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Voronovskaya type asymptotic expansions for error function based quasi-interpolation neural network operators

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we examine further the quasi-interpolation error function based neural network operators of one hidden layer. Based on fractional calculus theory we derive a fractional Voronovskaya type asymptotic expansion for the error of approximation of these operators the unit operator studying the univariate case. We treat also analogously the multivariate case.

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1 Background

We consider here the (Gauss) error special function ([1], [17])

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \quad x \in \mathbb{R}, \tag{1}$$

which is a sigmoidal type function and a strictly increasing function.

It has the basic properties

$$\operatorname{erf}(0) = 0, \quad \operatorname{erf}(-x) = -\operatorname{erf}(x), \quad \operatorname{erf}(+\infty) = 1, \quad \operatorname{erf}(-\infty) = -1.$$

We consider the activation function ([15])

$$\chi(x) = \frac{1}{4}(\operatorname{erf}(x+1) - \operatorname{erf}(x-1)), \text{ any } x \in \mathbb{R}, \tag{2}$$

which is an even positive function.

Next we follow [15] on χ. We got there $\chi(0) \simeq 0.4215$, and that χ is strictly decreasing on $[0,\infty)$ and strictly increasing on $(-\infty,0]$, and the x-axis is the horizontal asymptote on χ, i.e. χ is a bell symmetric function.

Theorem 1 *([15]) We have that*

$$\sum_{i=-\infty}^{\infty} \chi(x-i) = 1, \quad \text{all } x \in \mathbb{R}, \tag{3}$$

$$\sum_{i=-\infty}^{\infty} \chi(nx-i) = 1, \quad \text{all } x \in \mathbb{R},\ n \in \mathbb{N}, \tag{4}$$

and

$$\int_{-\infty}^{\infty} \chi(x)\, dx = 1, \tag{5}$$

that is $\chi(x)$ is a density function on $\mathbb{R}$.

We need the important

Theorem 2 *([15]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} \geq 3$. It holds*

$$\sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \chi(nx-k) < \frac{1}{2\sqrt{\pi}\left(n^{1-\alpha}-2\right) e^{\left(n^{1-\alpha}-2\right)^2}}. \tag{6}$$

Denote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.

Theorem 3 *([15]) Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k)} < \frac{1}{\chi(1)} \simeq 4.019, \quad \forall\, x \in [a,b]. \tag{7}$$

Also from [15] we get

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k) \neq 1, \tag{8}$$

at least for some $x \in [a,b]$.

For large enough n we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. In general it holds by (4) that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k) \leq 1. \tag{9}$$

We need the univariate neural network operator

Definition 4 *([15]) Let $f \in C([a,b])$, $n \in \mathbb{N}$. We set*

$$A_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \chi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k)}, \ \ \forall\, x \in [a.b], \tag{10}$$

A_n is a univariate neural network operator.

Notice here $0 < \alpha < 1$ and $n \in \mathbb{N}$ with $n^{1-\alpha} \geq 3$. Let the fixed $K, L > 0$; for the linear combination $\frac{K}{n^{\alpha}} + \frac{L}{(n^{1-\alpha}-2)e^{(n^{1-\alpha}-2)^2}}$, the dominant rate of convergence to zero, as $n \to \infty$, is $n^{-\alpha}$. The closer α is to 1, we get faster and better rate of convergence to zero.

We mention from [16] the following:

We define

$$Z(x_1,...,x_N) := Z(x) := \prod_{i=1}^{N} \chi(x_i), \quad x = (x_1,...,x_N) \in \mathbb{R}^N, \ N \in \mathbb{N}. \tag{11}$$

It has the properties:

(i) $Z(x) > 0, \ \forall\, x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} Z(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} Z(x_1-k_1,...,x_N-k_N) = 1, \tag{12}$$

where $k := (k_1,...,k_n) \in \mathbb{Z}^N$, $\forall\, x \in \mathbb{R}^N$,

hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z(nx-k) :=$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} Z(nx_1-k_1,...,nx_N-k_N) = 1, \tag{13}$$

$\forall\, x \in \mathbb{R}^N$; $n \in \mathbb{N}$,

and

(iv)

$$\int_{\mathbb{R}^N} Z(x)\, dx = 1, \tag{14}$$

that is Z is a multivariate density function.

Here $\|x\|_{\infty} := \max\{|x_1|, ..., |x_N|\}$, $x \in \mathbb{R}^N$, also set $\infty := (\infty, ..., \infty)$, $-\infty := (-\infty, ..., -\infty)$ upon the multivariate context, and

$$\begin{aligned} \lceil na \rceil &: = (\lceil na_1 \rceil, ..., \lceil na_N \rceil), \\ \lfloor nb \rfloor &: = (\lfloor nb_1 \rfloor, ..., \lfloor nb_N \rfloor), \end{aligned} \tag{15}$$

where $a := (a_1, ..., a_N)$, $b := (b_1, ..., b_N)$.

We obviously see that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\prod_{i=1}^{N} \chi(nx_i - k_i) \right) =$$

$$\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \cdots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} \left(\prod_{i=1}^{N} \chi(nx_i - k_i) \right) = \prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} \chi(nx_i - k_i) \right). \tag{16}$$

For $0 < \beta < 1$ and $n \in \mathbb{N}$, a fixed $x \in \mathbb{R}^N$, we have that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx - k) =$$

$$\sum_{\substack{k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_{\infty} \leq \frac{1}{n^{\beta}}}}^{\lfloor nb \rfloor} \chi(nx - k) + \sum_{\substack{k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\lfloor nb \rfloor} \chi(nx - k). \tag{17}$$

In the last two sums the counting is over disjoint vector sets of k's, because the condition $\left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}}$ implies that there exists at least one $\left| \frac{k_r}{n} - x_r \right| > \frac{1}{n^{\beta}}$, where $r \in \{1, ..., N\}$.

From [16] we need

(v)

$$\sum_{\substack{k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\lfloor nb \rfloor} Z(nx - k) \leq \frac{1}{2\sqrt{\pi}\,(n^{1-\beta} - 2)\, e^{(n^{1-\beta}-2)^2}}, \tag{18}$$

$0 < \beta < 1$, $n \in \mathbb{N}; n^{1-\beta} \geq 3$, $x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$,

(vi)

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)} < \frac{1}{(\chi(1))^N} \simeq (4.019)^N, \tag{19}$$

$\forall\, x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$, $n \in \mathbb{N}$, and

(vii)

$$\sum_{\substack{k=-\infty \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}}}^{\infty} Z(nx-k) \leq \frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^{2}}}, \tag{20}$$

$0<\beta<1$, $n\in\mathbb{N}: n^{1-\beta}\geq 3$, $x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

Also we get that

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)\neq 1, \tag{21}$$

for at least some $x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

Let $f\in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$ and $n\in\mathbb{N}$ such that $\lceil na_i\rceil\leq\lfloor nb_i\rfloor$, $i=1,...,N$.

We mention from [16] the multivariate positive linear neural network operator ($x:=(x_1,...,x_N)\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$)

$$H_n(f,x_1,...,x_N):=H_n(f,x):=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)Z(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)} \tag{22}$$

$$:=\frac{\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor}\sum_{k_2=\lceil na_2\rceil}^{\lfloor nb_2\rfloor}...\sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor} f\left(\frac{k_1}{n},...,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}\chi(nx_i-k_i)\right)}{\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor}\chi(nx_i-k_i)\right)}.$$

For large enough n we always obtain $\lceil na_i\rceil\leq\lfloor nb_i\rfloor$, $i=1,...,N$. Also $a_i\leq\frac{k_i}{n}\leq b_i$, iff $\lceil na_i\rceil\leq k_i\leq\lfloor nb_i\rfloor$, $i=1,...,N$.

Let fixed $j\in\mathbb{N}$, $0<\beta<1$, and $A,B>0$. For large enough $n\in\mathbb{N}: n^{1-\beta}\geq 3$, in the linear combination $\left(\frac{A}{n^{\beta j}}+\frac{B}{\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^{2}}}\right)$, the dominant rate of convergence, as $n\to\infty$, is $n^{-\beta j}$. The closer β is to 1 we get faster and better rate of convergence to zero.

By $AC^m\left(\prod\limits_{i=1}^{N}[a_i,b_i]\right)$, $m,N\in\mathbb{N}$, we denote the space of functions such that all partial derivatives of order $(m-1)$ are coordinatewise absolutely continuous functions, also $f\in C^{m-1}\left(\prod\limits_{i=1}^{N}[a_i,b_i]\right)$.

Let $f\in AC^m\left(\prod\limits_{i=1}^{N}[a_i,b_i]\right)$, $m,N\in\mathbb{N}$. Here f_α denotes a partial derivative of f, $\alpha:=(\alpha_1,...,\alpha_N)$, $\alpha_i\in\mathbb{Z}^+$, $i=1,...,N$, and $|\alpha|:=\sum\limits_{i=1}^{N}\alpha_i=l$, were $l=0,1,...,m$. We write also $f_\alpha:=\frac{\partial^\alpha f}{\partial x^\alpha}$ and we say it is order l.

We denote

$$\|f_\alpha\|_{\infty,m}^{\max} := \max_{|\alpha|=m} \{\|f_\alpha\|_\infty\}, \tag{23}$$

where $\|\cdot\|_\infty$ is the supremum norm.

We assume here that $\|f_\alpha\|_{\infty,m}^{\max} < \infty$.

We need

Definition 5 *Let $\nu > 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f \in AC^n([a,b])$ (space of functions f with $f^{(n-1)} \in AC([a,b])$, absolutely continuous functions). We call left Caputo fractional derivative (see [21], pp. 49-52) the function*

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt, \tag{24}$$

$\forall\, x \in [a,b]$, where Γ is the gamma function $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt$, $\nu > 0$. Notice $D_{*a}^{\nu} f \in L_1([a,b])$ and $D_{*a}^{\nu} f$ exists a.e.on $[a,b]$.

We set $D_{*a}^{0} f(x) = f(x)$, $\forall\, x \in [a,b]$.

Definition 6 *(see also [4], [22], [23]). Let $f \in AC^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. The right Caputo fractional derivative of order $\alpha > 0$ is given by*

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta)\, d\zeta, \tag{25}$$

$\forall\, x \in [a,b]$. We set $D_{b-}^{0} f(x) = f(x)$. Notice $D_{b-}^{\alpha} f \in L_1([a,b])$ and $D_{b-}^{\alpha} f$ exists a.e.on $[a,b]$.

Convention 7 *We assume that*

$$D_{*x_0}^{\alpha} f(x) = 0, \ \text{ for } x < x_0, \tag{26}$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0, \tag{27}$$

for all $x, x_0 \in (a,b]$.

We mention

Proposition 8 *(by [6]) Let $f \in C^n([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$. Then $D_{*a}^{\nu} f(x)$ is continuous in $x \in [a,b]$.*

Also we have

Proposition 9 *(by [6]) Let $f \in C^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D_{b-}^{\alpha} f(x)$ is continuous in $x \in [a,b]$.*

Theorem 10 *([6]) Let $f \in C^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $x, x_0 \in [a,b]$. Then $D^{\alpha}_{*x_0} f(x)$, $D^{\alpha}_{x_0-} f(x)$ are jointly continuous functions in (x, x_0) from $[a,b]^2$ into $\mathbb{R}$.*

We mention the left Caputo fractional Taylor formula with integral remainder.

Theorem 11 *([21], p. 54) Let $f \in AC^m([a,b])$, $[a,b] \subset \mathbb{R}$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then*

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x - J)^{\alpha - 1} D^{\alpha}_{*x_0} f(J)\, dJ, \tag{28}$$

$\forall\ x \geq x_0;\ x, x_0 \in [a,b]$.

Also we mention the right Caputo fractional Taylor formula.

Theorem 12 *([4]) Let $f \in AC^m([a,b])$, $[a,b] \subset \mathbb{R}$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then*

$$f(x) = \sum_{j=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{1}{\Gamma(\alpha)} \int_{x}^{x_0} (J - x)^{\alpha - 1} D^{\alpha}_{x_0-} f(J)\, dJ, \tag{29}$$

$\forall\ x \leq x_0;\ x, x_0 \in [a,b]$.

For more on fractional calculus related to this work see [3], [5] and [8].

Next we follow [9], pp. 284-286.

About Taylor formula -Multivariate Case and Estimates

Let Q be a compact convex subset of $\mathbb{R}^N$; $N \geq 2$; $z := (z_1, ..., z_N)$, $x_0 := (x_{01}, ..., x_{0N}) \in Q$.

Let $f : Q \to \mathbb{R}$ be such that all partial derivatives of order $(m-1)$ are coordinatewise absolutely continuous functions, $m \in \mathbb{N}$. Also $f \in C^{m-1}(Q)$. That is $f \in AC^m(Q)$. Each m^{th} order partial derivative is denoted by $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$, where $\alpha := (\alpha_1, ..., \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, ..., N$ and $|\alpha| := \sum_{i=1}^{N} \alpha_i = m$. Consider $g_z(t) := f(x_0 + t(z - x_0))$, $t \geq 0$. Then

$$g_z^{(j)}(t) = \left[\left(\sum_{i=1}^{N} (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^j f \right] (x_{01} + t(z_1 - x_{01}), ..., x_{0N} + t(z_N - x_{0N})), \tag{30}$$

for all $j = 0, 1, 2, ..., m$.

Example 13 *Let $m = N = 2$. Then*

$$g_z(t) = f(x_{01} + t(z_1 - x_{01}), x_{02} + t(z_2 - x_{02})), \quad t \in \mathbb{R},$$

and

$$g_z'(t) = (z_1 - x_{01}) \frac{\partial f}{\partial x_1}(x_0 + t(z - x_0)) + (z_2 - x_{02}) \frac{\partial f}{\partial x_2}(x_0 + t(z - x_0)). \tag{31}$$

Setting

$$(*) = (x_{01} + t(z_1 - x_{01}), x_{02} + t(z_2 - x_{02})) = (x_0 + t(z - x_0)),$$

we get

$$g_z''(t) = (z_1 - x_{01})^2 \frac{\partial f^2}{\partial x_1^2}(*) + (z_1 - x_{01})(z_2 - x_{02}) \frac{\partial f^2}{\partial x_2 \partial x_1}(*) +$$

$$(z_1 - x_{01})(z_2 - x_{02}) \frac{\partial f^2}{\partial x_1 \partial x_2}(*) + (z_2 - x_{02})^2 \frac{\partial f^2}{\partial x_2^2}(*). \tag{32}$$

Similarly, we have the general case of $m, N \in \mathbb{N}$ *for* $g_z^{(m)}(t)$.

We mention the following multivariate Taylor theorem.

Theorem 14 *([9]) Under the above assumptions we have*

$$f(z_1, ..., z_N) = g_z(1) = \sum_{j=0}^{m-1} \frac{g_z^{(j)}(0)}{j!} + R_m(z, 0), \tag{33}$$

where

$$R_m(z, 0) := \int_0^1 \left(\int_0^{t_1} ... \left(\int_0^{t_{m-1}} g_z^{(m)}(t_m)\, dt_m \right) ... \right) dt_1, \tag{34}$$

or

$$R_m(z, 0) = \frac{1}{(m-1)!} \int_0^1 (1 - \theta)^{m-1} g_z^{(m)}(\theta)\, d\theta. \tag{35}$$

Notice that $g_z(0) = f(x_0)$.

We make

Remark 15 *Assume here that*

$$\|f_\alpha\|_{\infty,Q,m}^{\max} := \max_{|\alpha|=m} \|f_\alpha\|_{\infty,Q} < \infty.$$

Then

$$\left\| g_z^{(m)} \right\|_{\infty,[0,1]} = \left\| \left[\left(\sum_{i=1}^{N} (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^m f \right] (x_0 + t(z - x_0)) \right\|_{\infty,[0,1]} \leq \tag{36}$$

$$\left(\sum_{i=1}^{N} |z_i - x_{0i}| \right)^m \|f_\alpha\|_{\infty,Q,m}^{\max},$$

that is

$$\left\| g_z^{(m)} \right\|_{\infty,[0,1]} \leq \left(\|z - x_0\|_{l_1} \right)^m \|f_\alpha\|_{\infty,Q,m}^{\max} < \infty. \tag{37}$$

Hence we get by (35) that

$$|R_m(z,0)| \leq \frac{\left\| g_z^{(m)} \right\|_{\infty,[0,1]}}{m!} < \infty. \tag{38}$$

And it holds

$$|R_m(z,0)| \leq \frac{\left(\|z - x_0\|_{l_1} \right)^m}{m!} \|f_\alpha\|_{\infty,Q,m}^{\max}, \tag{39}$$

$\forall\ z, x_0 \in Q$.

Inequality (39) will be an important tool in proving our multivariate main result.

In this article first we find fractional Voronskaya type asymptotic expansion for $A_n(f,x)$, $x \in [a,b]$, then we find multivariate Voronskaya type asymptotic expansion for $H_n(f,x)$, $x \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$; $n \in \mathbb{N}$.

Our considered neural networks here are of one hidden layer.

For other neural networks related work, see [2], [7], [10], [11], [12], [13], [14], [18], [19], [20]. For neural networks in general, read [24], [25] and [26].

2 Main Results

We present our first univariate main result

Theorem 16 *Let $\alpha > 0$, $N \in \mathbb{N}$, $N = \lceil \alpha \rceil$, $f \in AC^N([a,b])$, $0 < \beta < 1$, $x \in [a,b]$, $n \in \mathbb{N}$ large enough and $n^{1-\beta} \geq 3$. Assume that $\left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]}$, $\|D_{*x}^{\alpha} f\|_{\infty,[x,b]} \leq M$, $M > 0$. Then*

$$A_n(f,x) - f(x) = \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n\left((\cdot - x)^j \right)(x) + o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}} \right), \tag{40}$$

where $0 < \varepsilon \leq \alpha$.

If $N = 1$, the sum in (40) collapses.

The last (40) implies that

$$n^{\beta(\alpha-\varepsilon)} \left[A_n(f,x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n\left((\cdot - x)^j \right)(x) \right] \to 0, \tag{41}$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$.

When $N=1$, or $f^{(j)}(x)=0$, $j=1,...,N-1$, then we derive that

$$n^{\beta(\alpha-\varepsilon)}\left[A_n(f,x)-f(x)\right]\to 0$$

as $n\to\infty$, $0<\varepsilon\leq\alpha$. Of great interest is the case of $\alpha=\frac{1}{2}$.

Proof. From [21], p. 54; (28), we get by the left Caputo fractional Taylor formula that

$$f\left(\frac{k}{n}\right)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^j+\frac{1}{\Gamma(\alpha)}\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}D_{*x}^{\alpha}f(J)\,dJ, \quad (42)$$

for all $x\leq\frac{k}{n}\leq b$.

Also from [4]; (29), using the right Caputo fractional Taylor formula we get

$$f\left(\frac{k}{n}\right)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^j+\frac{1}{\Gamma(\alpha)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ, \quad (43)$$

for all $a\leq\frac{k}{n}\leq x$.

We call

$$W(x):=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\chi(nx-k). \quad (44)$$

Hence we have

$$\frac{f\left(\frac{k}{n}\right)\chi(nx-k)}{W(x)}=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\frac{\chi(nx-k)}{W(x)}\left(\frac{k}{n}-x\right)^j+ \quad (45)$$

$$\frac{\chi(nx-k)}{W(x)\Gamma(\alpha)}\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}D_{*x}^{\alpha}f(J)\,dJ,$$

all $x\leq\frac{k}{n}\leq b$, iff $\lceil nx\rceil\leq k\leq\lfloor nb\rfloor$, and

$$\frac{f\left(\frac{k}{n}\right)\chi(nx-k)}{W(x)}=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\frac{\chi(nx-k)}{W(x)}\left(\frac{k}{n}-x\right)^j+ \quad (46)$$

$$\frac{\chi(nx-k)}{W(x)\Gamma(\alpha)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ,$$

for all $a\leq\frac{k}{n}\leq x$, iff $\lceil na\rceil\leq k\leq\lfloor nx\rfloor$.

We have that $\lceil nx\rceil\leq\lfloor nx\rfloor+1$.

Therefore it holds

$$\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\frac{f\left(\frac{k}{n}\right)\chi(nx-k)}{W(x)}=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\frac{\chi(nx-k)\left(\frac{k}{n}-x\right)^j}{W(x)}+ \quad (47)$$

$$\frac{1}{\Gamma(\alpha)}\left(\frac{\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\chi(nx-k)}{W(x)}\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}D_{*x}^{\alpha}f(J)\,dJ\right),$$

and

$$\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}f\left(\frac{k}{n}\right)\frac{\chi(nx-k)}{W(x)}=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\frac{\chi(nx-k)}{W(x)}\left(\frac{k}{n}-x\right)^{j}+ \tag{48}$$

$$\frac{1}{\Gamma(\alpha)}\left(\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\frac{\chi(nx-k)}{W(x)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ\right).$$

Adding the last two equalities (47) and (48) we obtain

$$A_n(f,x)=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\frac{\chi(nx-k)}{W(x)}= \tag{49}$$

$$\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\frac{\chi(nx-k)}{W(x)}\left(\frac{k}{n}-x\right)^{j}+$$

$$\frac{1}{\Gamma(\alpha)W(x)}\left\{\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\chi(nx-k)\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ+\right.$$

$$\left.\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\chi(nx-k)\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)\right)dJ\right\}.$$

So we have derived

$$\theta(x):=A_n(f,x)-f(x)-\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}A_n\left((\cdot-x)^{j}\right)(x)=\theta_n^{*}(x), \tag{50}$$

where

$$\theta_n^{*}(x):=\frac{1}{\Gamma(\alpha)W(x)}\left\{\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\chi(nx-k)\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ\right.$$

$$\left.+\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\chi(nx-k)\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}D_{*x}^{\alpha}f(J)\,dJ\right\}. \tag{51}$$

We set

$$\theta_{1n}^{*}(x):=\frac{1}{\Gamma(\alpha)}\left(\frac{\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\chi(nx-k)}{W(x)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ\right), \tag{52}$$

and

$$\theta_{2n}^{*} := \frac{1}{\Gamma(\alpha)} \left(\frac{\sum_{k=\lfloor nx \rfloor +1}^{\lfloor nb \rfloor} \chi(nx-k)}{W(x)} \int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - J \right)^{\alpha-1} D_{*x}^{\alpha} f(J)\, dJ \right), \tag{53}$$

i.e.

$$\theta_{n}^{*}(x) = \theta_{1n}^{*}(x) + \theta_{2n}^{*}(x). \tag{54}$$

We assume $b-a > \frac{1}{n^{\beta}}$, $0 < \beta < 1$, which is always the case for large enough $n \in \mathbb{N}$, that is when $n > \left\lceil (b-a)^{-\frac{1}{\beta}} \right\rceil$. It is always true that either $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}}$ or $\left| \frac{k}{n} - x \right| > \frac{1}{n^{\beta}}$.

For $k = \lceil na \rceil, ..., \lfloor nx \rfloor$, we consider

$$\gamma_{1k} := \left| \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\alpha-1} D_{x-}^{\alpha} f(J)\, dJ \right| \leq \tag{55}$$

$$\int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\alpha-1} \left| D_{x-}^{\alpha} f(J) \right| dJ$$

$$\leq \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} \frac{\left(x - \frac{\kappa}{n} \right)^{\alpha}}{\alpha} \leq \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} \frac{(x-a)^{\alpha}}{\alpha}. \tag{56}$$

That is

$$\gamma_{1k} \leq \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} \frac{(x-a)^{\alpha}}{\alpha}, \tag{57}$$

for $k = \lceil na \rceil, ..., \lfloor nx \rfloor$.

Also we have in case of $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}}$ that

$$\gamma_{1k} \leq \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\alpha-1} \left| D_{x-}^{\alpha} f(J) \right| dJ \tag{58}$$

$$\leq \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} \frac{\left(x - \frac{\kappa}{n} \right)^{\alpha}}{\alpha} \leq \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} \frac{1}{n^{\alpha\beta}\alpha}.$$

So that, when $\left(x - \frac{k}{n} \right) \leq \frac{1}{n^{\beta}}$, we get

$$\gamma_{1k} \leq \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} \frac{1}{\alpha n^{a\beta}}. \tag{59}$$

Therefore

$$\left| \theta_{1n}^{*}(x) \right| \leq \frac{1}{\Gamma(\alpha)} \left(\frac{\sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \chi(nx-k)}{W(x)} \gamma_{1k} \right) = \frac{1}{\Gamma(\alpha)}.$$

$$\left\{ \frac{\sum_{\begin{cases} k=\lceil na\rceil \\ : \left|\frac{k}{n}-x\right| \le \frac{1}{n^\beta} \end{cases}}^{\lfloor nx\rfloor} \chi(nx-k)}{W(x)} \gamma_{1k} + \frac{\sum_{\begin{cases} k=\lceil na\rceil \\ : \left|\frac{k}{n}-x\right| > \frac{1}{n^\beta} \end{cases}}^{\lfloor nx\rfloor} \chi(nx-k)}{W(x)} \gamma_{1k} \right\}$$

$$\le \frac{1}{\Gamma(\alpha)} \left\{ \left(\frac{\sum_{\begin{cases} k=\lceil na\rceil \\ : \left|\frac{k}{n}-x\right| \le \frac{1}{n^\beta} \end{cases}}^{\lfloor nx\rfloor} \chi(nx-k)}{W(x)} \right) \left\| D_{x-}^\alpha f \right\|_{\infty,[a,x]} \frac{1}{\alpha n^{\alpha\beta}} + \right.$$

$$\left. \frac{1}{W(x)} \left(\sum_{\begin{cases} k=\lceil na\rceil \\ : \left|\frac{k}{n}-x\right| > \frac{1}{n^\beta} \end{cases}}^{\lfloor nx\rfloor} \chi(nx-k) \right) \left\| D_{x-}^\alpha f \right\|_{\infty,[a,x]} \frac{(x-a)^\alpha}{\alpha} \right\} \overset{\text{(by (6), (7))}}{\le}$$

$$\frac{\left\| D_{x-}^\alpha f \right\|_{\infty,[a,x]}}{\Gamma(\alpha+1)} \left\{ \frac{1}{n^{\alpha\beta}} + (4.019) \frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right) e^{\left(n^{1-\beta}-2\right)^2}} (x-a)^\alpha \right\}. \tag{60}$$

Therefore we proved

$$\left|\theta_{1n}^*(x)\right| \le \frac{\left\| D_{x-}^\alpha f \right\|_{\infty,[a,x]}}{\Gamma(\alpha+1)} \left\{ \frac{1}{n^{\alpha\beta}} + \frac{2.0095}{\sqrt{\pi}\left(n^{1-\beta}-2\right) e^{\left(n^{1-\beta}-2\right)^2}} (x-a)^\alpha \right\}. \tag{61}$$

But for large enough $n \in \mathbb{N}$ we get

$$\left|\theta_{1n}^*(x)\right| \le \frac{2\left\| D_{x-}^\alpha f \right\|_{\infty,[a,x]}}{\Gamma(\alpha+1)\, n^{\alpha\beta}}. \tag{62}$$

Similarly we have

$$\gamma_{2k} := \left| \int_x^{\frac{k}{n}} \left(\frac{k}{n}-J\right)^{\alpha-1} D_{*x}^\alpha f(J)\, dJ \right| \le$$

$$\int_x^{\frac{k}{n}} \left(\frac{k}{n}-J\right)^{\alpha-1} \left| D_{*x}^\alpha f(J) \right| dJ \le$$

$$\left\| D_{*x}^\alpha f \right\|_{\infty,[x,b]} \frac{\left(\frac{k}{n}-x\right)^\alpha}{\alpha} \le \left\| D_{*x}^\alpha f \right\|_{\infty,[x,b]} \frac{(b-x)^\alpha}{\alpha}. \tag{63}$$

That is

$$\gamma_{2k} \le \left\| D_{*x}^\alpha f \right\|_{\infty,[x,b]} \frac{(b-x)^\alpha}{\alpha}, \tag{64}$$

for $k = \lfloor nx \rfloor + 1, ..., \lfloor nb \rfloor$.

Also we have in case of $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}}$ that

$$\gamma_{2k} \leq \frac{\|D_{*x}^{\alpha} f\|_{\infty,[x,b]}}{\alpha n^{\alpha\beta}}. \tag{65}$$

Consequently it holds

$$|\theta_{2n}^{*}(x)| \leq \frac{1}{\Gamma(\alpha)} \left(\frac{\sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} \chi(nx-k)}{W(x)} \gamma_{2k} \right) =$$

$$\frac{1}{\Gamma(\alpha)} \left\{ \left(\frac{\sum_{\begin{cases} k = \lfloor nx \rfloor + 1 \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}} \end{cases}}^{\lfloor nb \rfloor} \chi(nx-k)}{W(x)} \right) \frac{\|D_{*x}^{\alpha} f\|_{\infty,[x,b]}}{\alpha n^{\alpha\beta}} + \right.$$

$$\left. \frac{1}{W(x)} \left(\sum_{\begin{cases} k = \lfloor nx \rfloor + 1 \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^{\beta}} \end{cases}}^{\lfloor nb \rfloor} \chi(nx-k) \right) \|D_{*x}^{\alpha} f\|_{\infty,[x,b]} \frac{(b-x)^{\alpha}}{\alpha} \right\} \leq$$

$$\frac{\|D_{*x}^{\alpha} f\|_{\infty,[x,b]}}{\Gamma(\alpha+1)} \left\{ \frac{1}{n^{\alpha\beta}} + \frac{2.0095}{\sqrt{\pi}\left(n^{1-\beta}-2\right) e^{\left(n^{1-\beta}-2\right)^{2}}} (b-x)^{\alpha} \right\}. \tag{66}$$

That is

$$|\theta_{2n}^{*}(x)| \leq \frac{\|D_{*x}^{\alpha} f\|_{\infty,[x,b]}}{\Gamma(\alpha+1)} \left\{ \frac{1}{n^{\alpha\beta}} + \frac{2.0095}{\sqrt{\pi}\left(n^{1-\beta}-2\right) e^{\left(n^{1-\beta}-2\right)^{2}}} (b-x)^{\alpha} \right\}. \tag{67}$$

But for large enough $n \in \mathbb{N}$ we get

$$|\theta_{2n}^{*}(x)| \leq \frac{2\|D_{*x}^{\alpha} f\|_{\infty,[x,b]}}{\Gamma(\alpha+1) n^{\alpha\beta}}. \tag{68}$$

Since $\left\|D_{x-}^{\alpha} f\right\|_{\infty,[a,x]}$, $\|D_{*x}^{\alpha} f\|_{\infty,[x,b]} \leq M$, $M > 0$, we derive

$$|\theta_{n}^{*}(x)| \leq |\theta_{1n}^{*}(x)| + |\theta_{2n}^{*}(x)| \overset{\text{(by (62), (68))}}{\leq} \frac{4M}{\Gamma(\alpha+1) n^{\alpha\beta}}. \tag{69}$$

That is for large enough $n \in \mathbb{N}$ we get

$$|\theta(x)| = |\theta_{n}^{*}(x)| \leq \left(\frac{4M}{\Gamma(\alpha+1)} \right) \left(\frac{1}{n^{\alpha\beta}} \right), \tag{70}$$

resulting to

$$|\theta(x)| = O\left(\frac{1}{n^{\alpha\beta}}\right), \tag{71}$$

and

$$|\theta(x)| = o(1). \tag{72}$$

And, letting $0 < \varepsilon \leq \alpha$, we derive

$$\frac{|\theta(x)|}{\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right)} \leq \left(\frac{4M}{\Gamma(\alpha+1)}\right)\left(\frac{1}{n^{\beta\varepsilon}}\right) \to 0, \tag{73}$$

as $n \to \infty$.

I.e.

$$|\theta(x)| = o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \tag{74}$$

proving the claim. ■

We present our second main result which is a multivariate one.

Theorem 17 *Let $0 < \beta < 1$, $x \in \prod_{i=1}^{N}[a_i, b_i]$, $n \in \mathbb{N}$ large enough and $n^{1-\beta} \geq 3$, $f \in AC^m\left(\prod_{i=1}^{N}[a_i, b_i]\right)$, $m, N \in \mathbb{N}$. Assume further that $\|f_\alpha\|_{\infty,m}^{\max} < \infty$. Then*

$$H_n(f,x) - f(x) =$$

$$\sum_{j=1}^{m-1}\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha(x)}{\prod_{i=1}^{N}\alpha_i!}\right)H_n\left(\prod_{i=1}^{N}(\cdot - x_i)^{\alpha_i}, x\right)\right) + o\left(\frac{1}{n^{\beta(m-\varepsilon)}}\right), \tag{75}$$

where $0 < \varepsilon \leq m$.

If $m = 1$, the sum in (75) collapses.

The last (75) implies that

$$n^{\beta(m-\varepsilon)}\left[H_n(f,x) - f(x) - \sum_{j=1}^{m-1}\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha(x)}{\prod_{i=1}^{N}\alpha_i!}\right)H_n\left(\prod_{i=1}^{N}(\cdot - x_i)^{\alpha_i}, x\right)\right)\right] \tag{76}$$

$$\to 0, \quad \text{as } n \to \infty,\ 0 < \varepsilon \leq m.$$

When $m = 1$, or $f_\alpha(x) = 0$, for $|\alpha| = j$, $j = 1, ..., m-1$, then we derive that

$$n^{\beta(m-\varepsilon)}\left[H_n(f,x) - f(x)\right] \to 0,$$

as $n \to \infty$, $0 < \varepsilon \leq m$.

Proof. Consider $g_z(t) := f(x_0 + t(z - x_0))$, $t \geq 0$; $x_0, z \in \prod_{i=1}^{N}[a_i, b_i]$. Then

$$g_z^{(j)}(t) = \left[\left(\sum_{i=1}^{N}(z_i - x_{0i})\frac{\partial}{\partial x_i}\right)^j f\right](x_{01} + t(z_1 - x_{01}), ..., x_{0N} + t(z_N - x_{0N})), \tag{77}$$

for all $j = 0, 1, ..., m$.

By (33) we have the multivariate Taylor's formula

$$f(z_1, ..., z_N) = g_z(1) = \sum_{j=0}^{m-1} \frac{g_z^{(j)}(0)}{j!} + \frac{1}{(m-1)!} \int_0^1 (1-\theta)^{m-1} g_z^{(m)}(\theta) \, d\theta. \tag{78}$$

Notice $g_z(0) = f(x_0)$. Also for $j = 0, 1, ..., m-1$, we have

$$g_z^{(j)}(0) = \sum_{\substack{\alpha := (\alpha_1, ..., \alpha_N),\ \alpha_i \in \mathbb{Z}^+, \\ i=1,...,N,\ |\alpha| := \sum_{i=1}^N \alpha_i = j}} \left(\frac{j!}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N (z_i - x_{0i})^{\alpha_i} \right) f_\alpha(x_0). \tag{79}$$

Furthermore

$$g_z^{(m)}(\theta) = \sum_{\substack{\alpha := (\alpha_1, ..., \alpha_N),\ \alpha_i \in \mathbb{Z}^+, \\ i=1,...,N,\ |\alpha| := \sum_{i=1}^N \alpha_i = m}} \left(\frac{m!}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N (z_i - x_{0i})^{\alpha_i} \right) f_\alpha(x_0 + \theta(z - x_0)), \tag{80}$$

$0 \leq \theta \leq 1$.

So we treat $f \in AC^m \left(\prod_{i=1}^N [a_i, b_i] \right)$ with $\|f_\alpha\|_{\infty,m}^{\max} < \infty$.

Thus, by (78) we have for $\frac{k}{n}, x \in \left(\prod_{i=1}^N [a_i, b_i] \right)$ that

$$f\left(\frac{k_1}{n}, ..., \frac{k_N}{n} \right) - f(x) = \sum_{j=1}^{m-1} \sum_{\substack{\alpha := (\alpha_1, ..., \alpha_N),\ \alpha_i \in \mathbb{Z}^+, \\ i=1,...,N,\ |\alpha| := \sum_{i=1}^N \alpha_i = j}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \left(\prod_{i=1}^N \left(\frac{k_i}{n} - x_i \right)^{\alpha_i} \right) f_\alpha(x) + R, \tag{81}$$

where

$$R := m \int_0^1 (1-\theta)^{m-1} \sum_{\substack{\alpha := (\alpha_1, ..., \alpha_N),\ \alpha_i \in \mathbb{Z}^+, \\ i=1,...,N,\ |\alpha| := \sum_{i=1}^N \alpha_i = m}} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \cdot \left(\prod_{i=1}^N \left(\frac{k_i}{n} - x_i \right)^{\alpha_i} \right) f_\alpha\left(x + \theta\left(\frac{k}{n} - x \right) \right) d\theta. \tag{82}$$

By (39) we obtain

$$|R| \leq \frac{\left(\left\| x - \frac{k}{n} \right\|_{l_1} \right)^m}{m!} \|f_\alpha\|_{\infty,m}^{\max}. \tag{83}$$

Notice here that

$$\left\| \frac{k}{n} - x \right\|_{\infty} \leq \frac{1}{n^{\beta}} \Leftrightarrow \left| \frac{k_i}{n} - x_i \right| \leq \frac{1}{n^{\beta}},\ i = 1, ..., N. \tag{84}$$

So, if $\left\| \frac{k}{n} - x \right\|_{\infty} \leq \frac{1}{n^{\beta}}$ we get that $\left\| x - \frac{k}{n} \right\|_{l_1} \leq \frac{N}{n^{\beta}}$, and

$$|R| \leq \frac{N^m}{n^{m\beta} m!} \left\| f_{\alpha} \right\|_{\infty, m}^{\max}. \tag{85}$$

Also we see that

$$\left\| x - \frac{k}{n} \right\|_{l_1} = \sum_{i=1}^{N} \left| x_i - \frac{k_i}{n} \right| \leq \sum_{i=1}^{N} (b_i - a_i) = \| b - a \|_{l_1},$$

therefore in general it holds

$$|R| \leq \frac{\left(\| b - a \|_{l_1} \right)^m}{m!} \left\| f_{\alpha} \right\|_{\infty, m}^{\max}. \tag{86}$$

Call

$$V(x) := \sum_{k = \lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k).$$

Hence we have

$$U_n(x) := \frac{\sum_{k = \lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) R}{V(x)} = \tag{87}$$

$$\frac{\sum_{\substack{k = \lceil na \rceil \\ : \left\| \frac{k}{n} - x \right\|_{\infty} \leq \frac{1}{n^{\beta}}}}^{\lfloor nb \rfloor} Z(nx - k) R}{V(x)} + \frac{\sum_{\substack{k = \lceil na \rceil \\ : \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\lfloor nb \rfloor} Z(nx - k) R}{V(x)}.$$

Consequently we obtain

$$|U_n(x)| \leq \left(\frac{\sum_{\substack{k = \lceil na \rceil \\ : \left\| \frac{k}{n} - x \right\|_{\infty} \leq \frac{1}{n^{\beta}}}}^{\lfloor nb \rfloor} Z(nx - k)}{V(x)} \right) \left(\frac{N^m}{n^{m\beta} m!} \left\| f_{\alpha} \right\|_{\infty, m}^{\max} \right) +$$

$$\frac{1}{V(x)} \left(\sum_{\substack{k = \lceil na \rceil \\ : \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}}}}^{\lfloor nb \rfloor} Z(nx - k) \right) \frac{\left(\| b - a \|_{l_1} \right)^m}{m!} \left\| f_{\alpha} \right\|_{\infty, m}^{\max} \overset{\text{(by (19), (18))}}{\leq}$$

$$\frac{N^m}{n^{m\beta}m!}\|f_\alpha\|_{\infty,m}^{\max}+(4.019)^N\frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}\frac{\left(\|b-a\|_{l_1}\right)^m}{m!}\|f_\alpha\|_{\infty,m}^{\max}. \tag{88}$$

Therefore we have found

$$|U_n(x)|\leq\frac{\|f_\alpha\|_{\infty,m}^{\max}}{m!}\left\{\frac{N^m}{n^{m\beta}}+(4.019)^N\frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}\left(\|b-a\|_{l_1}\right)^m\right\}. \tag{89}$$

For large enough $n\in\mathbb{N}$ we get

$$|U_n(x)|\leq\left(\frac{2\|f_\alpha\|_{\infty,m}^{\max}N^m}{m!}\right)\left(\frac{1}{n^{m\beta}}\right). \tag{90}$$

That is

$$|U_n(x)|=O\left(\frac{1}{n^{m\beta}}\right), \tag{91}$$

and

$$|U_n(x)|=o(1). \tag{92}$$

And, letting $0<\varepsilon\leq m$, we derive

$$\frac{|U_n(x)|}{\left(\frac{1}{n^{\beta(m-\varepsilon)}}\right)}\leq\left(\frac{2\|f_\alpha\|_{\infty,m}^{\max}N^m}{m!}\right)\frac{1}{n^{\beta\varepsilon}}\to 0, \tag{93}$$

as $n\to\infty$.

I.e.

$$|U_n(x)|=o\left(\frac{1}{n^{\beta(m-\varepsilon)}}\right). \tag{94}$$

By (81) we observe that

$$\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)Z(nx-k)}{V(x)}-f(x)=$$

$$\sum_{j=1}^{m-1}\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha(x)}{\prod_{i=1}^N\alpha_i!}\right)\right)\frac{\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx-k)\left(\prod_{i=1}^N\left(\frac{k_i}{n}-x_i\right)^{\alpha_i}\right)\right)}{V(x)}+ \tag{95}$$

$$\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx-k)R}{V(x)}.$$

The last says

$$H_n(f,x)-f(x)-\sum_{j=1}^{m-1}\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha(x)}{\prod_{i=1}^N\alpha_i!}\right)H_n\left(\prod_{i=1}^N(\cdot-x_i)^{\alpha_i},x\right)\right)=U_n(x). \tag{96}$$

The proof of the theorem is complete. ■

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