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Approximation by Perturbed Neural Network Operators

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Abstract

This article deals with the determination of the rate of convergence to the unit of each of three newly introduced here perturbed normalized neural network operators of one hidden layer. These are given through the modulus of continuity of the involved function or its high order derivative and that appears in the right-hand side of the associated Jackson type inequalities. The activation function is very general, especially it can derive from any sigmoid or bell-shaped function. The right hand sides of our convergence inequalities do not depend on the activation function. The sample functionals are of Stancu, Kantorovich and Quadrature types. We give applications for the first derivative of the involved function.

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1 Introduction

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle a_j \cdot x\rangle + b_j\right), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x\rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental network models, the activation function is the sigmoidal function of logistic type or other sigmoidal function or bell-shaped function.

It is well known that FNNs are universal approximators. Theoretically, any continuous function defined on a compact set can be approximated to any desired degree of accuracy by increasing the number of hidden neurons. It was proved by Cybenko [14] and Funahashi [16], that any continuous function can be approximated on a compact set with uniform topology by a network of the form $N_n(x)$, using any continuous, sigmoidal activation function. Hornik et al. in [19], have shown that any measurable function can be approached with such a network. Furthermore, these authors proved in [20], that any function of the Sobolev spaces can be approached with all derivatives. A variety of density results on FNN approximations to multivariate functions were later established by many authors using different methods, for more or less general situations: [21] by Leshno et al., [25] by Mhaskar and Micchelli, [11] by Chui and Li, [10] by Chen and Chen, [17] by Hahm and Hong, etc.

Usually these results only give theorems about the existence of an approximation. A related and important problem is that of complexity: determining the number of neurons required to guarantee that all functions belonging to a space can be approximated to the prescribed degree of accuracy ϵ.

Barron [6] shows that if the function is supposed to satisfy certain conditions expressed in terms of its Fourier transform, and if each of the neurons evaluates a sigmoidal activation function, then at most $O\left(\epsilon^{-2}\right)$ neurons are needed to achieve the order of approximation ϵ. Some other authors have published similar results on the complexity of FNN approximations: Mhaskar and Micchelli [26], Suzuki [29], Maiorov and Meir [22], Makovoz [23], Ferrari and Stengel [15], Xu and Cao [30], Cao et al. [7], etc.

P. Cardaliaguet and G. Euvrard were the first, see [8], to describe precisely and study neural network approximation operators to the unit operator. Namely they proved: be given $f : \mathbb{R} \to \mathbb{R}$ a continuous bounded function and b a centered bell-shaped function, then the functions

$$F_n(x) = \sum_{k=-n^2}^{n^2} \frac{f\left(\frac{k}{n}\right)}{In^{\alpha}} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right),$$

where $I := \int_{-\infty}^{\infty} b(t)\,dt$, $0 < \alpha < 1$, converge uniformly on compacta to f.

You see above that the weights $\frac{f\left(\frac{k}{n}\right)}{In^{\alpha}}$ are explicitly known, for the first time shown in [8].

Still the work [8] is qualitative and not quantitative.

The author in [1], [2] and [3], see chapters 2-5, was the first to establish neural network approximations to continuous functions with rates, that is quantitative works, by very specifically defined neural network operators of Cardaliagnet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" function are assumed to be of compact support. Also in [3] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see chapters 4-5 there.

Though the work in [1], [2], [3], was quantitative, the rate of convergence was not precisely determined.

Finally the author in [4], [5], by normalizing his operators he achieved to determine the exact rates of convergence.

In this article the author continuous and completes his related work, by introducing three new perturbed neural network operators of Cardaliaguet-Euvrard type.

The sample coefficients $f\left(\frac{k}{n}\right)$ are replaced by three suitable natural perturbations, what is actually happens in reality of a neural network operation.

The calculation of $f\left(\frac{k}{n}\right)$ at the neurons many times are not calculated as such, but rather in a distored way.

Next we justify why we take here the activation function to be of compact support, of course it helps us to conduct our study.

The activation function, same as transfer function or learning rule, is connected and associated to firing of neurons. Firing, which sends electric pulses or an output signal to other neurons, occurs when the activation level is above the threshold level set by the learning rule.

Each Neural Network firing is essentially of finite time duration. Essentially the firing in time decays, but in practice sends positive energy over a finite time interval.

Thus by using an activation function of compact support, in practice we do not alter much of the good results of our approximation.

To be more precise, we may take the compact support to be a large symmetric to the origin interval. This is what happens in real time with the firing of neurons.

For more about neural networks in general we refer to [9], [12], [13], [18], [24], [27].

2 Basics

Here the activation function $b : \mathbb{R} \to \mathbb{R}_+$ is of compact support $[-T, T]$, $T > 0$. That is $b(x) > 0$ for any $x \in [-T, T]$, and clearly b may have jump discontinuities. Also the shape of the graph of b could be anything. Typically in neural networks approximation we take b as a sigmoidal function or bell-shaped function, of course here of compact support $[-T, T]$, $T > 0$.

Example 1 *(i) b can be the characteristic function on* $[-1,1]$,

(ii) b can be that hat function over $[-1,1]$, *i.e.,*

$$b(x)=\begin{cases}1+x, & -1\leq x\leq 0,\\ 1-x, & 0<x\leq 1,\\ 0, & \text{elsewhere,}\end{cases}$$

(iii) the truncated sigmoidals

$$b(x)=\begin{cases}\frac{1}{1+e^{-x}} \text{ or } \tanh x \text{ or } \operatorname{erf}(x), & \text{for } x\in[-T,T], \text{ with large } T>0,\\ 0, & x\in\mathbb{R}-[-T,T],\end{cases}$$

(iv) the truncated Gompertz function

$$b(x)=\begin{cases}e^{-\alpha e^{-\beta x}}, & x\in[-T,T];\ \alpha,\beta>0;\ \text{large } T>0,\\ 0, & x\in\mathbb{R}-[-T,T],\end{cases}$$

The Gompertz functions are also sigmoidal functions, with wide applications to many applied fields, e.g. demography and tumor growth modeling, etc.

So the general function b we will be using here covers all kinds of activation functions in neural network approximations.

Here we consider functions $f:\mathbb{R}\to\mathbb{R}$ that are either continuous and bounded, or uniformly continuous.

Let here the parameters $\mu,\nu\geq 0$; $\mu_i,\nu_i\geq 0$, $i=1,...,r\in\mathbb{N}$; $w_i\geq 0$: $\sum_{i=1}^{r} w_i=1$; $0<\alpha<1$, $x\in\mathbb{R}$, $n\in\mathbb{N}$.

We use here the first modulus of continuity

$$\omega_1(f,\delta):=\sup_{\substack{x,y\in\mathbb{R}\\ |x-y|\leq\delta}}|f(x)-f(y)|,$$

and given that f is uniformly continuous we get $\lim_{\delta\to 0}\omega_1(f,\delta)=0$.

In this article mainly we study the pointwise convergence with rates over $\mathbb{R}$, to the unit operator, of the following one hidden layer normalized neural network perturbed operators,

(i)

$$(H_n^*(f))(x)=\frac{\sum_{k=-n^2}^{n^2}\left(\sum_{i=1}^{r} w_i f\left(\frac{k+\mu_i}{n+\nu_i}\right)\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}, \tag{1}$$

(ii) the Kantorovich type

$$(K_n^*(f))(x)=\frac{\sum_{k=-n^2}^{n^2}\left(\sum_{i=1}^{r} w_i (n+\nu_i)\int_{\frac{k+\mu_i}{n+\nu_i}}^{\frac{k+\mu_i+1}{n+\nu_i}} f(t)\,dt\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}, \tag{2}$$

and

(iii) the quadrature type

$$\left(M_n^*(f)\right)(x) = \frac{\sum_{k=-n^2}^{n^2} \left(\sum_{i=1}^{r} w_i f\left(\frac{k}{n} + \frac{i}{nr}\right) \right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}. \tag{3}$$

Similar operators defined for bell-shaped functions and sample coefficients $f\left(\frac{k}{n}\right)$ were studied initially in [8], [1], [2], [3], [4], [5], etc.

Here we study the generalized perturbed cases of these operators.

Operator K_n^* in the corresponding Signal Processing context, represents the natural called "time-jitter" error, where the sample information is calculated in a perturbed neighborhood of $\frac{k+\mu}{n+\nu}$ rather than exactly at the node $\frac{k}{n}$.

The perturbed sample coefficients $f\left(\frac{k+\mu}{n+\nu}\right)$ with $0 \leq \mu \leq \nu$, were first used by D. Stancu [28], in a totally different context, generalizing Bernstein operators approximation on $C([0,1])$.

The terms in the ratio of sums (1), (2), (3) are nonzero, iff

$$\left| n^{1-\alpha}\left(x - \frac{k}{n}\right) \right| \leq T, \text{ i.e. } \left| x - \frac{k}{n} \right| \leq \frac{T}{n^{1-\alpha}} \tag{4}$$

iff

$$nx - Tn^{\alpha} \leq k \leq nx + Tn^{\alpha}. \tag{5}$$

In order to have the desired order of the numbers

$$-n^2 \leq nx - Tn^{\alpha} \leq nx + Tn^{\alpha} \leq n^2, \tag{6}$$

it is sufficiently enough to assume that

$$n \geq T + |x|. \tag{7}$$

When $x \in [-T, T]$ it is enough to assume $n \geq 2T$, which implies (6).

Proposition 2 *([1]) Let $a \leq b$, $a, b \in \mathbb{R}$. Let $card(k)$ (≥ 0) be the maximum number of integers contained in $[a,b]$. Then*

$$\max(0, (b-a)-1) \leq card(k) \leq (b-a)+1. \tag{8}$$

Note 3 *We would like to establish a lower bound on $card(k)$ over the interval $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$. From Proposition 2 we get that*

$$card(k) \geq \max(2Tn^{\alpha} - 1, 0). \tag{9}$$

We obtain $card(k) \geq 1$, if

$$2Tn^{\alpha} - 1 \geq 1 \quad \textit{iff } n \geq T^{-\frac{1}{\alpha}}. \tag{10}$$

So to have the desired order (6) and card $(k) \geq 1$ *over* $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$, *we need to consider*

$$n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right). \tag{11}$$

Also notice that card $(k) \to +\infty$, *as* $n \to +\infty$.

Denote by $[\cdot]$ *the integral part of a number and by* $\lceil \cdot \rceil$ *its ceiling.*

So under assumption (11), the operators H_n^*, K_n^*, M_n^*, *collapse to*

(i)

$$\left(H_n^*(f)\right)(x) = \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{r} w_i f\left(\frac{k+\mu_i}{n+\nu_i}\right)\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}, \tag{12}$$

(ii)

$$\left(K_n^*(f)\right)(x) = \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{r} w_i (n+\nu_i) \int_{\frac{k+\mu_i}{n+\nu_i}}^{\frac{k+\mu_i+1}{n+\nu_i}} f(t)\,dt\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}, \tag{13}$$

and

(iii)

$$\left(M_n^*(f)\right)(x) = \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{r} w_i f\left(\frac{k}{n} + \frac{i}{nr}\right)\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}. \tag{14}$$

We make

Remark 4 *Let* k *as in (5). We observe that*

$$\left|\frac{k+\mu}{n+\nu} - x\right| \leq \left|\frac{k}{n+\nu} - x\right| + \frac{\mu}{n+\nu}. \tag{15}$$

Next we see

$$\left|\frac{k}{n+\nu} - x\right| \leq \left|\frac{k}{n+\nu} - \frac{k}{n}\right| + \left|\frac{k}{n} - x\right| \overset{(4)}{\leq} \frac{\nu|k|}{n(n+\nu)} + \frac{T}{n^{1-\alpha}}$$

(by $|k| \leq \max\left(|nx - Tn^{\alpha}|, |nx + Tn^{\alpha}|\right) \leq n|x| + Tn^{\alpha}$*)*

$$\leq \left(\frac{\nu}{n+\nu}\right)\left(|x| + \frac{T}{n^{1-\alpha}}\right) + \frac{T}{n^{1-\alpha}}. \tag{16}$$

Consequently it holds

$$\left|\frac{k+\mu}{n+\nu} - x\right| \leq \left(\frac{\nu}{n+\nu}\right)\left(|x| + \frac{T}{n^{1-\alpha}}\right) + \frac{T}{n^{1-\alpha}} + \frac{\mu}{n+\nu} = \tag{17}$$

$$\left(\frac{\nu\left|x\right|+\mu}{n+\nu}\right)+\left(1+\frac{\nu}{n+\nu}\right)\frac{T}{n^{1-\alpha}}.$$

Hence we obtain

$$\omega_1\left(f,\left|\frac{k+\mu}{n+\nu}-x\right|\right)\overset{(17)}{\leq}\omega_1\left(f,\left(\frac{\nu\left|x\right|+\mu}{n+\nu}\right)+\left(1+\frac{\nu}{n+\nu}\right)\frac{T}{n^{1-\alpha}}\right), \quad (18)$$

where $\mu,\nu\geq 0$, $0<\alpha<1$, *so that the dominant speed above is* $\frac{1}{n^{1-\alpha}}$.

Also, by change of variable method, the operator K_n^* *could be written conveniently as follows:*

(ii)'

$$\left(K_n^*\left(f\right)\right)(x)=$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{r}w_i\left(n+\nu_i\right)\int_0^{\frac{1}{n+\nu_i}}f\left(t+\frac{k+\mu_i}{n+\nu_i}\right)dt\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}. \quad (19)$$

3 Main Results

We present our first approximation result

Theorem 5 *Let* $x\in\mathbb{R}$, $T>0$ *and* $n\in\mathbb{N}$ *such that* $n\geq\max\left(T+\left|x\right|,T^{-\frac{1}{\alpha}}\right)$. *Then*

$$\left|\left(H_n^*\left(f\right)\right)(x)-f\left(x\right)\right|\leq\sum_{i=1}^{r}w_i\omega_1\left(f,\left(\frac{\nu_i\left|x\right|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)$$

$$\leq\max_{i\in\{1,\ldots,r\}}\left\{\omega_1\left(f,\left(\frac{\nu_i\left|x\right|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)\right\}. \quad (20)$$

Proof. We notice that

$$\left(H_n^*\left(f\right)\right)(x)-f\left(x\right)=\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{r}w_if\left(\frac{k+\mu_i}{n+\nu_i}\right)\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}-f\left(x\right) \quad (21)$$

$$=\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{r}w_if\left(\frac{k+\mu_i}{n+\nu_i}\right)\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-f(x)\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}=$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\left(\sum_{i=1}^{r}w_if\left(\frac{k+\mu_i}{n+\nu_i}\right)\right)-f\left(x\right)\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}= \quad (22)$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{r} w_i\left(f\left(\frac{k+\mu_i}{n+\nu_i}\right)-f(x)\right)\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}.$$

Hence it holds

$$\left|\left(H_n^*(f)\right)(x)-f(x)\right| \leq$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{r} w_i\left|f\left(\frac{k+\mu_i}{n+\nu_i}\right)-f(x)\right|\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)} \leq \tag{23}$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{r} w_i\omega_1\left(f,\left|\frac{k+\mu_i}{n+\nu_i}-x\right|\right)\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)} \overset{(18)}{\leq}$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{r} w_i\omega_1\left(f,\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)} = \tag{24}$$

$$\frac{\left[\sum_{i=1}^{r} w_i\omega_1\left(f,\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)\right]\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}$$

$$=\sum_{i=1}^{r} w_i\omega_1\left(f,\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right) \leq$$

$$\max_{i\in\{1,\ldots,r\}}\left\{\omega_1\left(f,\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)\right\}, \tag{25}$$

proving the claim. ■

Corollary 6 *Let* $x\in[-T^*,T^*]$, $T^*>0$, $n\in\mathbb{N}: n\geq\max\left(T+T^*,T^{-\frac{1}{\alpha}}\right)$, $T>0$. *Then*

$$\left\|H_n^*(f)-f\right\|_{\infty,[-T^*,T^*]}\leq\sum_{i=1}^{r} w_i\omega_1\left(f,\left(\frac{\nu_iT^*+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right) \tag{26}$$

$$\leq\max_{i\in\{1,\ldots,r\}}\left\{\omega_1\left(f,\left(\frac{\nu_iT^*+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)\right\}.$$

Proof. By (20). ■

We continue with

Theorem 7 *Let $x \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. Then*

$$|(K_n^*(f))(x) - f(x)| \leq \tag{27}$$

$$\max_{i \in \{1,\dots,r\}} \left\{ \omega_1 \left(f, \left(\frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \right\}.$$

Proof. Call

$$\delta_{n,k}(f) = \sum_{i=1}^{r} w_i (n + \nu_i) \int_0^{\frac{1}{n+\nu_i}} f\left(t + \frac{k + \mu_i}{n + \nu_i}\right) dt. \tag{28}$$

We observe that

$$\delta_{n,k}(f) - f(x) = \sum_{i=1}^{r} w_i (n + \nu_i) \int_0^{\frac{1}{n+\nu_i}} f\left(t + \frac{k + \mu_i}{n + \nu_i}\right) dt - f(x) =$$

$$\sum_{i=1}^{r} w_i (n + \nu_i) \int_0^{\frac{1}{n+\nu_i}} \left(f\left(t + \frac{k + \mu_i}{n + \nu_i}\right) - f(x) \right) dt. \tag{29}$$

Hence it holds

$$|\delta_{n,k}(f) - f(x)| \leq \sum_{i=1}^{r} w_i (n + \nu_i) \int_0^{\frac{1}{n+\nu_i}} \left| f\left(t + \frac{k + \mu_i}{n + \nu_i}\right) - f(x) \right| dt \leq$$

$$\sum_{i=1}^{r} w_i (n + \nu_i) \int_0^{\frac{1}{n+\nu_i}} \omega_1 \left(f, |t| + \left| \frac{k + \mu_i}{n + \nu_i} - x \right| \right) dt \leq \tag{30}$$

$$\sum_{i=1}^{r} w_i \omega_1 \left(f, \frac{1}{n + \nu_i} + \left| \frac{k + \mu_i}{n + \nu_i} - x \right| \right) \overset{(17)}{\leq}$$

$$\sum_{i=1}^{r} w_i \omega_1 \left(f, \left(\frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \leq \tag{31}$$

$$\max_{i \in \{1,\dots,r\}} \left\{ \omega_1 \left(f, \left(\frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \right\}.$$

We proved that

$$|\delta_{n,k}(f) - f(x)| \leq$$

$$\max_{i \in \{1,\dots,r\}} \left\{ \omega_1 \left(f, \left(\frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \right\}. \tag{32}$$

Therefore by (19) and (28) we get

$$(K_n^*(f))(x) - f(x) = \frac{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx + Tn^\alpha]} \delta_{n,k}(f)\, b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx + Tn^\alpha]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)} - f(x) =$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\delta_{n,k}(f)\,b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)-f(x)\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}$$

$$=\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\delta_{n,k}(f)-f(x)\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}. \tag{33}$$

Consequently we obtain

$$\left|\left(K_n^*(f)\right)(x)-f(x)\right|\leq\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left|\delta_{n,k}(f)-f(x)\right|b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)} \tag{34}$$

$$\overset{(32)}{\leq}\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right)\cdot$$

$$\frac{\max_{i\in\{1,\ldots,r\}}\left\{\omega_1\left(f,\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)\right\}}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}=$$

$$\max_{i\in\{1,\ldots,r\}}\left\{\omega_1\left(f,\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)\right\}, \tag{35}$$

proving the claim. ■

Corollary 8 *Let $x\in[-T^*,T^*]$, $T^*>0$, $n\in\mathbb{N}:n\geq\max\left(T+T^*,T^{-\frac{1}{\alpha}}\right)$, $T>0$. Then*

$$\left\|K_n^*(f)-f\right\|_{\infty,[-T^*,T^*]}\leq \tag{36}$$

$$\max_{i\in\{1,\ldots,r\}}\left\{\omega_1\left(f,\left(\frac{\nu_iT^*+\mu_i+1}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)\right\}.$$

Proof. By (27). ■

We also give

Theorem 9 *Let $x\in\mathbb{R}$, $T>0$ and $n\in\mathbb{N}$ such that $n\geq\max\left(T+|x|,T^{-\frac{1}{\alpha}}\right)$. Then*

$$\left|M_n^*(f)(x)-f(x)\right|\leq\omega_1\left(f,\frac{1}{n}+\frac{T}{n^{1-\alpha}}\right). \tag{37}$$

Proof. Let k as in (5). Set

$$\lambda_{nk}(f)=\sum_{i=1}^{r}w_if\left(\frac{k}{n}+\frac{i}{nr}\right),$$

then

$$\lambda_{nk}(f)-f(x)=\sum_{i=1}^{r}w_i\left(f\left(\frac{k}{n}+\frac{i}{nr}\right)-f(x)\right). \tag{38}$$

Then

$$|\lambda_{nk}(f) - f(x)| \leq \sum_{i=1}^{r} w_i \left| f\left(\frac{k}{n} + \frac{i}{nr}\right) - f(x) \right| \leq$$

$$\sum_{i=1}^{r} w_i \omega_1 \left(f, \left| \frac{k}{n} - x \right| + \frac{i}{nr} \right) \leq \tag{39}$$

$$\sum_{i=1}^{r} w_i \omega_1 \left(f, \frac{T}{n^{1-\alpha}} + \frac{1}{n} \right) = \omega_1 \left(f, \frac{1}{n} + \frac{T}{n^{1-\alpha}} \right).$$

Hence it holds

$$|\lambda_{nk}(f) - f(x)| \leq \omega_1 \left(f, \frac{1}{n} + \frac{T}{n^{1-\alpha}} \right). \tag{40}$$

By (14) we can write and use next

$$(M_n^*(f))(x) = \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \lambda_{nk}(f)\, b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}. \tag{41}$$

That is we have

$$M_n^*(f)(x) - f(x) = \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} (\lambda_{nk}(f) - f(x))\, b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}. \tag{42}$$

Hence we easily derive by (40), as before, that

$$|M_n^*(f)(x) - f(x)| \leq \omega_1 \left(f, \frac{1}{n} + \frac{T}{n^{1-\alpha}} \right), \tag{43}$$

proving the claim. ■

Corollary 10 *Let* $x \in [-T^*, T^*]$, $T^* > 0$, $n \in \mathbb{N} : n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}}\right)$, $T > 0$. *Then*

$$\|M_n^*(f) - f\|_{\infty,[-T^*,T^*]} \leq \omega_1 \left(f, \frac{1}{n} + \frac{T}{n^{1-\alpha}} \right). \tag{44}$$

Proof. By (37). ■

Theorems 5, 7, 9 and Corollaries 6, 8, 10 given that f is uniformly continuous, produce the pointwise and uniform convergences with rates, at speed $\frac{1}{n^{1-\alpha}}$, of neural network operators H_n^*, K_n^*, M_n^* to the unit operator. Notice that the right hand sides of inequalities (20), (26), (27), (36), (37) and (44) do not depend on b.

We proceed to the following results where we use the smoothness of a derivative of f.

Theorem 11 *Let $x \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + |x|, T^{-\frac{1}{\alpha}}\right)$. Let $f \in C^N(\mathbb{R})$, $N \in \mathbb{N}$, such that $f^{(N)}$ is uniformly continuous or is continuous and bounded. Then*

$$|(H_n^*(f))(x) - f(x)| \leq$$

$$\sum_{j=1}^{N} \frac{\left|f^{(j)}(x)\right|}{j!} \left\{ \sum_{i=1}^{r} w_i \left[\left(\frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right]^j \right\} + \tag{45}$$

$$\sum_{i=1}^{r} w_i \omega_1 \left(f^{(N)}, \left(\frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \cdot$$

$$\frac{\left(\left(\frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right)^N}{N!}.$$

Inequality (45) implies the pointwise convergence with rates of $(H_n^*(f))(x)$to $f(x)$, as $n \to \infty$, at speed $\frac{1}{n^{1-\alpha}}$.

Proof. Let k as in (5). We observe that

$$w_i f\left(\frac{k + \mu_i}{n + \nu_i} \right) = \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} w_i \left(\frac{k + \mu_i}{n + \nu_i} - x \right)^j + \tag{46}$$

$$w_i \int_x^{\frac{k+\mu_i}{n+\nu_i}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{k+\mu_i}{n+\nu_i} - t \right)^{N-1}}{(N-1)!} dt, \quad i = 1, ..., r.$$

Call

$$V(x) = \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx + Tn^{\alpha}]} b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right). \tag{47}$$

Hence

$$\frac{\left(\sum_{i=1}^{r} w_i f\left(\frac{k+\mu_i}{n+\nu_i} \right) \right) b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}{V(x)} =$$

$$\sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} \left(\sum_{i=1}^{r} w_i \left(\frac{k + \mu_i}{n + \nu_i} - x \right)^j \right) \frac{b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}{V(x)} +$$

$$\frac{b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}{V(x)} \left(\sum_{i=1}^{r} w_i \int_x^{\frac{k+\mu_i}{n+\nu_i}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{k+\mu_i}{n+\nu_i} - t \right)^{N-1}}{(N-1)!} dt \right). \tag{48}$$

Therefore it holds (see (12))

$$(H_n^*(f))(x) - f(x) = \tag{49}$$

$$\sum_{j=1}^{N}\frac{f^{(j)}(x)}{j!}\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left\{\left(\sum_{i=1}^{r}w_i\left(\frac{k+\mu_i}{n+\nu_i}-x\right)^{j}\right)\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\right\}\right)+R,$$

where

$$R=\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\cdot \tag{50}$$

$$\left(\sum_{i=1}^{r}w_i\int_{x}^{\frac{k+\mu_i}{n+\nu_i}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k+\mu_i}{n+\nu_i}-t\right)^{N-1}}{(N-1)!}dt\right).$$

So that

$$|(H_n^{*}(f))(x)-f(x)|\leq \tag{51}$$

$$\sum_{j=1}^{N}\frac{\left|f^{(j)}(x)\right|}{j!}\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left\{\left(\sum_{i=1}^{r}w_i\left|\frac{k+\mu_i}{n+\nu_i}-x\right|^{j}\right)\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\right\}\right)+|R|\leq$$

$$\sum_{j=1}^{N}\frac{\left|f^{(j)}(x)\right|}{j!}\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left\{\left(\sum_{i=1}^{r}w_i\left[\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right]^{j}\right)\right.\right. \tag{52}$$

$$\left.\left.\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\right\}\right)+|R|=$$

$$\sum_{j=0}^{N}\frac{\left|f^{(j)}(x)\right|}{j!}\left\{\sum_{i=1}^{r}w_i\left[\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right]^{j}\right\}+|R|. \tag{53}$$

So that thus far we have

$$|(H_n^{*}(f))(x)-f(x)|\leq$$

$$\sum_{j=0}^{N}\frac{\left|f^{(j)}(x)\right|}{j!}\left\{\sum_{i=1}^{r}w_i\left[\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right]^{j}\right\}+|R|. \tag{54}$$

Furthermore we see

$$|R|\leq\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\cdot \tag{55}$$

$$\left(\sum_{i=1}^{r}w_i\left|\int_{x}^{\frac{k+\mu_i}{n+\nu_i}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k+\mu_i}{n+\nu_i}-t\right)^{N-1}}{(N-1)!}dt\right|\right)\leq$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\gamma, \tag{56}$$

where

$$\gamma := \sum_{i=1}^{r} w_i \left| \int_x^{\frac{k+\mu_i}{n+\nu_i}} \left| f^{(N)}(t) - f^{(N)}(x) \right| \frac{\left| \frac{k+\mu_i}{n+\nu_i} - t \right|^{N-1}}{(N-1)!} dt \right|. \tag{57}$$

(i) Let $x \leq \frac{k+\mu_i}{n+\nu_i}$, then

$$\varepsilon_i := \int_x^{\frac{k+\mu_i}{n+\nu_i}} \left| f^{(N)}(t) - f^{(N)}(x) \right| \frac{\left| \frac{k+\mu_i}{n+\nu_i} - t \right|^{N-1}}{(N-1)!} dt \leq \tag{58}$$

$$\omega_1 \left(f^{(N)}, \left| \frac{k+\mu_i}{n+\nu_i} - x \right| \right) \int_x^{\frac{k+\mu_i}{n+\nu_i}} \frac{\left(\frac{k+\mu_i}{n+\nu_i} - t \right)^{N-1}}{(N-1)!} dt =$$

$$\omega_1 \left(f^{(N)}, \left| \frac{k+\mu_i}{n+\nu_i} - x \right| \right) \frac{\left| \left(\frac{k+\mu_i}{n+\nu_i} - t \right) \right|^{N}}{N!} \overset{(17)}{\leq}$$

$$\omega_1 \left(f^{(N)}, \left(\frac{\nu_i |x| + \mu_i}{n+\nu_i} \right) + \left(1 + \frac{\nu_i}{n+\nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \cdot$$

$$\frac{\left[\left(\frac{\nu_i |x| + \mu_i}{n+\nu_i} \right) + \left(1 + \frac{\nu_i}{n+\nu_i} \right) \frac{T}{n^{1-\alpha}} \right]^N}{N!}. \tag{59}$$

So when $x \leq \frac{k+\mu_i}{n+\nu_i}$, we got

$$\varepsilon_i \leq \omega_1 \left(f^{(N)}, \left(\frac{\nu_i |x| + \mu_i}{n+\nu_i} \right) + \left(1 + \frac{\nu_i}{n+\nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \cdot$$

$$\frac{\left[\left(\frac{\nu_i |x| + \mu_i}{n+\nu_i} \right) + \left(1 + \frac{\nu_i}{n+\nu_i} \right) \frac{T}{n^{1-\alpha}} \right]^N}{N!}. \tag{60}$$

ii) Let $x > \frac{k+\mu_i}{n+\nu_i}$, then

$$\rho_i := \int_{\frac{k+\mu_i}{n+\nu_i}}^{x} \left| f^{(N)}(t) - f^{(N)}(x) \right| \frac{\left(t - \frac{k+\mu_i}{n+\nu_i} \right)^{N-1}}{(N-1)!} dt \leq$$

$$\omega_1 \left(f^{(N)}, \left| \frac{k+\mu_i}{n+\nu_i} - x \right| \right) \frac{\left(x - \frac{k+\mu_i}{n+\nu_i} \right)^{N}}{N!} = \tag{61}$$

$$\omega_1 \left(f^{(N)}, \left| \frac{k+\mu_i}{n+\nu_i} - x \right| \right) \frac{\left| \frac{k+\mu_i}{n+\nu_i} - x \right|^{N}}{N!} \leq$$

$$\omega_1 \left(f^{(N)}, \left(\frac{\nu_i |x| + \mu_i}{n+\nu_i} \right) + \left(1 + \frac{\nu_i}{n+\nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \cdot$$

$$\frac{\left(\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!}. \tag{62}$$

Hence when $x > \frac{k+\mu_i}{n+\nu_i}$, then

$$\rho_i \leq \omega_1\left(f^{(N)}, \left(\frac{\nu_i |x| + \mu_i}{n+\nu_i}\right) + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right) \cdot$$

$$\frac{\left(\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!}. \tag{63}$$

Notice in (60) and (63) we obtained the same upper bound. Hence it holds

$$\gamma \leq \sum_{i=1}^{r} w_i \omega_1\left(f^{(N)}, \left(\frac{\nu_i |x| + \mu_i}{n+\nu_i}\right) + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right) \cdot$$

$$\frac{\left(\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!} =: E. \tag{64}$$

Thus

$$|R| \leq E, \tag{65}$$

proving the claim. ∎

Corollary 12 *All as in Theorem 11, plus $f^{(j)}(x) = 0$, $j = 1, ..., N$. Then*

$$|(H_n^*(f))(x) - f(x)| \leq$$

$$\sum_{i=1}^{r} w_i \omega_1\left(f^{(N)}, \left(\frac{\nu_i |x| + \mu_i}{n+\nu_i}\right) + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right) \cdot$$

$$\frac{\left(\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!}. \tag{66}$$

Proof. By (49), (50), (64) and (65). ∎

In (66) notice the extremely high speed of convergence $\frac{1}{n^{(1-\alpha)(N+1)}}$.

The uniform convergence with rates follows from

Corollary 13 *Let $x \in [-T^*, T^*]$, $T^* > 0$; $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}}\right)$. Let $f \in C^N(\mathbb{R})$, $N \in \mathbb{N}$, such that $f^{(N)}$ is uniformly continuous or is continuous and bounded. Then*

$$\|H_n^*(f) - f\|_{\infty,[-T^*,T^*]} \leq \tag{67}$$

$$\sum_{j=1}^{N} \frac{\left\|f^{(j)}\right\|_{\infty,[-T^*,T^*]}}{j!}\left\{\sum_{i=1}^{r} w_i \left[\left(\frac{\nu_i T^* + \mu_i}{n+\nu_i}\right) + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right]^j\right\} +$$

$$\sum_{i=1}^{r} w_i \omega_1 \left(f^{(N)}, \left(\frac{\nu_i T^* + \mu_i}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \cdot$$

$$\frac{\left(\left(\frac{\nu_i T^* + \mu_i}{n+\nu_i} \right) + \left(1 + \frac{\nu_i}{n+\nu_i} \right) \frac{T}{n^{1-\alpha}} \right)^N}{N!}.$$

Proof. By (45). ■

Corollary 14 *All as in Theorem 11, case of $N = 1$. It holds*

$$\left| \left(H_n^* (f) \right) (x) - f (x) \right| \leq$$

$$\left| f' (x) \right| \left\{ \sum_{i=1}^{r} w_i \left[\left(\frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right] \right\} + \tag{68}$$

$$\sum_{i=1}^{r} w_i \omega_1 \left(f', \left(\frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \cdot$$

$$\left(\left(\frac{\nu_i |x| + \mu_i}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right).$$

We continue with

Theorem 15 *Same assumptions as in Theorem 11, with $0 < \alpha < 1$. Then*

$$\left| \left(K_n^* (f) \right) (x) - f (x) \right| \leq \sum_{j=1}^{N} \frac{\left| f^{(j)} (x) \right|}{j!} \cdot \tag{69}$$

$$\left(\sum_{i=1}^{r} w_i \left(\left(\frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right)^j \right) +$$

$$\sum_{i=1}^{r} w_i \left\{ \omega_1 \left(f^{(N)}, \left(\frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \cdot \right.$$

$$\left. \frac{\left(\left(\frac{\nu_i |x| + \mu_i + 1}{n+\nu_i} \right) + \left(1 + \frac{\nu_i}{n+\nu_i} \right) \frac{T}{n^{1-\alpha}} \right)^N}{N!} \right\}.$$

Inequality (69) implies the pointwise convergence with rates of $\left(K_n^* (f) \right) (x)$ to $f (x)$, as $n \to \infty$, at the speed $\frac{1}{n^{1-\alpha}}$.

Proof. Let k as in (5). We observe that

$$f \left(t + \frac{k + \mu_i}{n + \nu_i} \right) = \sum_{j=0}^{N} \frac{f^{(j)} (x)}{j!} \left(t + \frac{k + \mu_i}{n + \nu_i} - x \right)^j +$$

$$\int_x^{t+\frac{k+\mu_i}{n+\nu_i}} \left(f^{(N)}(z) - f^{(N)}(x)\right) \frac{\left(t + \frac{k+\mu_i}{n+\nu_i} - z\right)^{N-1}}{(N-1)!} dz, \tag{70}$$

$i = 1, ..., r$.

That is

$$\int_0^{\frac{1}{n+\nu_i}} f\left(t + \frac{k+\mu_i}{n+\nu_i}\right) dt =$$

$$\sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} \int_0^{\frac{1}{n+\nu_i}} \left(t + \frac{k+\mu_i}{n+\nu_i} - x\right)^j dt +$$

$$\int_0^{\frac{1}{n+\nu_i}} \left(\int_x^{t+\frac{k+\mu_i}{n+\nu_i}} \left(f^{(N)}(z) - f^{(N)}(x)\right) \frac{\left(t + \frac{k+\mu_i}{n+\nu_i} - z\right)^{N-1}}{(N-1)!} dz\right) dt, \tag{71}$$

$i = 1, ..., r$.

Furthermore we have

$$\sum_{i=1}^{r} w_i (n+\nu_i) \int_0^{\frac{1}{n+\nu_i}} f\left(t + \frac{k+\mu_i}{n+\nu_i}\right) dt =$$

$$\sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} \left(\sum_{i=1}^{r} w_i (n+\nu_i) \int_0^{\frac{1}{n+\nu_i}} \left(t + \frac{k+\mu_i}{n+\nu_i} - x\right)^j dt\right) +$$

$$\sum_{i=1}^{r} w_i (n+\nu_i) \int_0^{\frac{1}{n+\nu_i}} \left(\int_x^{t+\frac{k+\mu_i}{n+\nu_i}} \left(f^{(N)}(z) - f^{(N)}(x)\right) \frac{\left(t + \frac{k+\mu_i}{n+\nu_i} - z\right)^{N-1}}{(N-1)!} dz\right) dt. \tag{72}$$

Call

$$V(x) = \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right). \tag{73}$$

Consequently we get

$$(K_n^*(f))(x) =$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{r} w_i (n+\nu_i) \int_0^{\frac{1}{n+\nu_i}} f\left(t + \frac{k+\mu_i}{n+\nu_i}\right) dt\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)}$$

$$= \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)} \cdot$$

$$\left(\sum_{i=1}^{r} w_i (n+\nu_i) \int_0^{\frac{1}{n+\nu_i}} \left(t + \frac{k+\mu_i}{n+\nu_i} - x\right)^j dt\right) +$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \left(\sum_{i=1}^{r} w_i (n+\nu_i)\right.$$

$$\left.\int_0^{\frac{1}{n+\nu_i}} \left(\int_x^{t+\frac{k+\mu_i}{n+\nu_i}} \left(f^{(N)}(z) - f^{(N)}(x)\right) \frac{\left(t+\frac{k+\mu_i}{n+\nu_i}-z\right)^{N-1}}{(N-1)!} dz\right) dt\right). \tag{74}$$

Therefore it holds

$$\left(K_n^*(f)\right)(x) - f(x) = \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \cdot$$

$$\left(\sum_{i=1}^{r} w_i (n+\nu_i) \int_0^{\frac{1}{n+\nu_i}} \left(t+\frac{k+\mu_i}{n+\nu_i}-x\right)^j dt\right) + R, \tag{75}$$

where

$$R = \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \left(\sum_{i=1}^{r} w_i (n+\nu_i)\right.$$

$$\left.\int_0^{\frac{1}{n+\nu_i}} \left(\int_x^{t+\frac{k+\mu_i}{n+\nu_i}} \left(f^{(N)}(z) - f^{(N)}(x)\right) \frac{\left(t+\frac{k+\mu_i}{n+\nu_i}-z\right)^{N-1}}{(N-1)!} dz\right) dt\right). \tag{76}$$

We derive that

$$\left|\left(K_n^*(f)\right)(x) - f(x)\right| \leq \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \cdot \tag{77}$$

$$\left(\sum_{i=1}^{r} w_i (n+\nu_i) \int_0^{\frac{1}{n+\nu_i}} |t+\frac{k+\mu_i}{n+\nu_i}-x|^j dt\right) + |R| \overset{(17)}{\leq}$$

$$\sum_{j=1}^{N} \frac{\left|f^{(j)}(x)\right|}{j!} \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \cdot$$

$$\sum_{i=1}^{r} w_i [\left(\frac{\nu_i |x| + \mu_i + 1}{n+\nu_i}\right) + \left(1+\frac{\nu_i}{n+\nu_i}\right) \frac{T}{n^{1-\alpha}}]^j$$

$$+ |R|. \tag{78}$$

Above we used

$$|t+\frac{k+\mu_i}{n+\nu_i}-x|$$

$$\leq [\left(\frac{\nu_i |x| + \mu_i + 1}{n+\nu_i}\right) + \left(1+\frac{\nu_i}{n+\nu_i}\right) \frac{T}{n^{1-\alpha}}]. \tag{79}$$

We have found that

$$\left|\left(K_n^*(f)\right)(x)-f(x)\right| \leq$$

$$\sum_{j=1}^{N}\frac{\left|f^{(j)}(x)\right|}{j!}\sum_{i=1}^{r}w_i[\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}]^j$$

$$+|R|. \tag{80}$$

Notice that

$$[\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}]\to 0, as\ n\to\infty,\ 0<\alpha<1. \tag{81}$$

We observe that

$$|R| \overset{(76)}{\leq} \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\left(\sum_{i=1}^{r}w_i(n+\nu_i)\right. \tag{82}$$

$$\left.\int_0^{\frac{1}{n+\nu_i}}\left|\int_x^{t+\frac{k+\mu_i}{n+\nu_i}}\left|f^{(N)}(z)-f^{(N)}(x)\right|\frac{\left|t+\frac{k+\mu_i}{n+\nu_i}-z\right|^{N-1}}{(N-1)!}dz\right|dt\right)=:(\xi).$$

We distinguish the cases:

(i) if $t+\frac{k+\mu_i}{n+\nu_i}\geq x$, then

$$\theta_i:=\left|\int_x^{t+\frac{k+\mu_i}{n+\nu_i}}\left|f^{(N)}(t)-f^{(N)}(x)\right|\frac{\left|t+\frac{k+\mu_i}{n+\nu_i}-z\right|^{N-1}}{(N-1)!}dz\right|= \tag{83}$$

$$\int_x^{t+\frac{k+\mu_i}{n+\nu_i}}\left|f^{(N)}(t)-f^{(N)}(x)\right|\frac{\left(t+\frac{k+\mu_i}{n+\nu_i}-z\right)^{N-1}}{(N-1)!}dz\leq$$

$$\omega_1\left(f^{(N)},|t|+\left|\frac{k+\mu_i}{n+\nu_i}-x\right|\right)\frac{\left(|t|+\left|\frac{k+\mu_i}{n+\nu_i}-x\right|\right)^N}{N!}\overset{(17)}{\leq}$$

$$\omega_1\left(f^{(N)},\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)\cdot$$

$$\frac{\left(\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!}. \tag{84}$$

That is, if $t+\frac{k+\mu_i}{n+\nu_i}\geq x$, we proved that

$$\theta_i\leq\omega_1\left(f^{(N)},\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)\cdot$$

$$\frac{\left(\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!}. \tag{85}$$

ii) if $t+\frac{k+\mu_i}{n+\nu_i}<x$, then

$$\theta_i := \int_{t+\frac{k+\mu_i}{n+\nu_i}}^{x} \left|f^{(N)}(z)-f^{(N)}(x)\right| \frac{\left(z-\left(t+\frac{k+\mu_i}{n+\nu_i}\right)\right)^{N-1}}{(N-1)!}dz \leq$$

$$\omega_1\left(f^{(N)},|t|+\left|\frac{k+\mu_i}{n+\nu_i}-x\right|\right)\frac{\left(|t|+\left|\frac{k+\mu_i}{n+\nu_i}-x\right|\right)^N}{N!} \overset{(17)}{\leq}$$

$$\omega_1\left(f^{(N)},\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)\cdot$$

$$\frac{\left(\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!}, \tag{86}$$

same estimate as in (85).

Therefore we derive (see (82))

$$(\xi) \leq \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\cdot$$

$$\left(\sum_{i=1}^{r} w_i\left\{\omega_1\left(f^{(N)},\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)\cdot \right.\right. \tag{87}$$

$$\left.\left.\frac{\left(\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!}\right\}\right).$$

Clearly we have found the estimate

$$|R| \leq \left(\sum_{i=1}^{r} w_i\left\{\omega_1\left(f^{(N)},\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)\cdot \right.\right. \tag{88}$$

$$\left.\left.\frac{\left(\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}\right)+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!}\right\}\right).$$

Based on (80) and (88) we derive (69). ■

Corollary 16 *All as in Theorem 15, plus* $f^{(j)}(x)=0$, $j=1,...,N$; $0<\alpha<1$. *Then*

$$|(K_n^*(f))(x)-f(x)| \leq$$

$$\sum_{i=1}^{r} w_i \left\{ \omega_1 \left(f^{(N)}, \left(\frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \cdot \right. \tag{89}$$

$$\left. \frac{\left(\left(\frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right)^N}{N!} \right\}.$$

Proof. By (75), (76) and (88). ■

In (89) notice the extremely high speed of convergence $\frac{1}{n^{(1-\alpha)(N+1)}}$.

The uniform convergence with rates follows from

Corollary 17 *Let $x \in [-T^*, T^*]$, $T^* > 0$; $T > 0$ and $n \in \mathbb{N}$ such that $n \geq \max\left(T + T^*, T^{-\frac{1}{\alpha}} \right)$, $0 < \alpha < 1$. Let $f \in C^N(\mathbb{R})$, $N \in \mathbb{N}$, such that $f^{(N)}$ is uniformly continuous or is continuous and bounded. Then*

$$\| K_n^*(f) - f \|_{\infty, [-T^*, T^*]} \leq \sum_{j=1}^{N} \frac{\| f^{(j)} \|_{\infty, [-T^*, T^*]}}{j!} \cdot \tag{90}$$

$$\left(\sum_{i=1}^{r} w_i \left[\left(\frac{\nu_i T^* + \mu_i + 1}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right]^j \right) +$$

$$\sum_{i=1}^{r} w_i \left\{ \omega_1 \left(f^{(N)}, \left(\frac{\nu_i T^* + \mu_i + 1}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \cdot \right.$$

$$\left. \frac{\left(\left(\frac{\nu_i T^* + \mu_i + 1}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right)^N}{N!} \right\}.$$

Proof. By (69). ■

Corollary 18 *All as in Theorem 15, case of $N = 1$. It holds*

$$|(K_n^*(f))(x) - f(x)| \leq$$

$$|f'(x)| \left(\sum_{i=1}^{r} w_i \left(\left(\frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \right) + \tag{91}$$

$$\sum_{i=1}^{r} w_i \left\{ \omega_1 \left(f', \left(\frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \cdot \right.$$

$$\left. \left(\left(\frac{\nu_i |x| + \mu_i + 1}{n + \nu_i} \right) + \left(1 + \frac{\nu_i}{n + \nu_i} \right) \frac{T}{n^{1-\alpha}} \right) \right\}.$$

Proof. By (69). ■

We also present

Theorem 19 *Let all as in Theorem 11. Then*

$$\left|\left(M_n^*(f)\right)(x)-f(x)\right| \leq \sum_{j=1}^{N} \frac{\left|f^{(j)}(x)\right|}{j!}\left[\frac{T}{n^{1-\alpha}}+\frac{1}{n}\right]^j + \tag{92}$$

$$\omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}+\frac{1}{n}\right) \frac{\left(\frac{T}{n^{1-\alpha}}+\frac{1}{n}\right)^N}{N!}.$$

Inequality (92) implies the pointwise convergence with rates of $\left(M_n^*(f)\right)(x)$ *to* $f(x)$, *as* $n \to \infty$, *at the speed* $\frac{1}{n^{1-\alpha}}$.

Proof. Let k as in (5). Again by Taylor's formula we have that

$$\sum_{i=1}^{r} w_i f\left(\frac{k}{n}+\frac{i}{nr}\right) = \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} \sum_{i=1}^{r} w_i \left(\frac{k}{n}+\frac{i}{nr}-x\right)^j + \tag{93}$$

$$\sum_{i=1}^{r} w_i \int_x^{\frac{k}{n}+\frac{i}{nr}} \left(f^{(N)}(t)-f^{(N)}(x)\right) \frac{\left(\frac{k}{n}+\frac{i}{nr}-t\right)^{N-1}}{(N-1)!} dt.$$

Call

$$V(x) = \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right). \tag{94}$$

Then

$$\left(M_n^*(f)\right)(x) = \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{r} w_i f\left(\frac{k}{n}+\frac{i}{nr}\right)\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}$$

$$= \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \cdot$$

$$\sum_{i=1}^{r} w_i \left(\frac{k}{n}+\frac{i}{nr}-x\right)^j + \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \cdot$$

$$\sum_{i=1}^{r} w_i \int_x^{\frac{k}{n}+\frac{i}{nr}} \left(f^{(N)}(t)-f^{(N)}(x)\right) \frac{\left(\frac{k}{n}+\frac{i}{nr}-t\right)^{N-1}}{(N-1)!} dt. \tag{95}$$

Therefore we get

$$\left(M_n^*(f)\right)(x)-f(x) = \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \cdot$$

$$\sum_{i=1}^{r} w_i \left(\frac{k}{n}+\frac{i}{nr}-x\right)^j + R, \tag{96}$$

where

$$R = \frac{\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)} \cdot$$

$$\sum_{i=1}^{r} w_i \int_{x}^{\frac{k}{n}+\frac{i}{nr}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{k}{n} + \frac{i}{nr} - t\right)^{N-1}}{(N-1)!} dt. \tag{97}$$

Hence it holds

$$|(M_n^*(f))(x) - f(x)| \leq \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} \frac{\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)} \cdot \tag{98}$$

$$\sum_{i=1}^{r} w_i \left[\left|\frac{k}{n} - x\right| + \frac{i}{nr}\right]^j + |R| \leq$$

$$\sum_{j=1}^{N} \frac{\left|f^{(j)}(x)\right|}{j!} \frac{\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)} \cdot$$

$$\sum_{i=1}^{r} w_i \left[\frac{T}{n^{1-\alpha}} + \frac{1}{n}\right]^j + |R| = \sum_{j=1}^{N} \frac{\left|f^{(j)}(x)\right|}{j!} \left[\frac{T}{n^{1-\alpha}} + \frac{1}{n}\right]^j + |R|. \tag{99}$$

We have proved that

$$|(M_n^*(f))(x) - f(x)| \leq \sum_{j=1}^{N} \frac{\left|f^{(j)}(x)\right|}{j!} \left[\frac{T}{n^{1-\alpha}} + \frac{1}{n}\right]^j + |R|. \tag{100}$$

Next we observe it holds

$$|R| \leq \frac{\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)} \cdot \tag{101}$$

$$\sum_{i=1}^{r} w_i \left| \int_{x}^{\frac{k}{n}+\frac{i}{nr}} \left|f^{(N)}(t) - f^{(N)}(x)\right| \frac{\left|\frac{k}{n} + \frac{i}{nr} - t\right|^{N-1}}{(N-1)!} dt \right|.$$

Call

$$\varepsilon_i := \left| \int_{x}^{\frac{k}{n}+\frac{i}{nr}} \left|f^{(N)}(t) - f^{(N)}(x)\right| \frac{\left|\frac{k}{n} + \frac{i}{nr} - t\right|^{N-1}}{(N-1)!} dt \right|. \tag{102}$$

We distinguish the cases:
(i) if $\frac{k}{n} + \frac{i}{nr} \geq x$, then

$$\varepsilon_i := \int_{x}^{\frac{k}{n}+\frac{i}{nr}} \left|f^{(N)}(t) - f^{(N)}(x)\right| \frac{\left(\frac{k}{n} + \frac{i}{nr} - t\right)^{N-1}}{(N-1)!} dt \leq$$

$$\omega_1\left(f^{(N)}, \left|\frac{k}{n} - x\right| + \frac{1}{n}\right) \frac{\left(\frac{k}{n} + \frac{i}{nr} - x\right)^N}{N!} \leq \tag{103}$$

$$\omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}+\frac{1}{n}\right)\frac{\left(\frac{T}{n^{1-\alpha}}+\frac{1}{n}\right)^N}{N!}. \tag{104}$$

Thus

$$\varepsilon_i \leq \omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}+\frac{1}{n}\right)\frac{\left(\frac{T}{n^{1-\alpha}}+\frac{1}{n}\right)^N}{N!}. \tag{105}$$

ii) if $\frac{k}{n}+\frac{i}{nr}<x$, then

$$\varepsilon_i := \int_{\frac{k}{n}+\frac{i}{nr}}^{x}\left|f^{(N)}(t)-f^{(N)}(x)\right|\frac{\left(t-\left(\frac{k}{n}+\frac{i}{nr}\right)\right)^{N-1}}{(N-1)!}dt \leq$$

$$\omega_1\left(f^{(N)}, \left(x-\left(\frac{k}{n}+\frac{i}{nr}\right)\right)\right)\frac{\left(x-\left(\frac{k}{n}+\frac{i}{nr}\right)\right)^N}{N!} \leq \tag{106}$$

$$\omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}+\frac{1}{n}\right)\frac{\left(\frac{T}{n^{1-\alpha}}+\frac{1}{n}\right)^N}{N!}.$$

So we obtain again (105).

Clearly now by (101) we derive that

$$|R| \leq \omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}+\frac{1}{n}\right)\frac{\left(\frac{T}{n^{1-\alpha}}+\frac{1}{n}\right)^N}{N!}, \tag{107}$$

proving the claim. ■

Corollary 20 *All as in Theorem 19, plus* $f^{(j)}(x)=0$, $j=1,...,N$. *Then*

$$|(M_n^*(f))(x)-f(x)| \leq \omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}+\frac{1}{n}\right)\frac{\left(\frac{T}{n^{1-\alpha}}+\frac{1}{n}\right)^N}{N!}. \tag{108}$$

Proof. By (92). ■

In (108) notice the extremely high speed of convergence $\frac{1}{n^{(1-\alpha)(N+1)}}$.

Uniform convergence estimate follows

Corollary 21 *All here as in Corollary 13. Then*

$$\|M_n^*(f)-f\|_{\infty,[-T^*,T^*]} \leq \sum_{j=1}^{N}\frac{\left\|f^{(j)}\right\|_{\infty,[-T^*,T^*]}}{j!}\left(\frac{T}{n^{1-\alpha}}+\frac{1}{n}\right)^j \tag{109}$$

$$+\omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}+\frac{1}{n}\right)\frac{\left(\frac{T}{n^{1-\alpha}}+\frac{1}{n}\right)^N}{N!}.$$

Proof. By (92). ■

Corollary 22 *All as in Theorem 19, $N=1$ case. It holds*

$$\left|\left(M_n^*(f)\right)(x)-f(x)\right| \leq \left[\left|f'(x)\right|\right.$$
$$\left.+\omega_1\left(f', \frac{T}{n^{1-\alpha}}+\frac{1}{n}\right)\right]\left(\frac{T}{n^{1-\alpha}}+\frac{1}{n}\right). \tag{110}$$

Proof. By (92). ■

Note 23 *We also observe that all the right hand sides of convergence inequalities (45), (66), (67), (68), (69), (89), (90), (91), (92), (108), (109), (110), are independent of b.*

Note 24 *We observe that*

$$H_n^*(1)=K_n^*(1)=M_n^*(1)=1, \tag{111}$$

thus unitary operators.

Also, given that f is bounded, we get

$$\left\|H_n^*(f)\right\|_{\infty,\mathbb{R}} \leq \|f\|_{\infty,\mathbb{R}}, \tag{112}$$

$$\left\|K_n^*(f)\right\|_{\infty,\mathbb{R}} \leq \|f\|_{\infty,\mathbb{R}}, \tag{113}$$

and

$$\left\|M_n^*(f)\right\|_{\infty,\mathbb{R}} \leq \|f\|_{\infty,\mathbb{R}}. \tag{114}$$

Operators H_n^*, K_n^*, M_n^* are positive linear operators, and of course bounded operators directly by (112)-(114).

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