x) - f(x) - \sum_{l=1}^{N-1}\left(\sum_{\widetilde{\alpha}:|\widetilde{\alpha}|=l}\left(\frac{f_{\widetilde{\alpha}}(x)}{\prod_{j=1}^{d}\alpha_j!}\right)M_n^*\left(\prod_{j=1}^{d}(\cdot - x_j)^{\alpha_j}, x\right)\right) = R^*(x). \tag{195}$$

Based on (194) and (195) we derive (170). ■

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