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Approximations by Multivariate Perturbed Neural Network Operators

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Abstract

This article deals with the determination of the rate of convergence to the unit of each of three newly introduced here multivariate perturbed normalized neural network operators of one hidden layer. These are given through the multivariate modulus of continuity of the involved multivariate function or its high order partial derivatives and that appears in the right-hand side of the associated multivariate Jackson type inequalities. The multivariate activation function is very general, especially it can derive from any multivariate sigmoid or multivariate bell-shaped function. The right hand sides of our convergence inequalities do not depend on the activation function. The sample functionals are of multivariate Stancu, Kantorovich and Quadrature types. We give applications for the first partial derivatives of the involved function.

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1 Introduction

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle a_j \cdot x\rangle + b_j\right), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental network models, the activation function is the sigmoid function of logistic type or other sigmoid function or bell-shaped function.

It is well known that FNNs are universal approximators. Theoretically, any continuous function defined on a compact set can be approximated to any desired degree of accuracy by increasing the number of hidden neurons. It was proved by Cybenko [15] and Funahashi [17], that any continuous function can be approximated on a compact set with uniform topology by a network of the form $N_n(x)$, using any continuous, sigmoid activation function. Hornik et al. in [20], have shown that any measurable function can be approached with such a network. Furthermore, these authors proved in [21], that any function of the Sobolev spaces can be approached with all derivatives. A variety of density results on FNN approximations to multivariate functions were later established by many authors using different methods, for more or less general situations: [22] by Leshno et al., [26] by Mhaskar and Micchelli, [12] by Chui and Li, [11] by Chen and Chen, [18] by Hahm and Hong, etc.

Usually these results only give theorems about the existence of an approximation. A related and important problem is that of complexity: determining the number of neurons required to guarantee that all functions belonging to a space can be approximated to the prescribed degree of accuracy ϵ.

Barron [7] shows that if the function is supposed to satisfy certain conditions expressed in terms of its Fourier transform, and if each of the neurons evaluates a sigmoid activation function, then at most $O\left(\epsilon^{-2}\right)$ neurons are needed to achieve the order of approximation ϵ. Some other authors have published similar results on the complexity of FNN approximations: Mhaskar and Micchelli [27], Suzuki [30], Maiorov and Meir [23], Makovoz [24], Ferrari and Stengel [16], Xu and Cao [31], Cao et al. [8], etc.

P. Cardaliaguet and G. Euvrard were the first, see [9], to describe precisely and study neural network approximation operators to the unit operator. Namely they proved: be given $f : \mathbb{R} \to \mathbb{R}$ a continuous bounded function and b a centered bell-shaped function, then the functions

$$F_n(x) = \sum_{k=-n^2}^{n^2} \frac{f\left(\frac{k}{n}\right)}{In^{\alpha}} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right),$$

where $I := \int_{-\infty}^{\infty} b(t)\,dt$, $0 < \alpha < 1$, converge uniformly on compacta to f.

You see above that the weights $\frac{f\left(\frac{k}{n}\right)}{In^{\alpha}}$ are explicitly known, for the first time shown in [9].

Furthermore the authors in [9] proved that: let $f : \mathbb{R}^p \to \mathbb{R}$, $p \in \mathbb{N}$, be a continuous bounded function and b a p-dimensional bell-shaped function. Then the functions

$$G_n(x) = \sum_{k_1=-n^2}^{n^2} \cdots \sum_{k_p=-n^2}^{n^2} \frac{f\left(\frac{k_1}{n}, \ldots, \frac{k_p}{n}\right)}{I n^{\alpha p}} b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), \ldots, n^{1-\alpha}\left(x_p - \frac{k_p}{n}\right)\right),$$

where I is the integral of b on $\mathbb{R}^p$ and $0 < \alpha < 1$, converge uniformly on compacta to f.

Still the work [9] is qualitative and not quantitative.

The author in [1], [2] and [3], see chapters 2-5, was the first to establish neural network approximations to continuous functions with rates, that is quantitative works, by very specifically defined neural network operators of Cardaliagnet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative or partial derivatives, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support. Also in [3] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see chapters 4-5 there.

Though the work in [1], [2], [3], was quantitative, the rate of convergence was not precisely determined.

Finally the author in [4], [5], by normalizing his operators he achieved to determine the exact rates of convergence.

In this article the author continuous and completes his related work, by introducing three new multivariate perturbed neural network operators of Cardaliaguet-Euvrard type. This started with the univariate treatment in [6].

The sample coefficients $f\left(\frac{k}{n}\right)$ are replaced by three suitable natural perturbations, what is actuality happens in reality of neural network operations.

The calculuation of $f\left(\frac{k}{n}\right)$ at the neurons many times are not calculated as such, but rather in a distored way.

Next we justify why we take here the multivariate activation function to be of compact support, of course it helps us to conduct our study.

The multivariate activation function, same as transfer function or learning rule, is connected and associated to firing of neurons. Firing, which sends electric pulses or an output signal to other neurons, occurs when the activation level is above the threshold level set by the learning rule.

Each Neural Network firing is essentially of finite time duration. Essentially the firing in time decays, but in practice sends positive energy over a finite time interval.

Thus by using an activation function of compact support, in practice we do not alter much of the good results of our approximation.

To be more precise, we may take the compact support to be a large symmetric to the origin box. This is what happens in real time with the firing of neurons.

For more about neural networks in general we refer to [10], [13], [14], [19], [25], [28].

2 Basics

Here the activation function $b : \mathbb{R}^d \to \mathbb{R}_+$, $d \in \mathbb{N}$, is of compact support $B := \prod_{j=1}^{d} [-T_j, T_j]$, $T_j > 0$, $j = 1, ..., d$. That is $b(x) > 0$ for any $x \in B$, and clearly b may have jump discontinuities. Also the shape of the graph of b is immaterial.

Typically in neural networks approximation we take b to be a d-dimensional bell-shaped function (i.e. per coordinate is a centered bell-shaped function), or a product of univariate centered bell-shaped functions, or a product of sigmoid functions, in our case all of them are of compact support B.

Example 1 *Take $b(x) = \beta(x_1)\beta(x_2)...\beta(x_d)$, where β is any of the following functions, $j = 1, ..., d$:*

(i) $\beta(x_j)$ is the characteristic function on $[-1, 1]$,

(ii) $\beta(x_j)$ is the hat function over $[-1, 1]$, that is,

$$\beta(x_j) = \begin{cases} 1 + x_j, & -1 \leq x_j \leq 0, \\ 1 - x_j, & 0 < x_j \leq 1, \\ 0, & \text{elsewhere}, \end{cases}$$

(iii) the truncated sigmoids

$$\beta(x_j) = \begin{cases} \frac{1}{1+e^{-x_j}} \text{ or } \tanh x_j \text{ or } \operatorname{erf}(x_j), & \text{for } x_j \in [-T_j, T_j], \text{ with large } T_j > 0, \\ 0, \quad x_j \in \mathbb{R} - [-T_j, T_j], \end{cases}$$

(iv) the truncated Gompertz function

$$\beta(x_j) = \begin{cases} e^{-\alpha e^{-\beta x_j}}, & x_j \in [-T_j, T_j];\ \alpha, \beta > 0;\ \text{large } T_j > 0, \\ 0, & x \in \mathbb{R} - [-T_j, T_j], \end{cases}$$

The Gompertz functions are also sigmoid functions, with wide applications to many applied fields, e.g. demography and tumor growth modeling, etc.

Thus the general activation function b we will be using here includes all kinds of activation functions in neural network approximations.

Here we consider functions $f : \mathbb{R}^d \to \mathbb{R}$ that either continuous and bounded, or uniformly continuous.

Let here the parameters: $0 < \alpha < 1$, $x = (x_1, ..., x_d) \in \mathbb{R}^d$, $n \in \mathbb{N}$; $r = (r_1, ..., r_d) \in \mathbb{N}^d$, $i = (i_1, ..., i_d) \in \mathbb{N}^d$, with $i_j = 1, 2, ..., r_j$, $j = 1, ..., d$; also let

$w_i = w_{i_1,...,i_d} \geq 0$, such that $\sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} ... \sum_{i_d=1}^{r_d} w_{i_1,...,i_d} = 1$, in brief written as $\sum_{i=1}^{r} w_i = 1$. We further consider the parameters $k = (k_1, ..., k_d) \in \mathbb{Z}^d$; $\mu_i = (\mu_{i_1}, ..., \mu_{i_d}) \in \mathbb{R}_+^d$, $\nu_i = (\nu_{i_1}, ..., \nu_{i_d}) \in \mathbb{R}_+^d$; and $\lambda_i = \lambda_{i_1,...,i_d}$, $\rho_i = \rho_{i_1,...,i_d} \geq 0$; $\mu, \nu \geq 0$. Call $\nu_i^{\min} = \min\{\nu_{i_1}, ..., \nu_{i_d}\}$.

We use here the first modulus of continuity, with $\delta > 0$,

$$\omega_1(f, \delta) := \sup_{\substack{x, y \in \mathbb{R}^d \\ \|x - y\|_\infty \leq \delta}} |f(x) - f(y)|,$$

where $\|x\|_\infty = \max(|x_1|, ..., |x_d|)$.

Given that f is uniformly continuous we get $\lim_{\delta \to 0} \omega_1(f, \delta) = 0$.

This article is a continuation of [6] at the multivariate level.

So in this article mainly we study the pointwise convergence with rates over $\mathbb{R}^d$, to the unit operator, of the following one hidden layer multivariate normalized neural network perturbed operators,

(i) the Stancu type (see [29])

$$(H_n^*(f))(x) = (H_n^*(f))(x_1, ..., x_d) = \tag{1}$$

$$\frac{\sum_{k=-n^2}^{n^2} \left(\sum_{i=1}^{r} w_i f\left(\frac{k+\mu_i}{n+\nu_i}\right)\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)} =$$

$$\frac{\sum_{k_1=-n^2}^{n^2} ... \sum_{k_d=-n^2}^{n^2} \left(\sum_{i_1=1}^{r} ... \sum_{i_d=1}^{r} w_{i_1,...,i_d} f\left(\frac{k_1+\mu_{i_1}}{n+\nu_{i_1}}, ..., \frac{k_d+\mu_{i_d}}{n+\nu_{i_d}}\right)\right) \cdot}{\sum_{k_1=-n^2}^{n^2} ... \sum_{k_d=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}$$

$$b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right),$$

(ii) the Kantorovich type

$$(K_n^*(f))(x) = \tag{2}$$

$$\frac{\sum_{k=-n^2}^{n^2} \left(\sum_{i=1}^{r} w_i (n + \rho_i)^d \int_0^{\frac{1}{n+\rho_i}} f\left(t + \frac{k+\lambda_i}{n+\rho_i}\right) dt\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)} =$$

$$\sum_{k_1=-n^2}^{n^2} ... \sum_{k_d=-n^2}^{n^2} \left(\sum_{i_1=1}^{r} ... \sum_{i_d=1}^{r} w_{i_1,...,i_d} \left(n + \rho_{i_1,...,i_d}\right)^d \cdot\right.$$

$$\frac{\int ... \int ... \int_0^{\frac{1}{n+\rho_{i_1,...,i_d}}} f\left(t_1 + \frac{k_1+\lambda_{i_1,...,i_d}}{n+\rho_{i_1,...,i_d}}, ..., t_d + \frac{k_d+\lambda_{i_1,...,i_d}}{n+\rho_{i_1,...,i_d}}\right) dt_1...dt_d\right) \cdot}{\sum_{k_1=-n^2}^{n^2} ... \sum_{k_d=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)} \tag{3}$$

$$b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right),$$

and

(iii) the quadrature type

$$(M_n^*(f))(x) = \frac{\sum_{k=-n^2}^{n^2} \left(\sum_{i=1}^{r} w_i f\left(\frac{k}{n} + \frac{i}{nr}\right)\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)} = \tag{4}$$

$$\frac{\sum_{k_1=-n^2}^{n^2} ... \sum_{k_d=-n^2}^{n^2} \left(\sum_{i_1=1}^{r_1} ... \sum_{i_d=1}^{r_d} w_{i_1,...,i_d} f\left(\frac{k_1}{n} + \frac{i_1}{nr_1}, ..., \frac{k_d}{n} + \frac{i_d}{nr_d}\right)\right) \cdot}{\sum_{k_1=-n^2}^{n^2} ... \sum_{k_d=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}$$

$$b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right).$$

Similar operators defined for d-dimensional bell-shaped activation functions and sample coefficients $f\left(\frac{k}{n}\right) = f\left(\frac{k_1}{n}, ..., \frac{k_d}{n}\right)$ were studied initially in [9], [1], [2], [3], [4], [5], etc.

Here we study the multivariate generalized perturbed cases of these operators.

Operator K_n^* in the corresponding Signal Processing context, represents the natural so called "time-jitter" error, where the sample information is calculated in a perturbed neighborhood of $\frac{k+\mu}{n+\nu}$ rather than exactly at the node $\frac{k}{n}$.

The perturbed sample coefficients $f\left(\frac{k+\mu}{n+\nu}\right)$ with $0 \leq \mu \leq \nu$, were first used by D. Stancu [29], in a totally different context, generalizing Bernstein operators approximation on $C([0,1])$.

The terms in the ratio of sums (1)-(4) can be nonzero, iff simultaneously

$$\left|n^{1-\alpha}\left(x_j - \frac{k_j}{n}\right)\right| \leq T_j, \text{ all } j = 1, ..., d, \tag{5}$$

i.e. $\left|x_j - \frac{k_j}{n}\right| \leq \frac{T_j}{n^{1-\alpha}}$, all $j = 1, ..., d$, iff

$$nx_j - T_j n^{\alpha} \leq k_j \leq nx_j + T_j n^{\alpha}, \quad \text{all } j = 1, ..., d. \tag{6}$$

To have the order

$$-n^2 \leq nx_j - T_j n^{\alpha} \leq k_j \leq nx_j + T_j n^{\alpha} \leq n^2, \tag{7}$$

we need $n \geq T_j + |x_j|$, all $j = 1, ..., d$. So (7) is true when we take

$$n \geq \max_{j \in \{1,...,d\}} \left(T_j + |x_j|\right). \quad (8)$$

When $x \in B$ in order to have (7) it is enough to assume that $n \geq 2T^*$, where $T^* := \max\{T_1, ..., T_d\} > 0$. Consider

$$\widetilde{I}_j := \left[nx_j - T_j n^{\alpha}, nx_j + T_j n^{\alpha}\right], \ \ j = 1, ..., d, \ n \in \mathbb{N}.$$

The length of $\widetilde{I}_j$ is $2T_j n^{\alpha}$. By Proposition 1 of [1], we get that the cardinality of $k_j \in \mathbb{Z}$ that belong to $\widetilde{I}_j := card\left(k_j\right) \geq \max\left(2T_j n^{\alpha} - 1, 0\right)$, any $j \in \{1, ..., d\}$. In order to have $card\left(k_j\right) \geq 1$, we need $2T_j n^{\alpha} - 1 \geq 1$ iff $n \geq T_j^{-\frac{1}{\alpha}}$, any $j \in \{1, ..., d\}$.

Therefore, a sufficient condition in order to obtain the order (7) along with the interval $\widetilde{I}_j$ to contain at least one integer for all $j = 1, ..., d$ is that

$$n \geq \max_{j \in \{1,...,d\}} \left\{T_j + |x_j|, T_j^{-\frac{1}{\alpha}}\right\}. \quad (9)$$

Clearly as $n \to +\infty$ we get that $card\left(k_j\right) \to +\infty$, all $j = 1, ..., d$. Also notice that $card\left(k_j\right)$ equals to the cardinality of integers in $\left[\lceil nx_j - T_j n^{\alpha}\rceil, [nx_j + T_j n^{\alpha}]\right]$ for all $j = 1, ..., d$. Here $[\cdot]$ denotes the integral part of the number while $\lceil\cdot\rceil$ denotes its ceiling.

From now on, in this article we assume (9).

We denote by $T = (T_1, ..., T_d)$, $[nx + Tn^{\alpha}] = ([nx_1 + T_1 n^{\alpha}], ..., [nx_d + T_d n^{\alpha}])$, and $\lceil nx - Tn^{\alpha}\rceil = (\lceil nx_1 - T_1 n^{\alpha}\rceil, ..., \lceil nx_d - T_d n^{\alpha}\rceil)$. Furthermore it holds

(i)

$$\left(H_n^*(f)\right)(x) = \left(H_n^*(f)\right)(x_1, ..., x_d) = \quad (10)$$

$$\frac{\sum_{k=\lceil nx - Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{r} w_i f\left(\frac{k+\mu_i}{n+\nu_i}\right)\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=\lceil nx - Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)} =$$

$$\frac{\sum_{k_1=\lceil nx_1 - T_1 n^{\alpha}\rceil}^{[nx_1+T_1 n^{\alpha}]} \cdots \sum_{k_d=\lceil nx_d - T_d n^{\alpha}\rceil}^{[nx_d+T_d n^{\alpha}]} \left(\sum_{i_1=1}^{r_1} \cdots \sum_{i_d=1}^{r_d} w_{i_1,...,i_d} f\left(\frac{k_1+\mu_{i_1}}{n+\nu_{i_1}}, ..., \frac{k_d+\mu_{i_d}}{n+\nu_{i_d}}\right)\right) \cdot}{\sum_{k_1=\lceil nx_1 - T_1 n^{\alpha}\rceil}^{[nx_1+T_1 n^{\alpha}]} \cdots \sum_{k_d=\lceil nx_d - T_d n^{\alpha}\rceil}^{[nx_d+T_d n^{\alpha}]} b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}$$

$$b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), ..., n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right),$$

(ii)

$$\left(K_n^*(f)\right)(x) = \quad (11)$$

$$\frac{\sum_{k=\lceil nx - Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{r} w_i \left(n + \rho_i\right) \int_0^{\frac{1}{n+\rho_i}} f\left(t + \frac{k+\lambda_i}{n+\rho_i}\right) dt\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{\sum_{k=\lceil nx - Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)} =$$

$$\sum_{k_1=\lceil nx_1-T_1n^{\alpha}\rceil}^{[nx_1+T_1n^{\alpha}]} \dots \sum_{k_d=\lceil nx_d-T_dn^{\alpha}\rceil}^{[nx_d+T_dn^{\alpha}]} \left(\sum_{i_1=1}^{r_1} \dots \sum_{i_d=1}^{r_d} w_{i_1,\dots,i_d} \left(n+\rho_{i_1,\dots,i_d}\right)^d \cdot \right.$$

$$\frac{\left. \int \dots \int \dots \int_0^{\frac{1}{n+\rho_{i_1,\dots,i_d}}} f\left(t_1 + \frac{k_1+\lambda_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}, \dots, t_d + \frac{k_d+\lambda_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}\right) dt_1 \dots dt_d \right) \cdot}{\sum_{k_1=\lceil nx_1-T_1n^{\alpha}\rceil}^{[nx_1+T_1n^{\alpha}]} \dots \sum_{k_d=\lceil nx_d-T_dn^{\alpha}\rceil}^{[nx_d+T_dn^{\alpha}]} b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right), \dots, n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)} \tag{12}$$

$$b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right), \dots, n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right),$$

and

(iii)

$$\left(M_n^*(f)\right)(x) = \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{r} w_i f\left(\frac{k}{n}+\frac{i}{nr}\right)\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)} = \tag{13}$$

$$\frac{\sum_{k_1=\lceil nx_1-T_1n^{\alpha}\rceil}^{[nx_1+T_1n^{\alpha}]} \dots \sum_{k_d=\lceil nx_d-T_dn^{\alpha}\rceil}^{[nx_d+T_dn^{\alpha}]} \left(\sum_{i_1=1}^{r_1} \dots \sum_{i_d=1}^{r_d} w_{i_1,\dots,i_d} f\left(\frac{k_1}{n}+\frac{i_1}{nr_1}, \dots, \frac{k_d}{n}+\frac{i_d}{nr_d}\right)\right) \cdot}{\sum_{k_1=\lceil nx_1-T_1n^{\alpha}\rceil}^{[nx_1+T_1n^{\alpha}]} \dots \sum_{k_d=\lceil nx_d-T_dn^{\alpha}\rceil}^{[nx_d+T_dn^{\alpha}]} b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right), \dots, n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)}$$

$$b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right), \dots, n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right).$$

So if $\left|n^{1-\alpha}\left(x_j-\frac{k_j}{n}\right)\right| \leq T_j$, all $j=1,\dots,d$, we get that

$$\left\|x-\frac{k}{n}\right\|_{\infty} \leq \frac{T^*}{n^{1-\alpha}}. \tag{14}$$

For convinience we call

$$V(x) = \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) =$$

$$\sum_{k_1=\lceil nx_1-T_1n^{\alpha}\rceil}^{[nx_1+T_1n^{\alpha}]} \dots \sum_{k_d=\lceil nx_d-T_dn^{\alpha}\rceil}^{[nx_d+T_dn^{\alpha}]} b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right), \dots, n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right). \tag{15}$$

We make

Remark 2 *Here always k is as in (7).*

I) We observe that

$$\left\|\frac{k+\mu_i}{n+\nu_i}-x\right\|_{\infty} \leq \left\|\frac{k}{n+\nu_i}-x\right\|_{\infty} + \left\|\frac{\mu_i}{n+\nu_i}\right\|_{\infty} \tag{16}$$

$$\leq \left\| \frac{k}{n+\nu_i} - x \right\|_\infty + \frac{\|\mu_i\|_\infty}{n+\nu_i^{\min}}.$$

Next we see

$$\left\| \frac{k}{n+\nu_i} - x \right\|_\infty \leq \left\| \frac{k}{n+\nu_i} - \frac{k}{n} \right\|_\infty + \left\| \frac{k}{n} - x \right\|_\infty \overset{(14)}{\leq} \left\| \frac{\nu_i k}{n(n+\nu_i)} \right\|_\infty + \frac{T^*}{n^{1-\alpha}} \tag{17}$$

$$\leq \|k\|_\infty \frac{\|\nu_i\|_\infty}{n\left(n+\nu_i^{\min}\right)} + \frac{T^*}{n^{1-\alpha}} =: (*).$$

We notice for $j = 1, ..., d$ *we get that*

$$|k_j| \leq n|x_j| + T_j n^\alpha.$$

Therefore

$$\|k\|_\infty \leq \|n|x| + Tn^\alpha\|_\infty \leq n\|x\|_\infty + T^* n^\alpha, \tag{18}$$

where $|x| = (|x_1|, ..., |x|_d)$.

Thus

$$(*) \leq (n\|x\|_\infty + T^* n^\alpha) \frac{\|\nu_i\|_\infty}{n\left(n+\nu_i^{\min}\right)} + \frac{T^*}{n^{1-\alpha}} = \tag{19}$$

$$\left(\|x\|_\infty + T^* n^{\alpha-1}\right) \frac{\|\nu_i\|_\infty}{\left(n+\nu_i^{\min}\right)} + \frac{T^*}{n^{1-\alpha}}.$$

So we get

$$\left\| \frac{k}{n+\nu_i} - x \right\|_\infty \leq \left(\|x\|_\infty + \frac{T^*}{n^{1-\alpha}} \right) \frac{\|\nu_i\|_\infty}{\left(n+\nu_i^{\min}\right)} + \frac{T^*}{n^{1-\alpha}}. \tag{20}$$

Consequently we obtain

$$\left\| \frac{k+\mu_i}{n+\nu_i} - x \right\|_\infty \leq \left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n+\nu_i^{\min}} \right) + \left(1 + \frac{\|\nu_i\|_\infty}{\left(n+\nu_i^{\min}\right)} \right) \frac{T^*}{n^{1-\alpha}}. \tag{21}$$

Hence we derive

$$\omega_1\left(f, \left\| \frac{k+\mu_i}{n+\nu_i} - x \right\|_\infty \right) \leq$$

$$\omega_1\left(f, \left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n+\nu_i^{\min}} \right) + \left(1 + \frac{\|\nu_i\|_\infty}{\left(n+\nu_i^{\min}\right)} \right) \frac{T^*}{n^{1-\alpha}} \right), \tag{22}$$

with dominant speed of convergence $\frac{1}{n^{1-\alpha}}$.

II) We also have for

$$0 \leq t_j \leq \frac{1}{n+\rho_{i_1,...,i_d}}, \quad j = 1, ..., d, \tag{23}$$

that

$$\left\|t+\frac{k+\lambda_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}-x\right\|_\infty \le \frac{1}{n+\rho_{i_1,\dots,i_d}}+\left\|\frac{k+\lambda_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}-x\right\|_\infty \le \tag{24}$$

$$\frac{1+\lambda_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}+\left\|\frac{k}{n+\rho_{i_1,\dots,i_d}}-x\right\|_\infty \le$$

$$\frac{1+\lambda_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}+\left\|\frac{k}{n+\rho_{i_1,\dots,i_d}}-\frac{k}{n}\right\|_\infty+\left\|\frac{k}{n}-x\right\|_\infty \le$$

$$\frac{1+\lambda_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}+\frac{T^*}{n^{1-\alpha}}+\frac{\rho_{i_1,\dots,i_d}\|k\|_\infty}{\left(n+\rho_{i_1,\dots,i_d}\right)n} \le$$

$$\frac{1+\lambda_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}+\frac{T^*}{n^{1-\alpha}}+\frac{\rho_{i_1,\dots,i_d}}{n\left(n+\rho_{i_1,\dots,i_d}\right)}\left(n\|x\|_\infty+T^*n^\alpha\right)= \tag{25}$$

$$\frac{1+\lambda_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}+\frac{T^*}{n^{1-\alpha}}+\left(\frac{\rho_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}\right)\left(\|x\|_\infty+\frac{T^*}{n^{1-\alpha}}\right). \tag{26}$$

We have found that

$$\left\|t+\frac{k+\lambda_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}-x\right\|_\infty \le \tag{27}$$

$$\left(\frac{\rho_{i_1,\dots,i_d}\|x\|_\infty+\lambda_{i_1,\dots,i_d}+1}{n+\rho_{i_1,\dots,i_d}}\right)+\left(1+\frac{\rho_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}\right)\frac{T^*}{n^{1-\alpha}}.$$

So when $0\le t_j\le \frac{1}{n+\rho_{i_1,\dots,i_d}}$, $j=1,...,d$, *we get that*

$$\omega_1\left(f,\left\|t+\frac{k+\lambda_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}-x\right\|_\infty\right)\le \tag{28}$$

$$\omega_1\left(f,\left(\frac{\rho_{i_1,\dots,i_d}\|x\|_\infty+\lambda_{i_1,\dots,i_d}+1}{n+\rho_{i_1,\dots,i_d}}\right)+\left(1+\frac{\rho_{i_1,\dots,i_d}}{n+\rho_{i_1,\dots,i_d}}\right)\frac{T^*}{n^{1-\alpha}}\right),$$

with dominant speed $\frac{1}{n^{1-\alpha}}$.

III) We observe that

$$\left\|\frac{k}{n}+\frac{i}{nr}-x\right\|_\infty \le \left\|\frac{k}{n}-x\right\|_\infty+\frac{1}{n}\left\|\frac{i}{r}\right\|_\infty \le \frac{T^*}{n^{1-\alpha}}+\frac{1}{n}. \tag{29}$$

Hence

$$\omega_1\left(f,\left\|\frac{k}{n}+\frac{i}{nr}-x\right\|_\infty\right)\le \omega_1\left(f,\frac{T^*}{n^{1-\alpha}}+\frac{1}{n}\right), \tag{30}$$

with dominant speed $\frac{1}{n^{1-\alpha}}$.

3 Main Results

We present our first approximation result

Theorem 3 *Let* $x \in \mathbb{R}^d$ *and* $n \in \mathbb{N}$ *such that* $n \geq \max_{j \in \{1,...,d\}} \left(T_j + |x_j|, T_j^{-\frac{1}{\alpha}} \right)$, $T_j > 0$, $0 < \alpha < 1$. *Then*

$$\left| \left(H_n^* (f) \right) (x) - f(x) \right| \leq$$

$$\sum_{i=1}^{r} w_i \omega_1 \left(f, \left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n + \nu_i^{\min}} \right) + \left(1 + \frac{\|\nu_i\|_\infty}{n + \nu_i^{\min}} \right) \frac{T^*}{n^{1-\alpha}} \right) = \tag{31}$$

$$\sum_{i_1=1}^{r_1} ... \sum_{i_d=1}^{r_d} w_{i_1,...,i_d} \omega_1 \left(f, \left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n + \nu_i^{\min}} \right) + \left(1 + \frac{\|\nu_i\|_\infty}{n + \nu_i^{\min}} \right) \frac{T^*}{n^{1-\alpha}} \right),$$

where $i = (i_1, ..., i_d)$.

Proof. We notice that

$$\left(H_n^* (f) \right) (x) - f(x) = \frac{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} \left(\sum_{i=1}^{r} w_i f\left(\frac{k+\mu_i}{n+\nu_i} \right) \right) b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)} - f(x) \tag{32}$$

$$= \frac{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} \left(\sum_{i=1}^{r} w_i f\left(\frac{k+\mu_i}{n+\nu_i} \right) \right) b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right) - f(x) \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)} =$$

$$\frac{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} \left(\left(\sum_{i=1}^{r} w_i f\left(\frac{k+\mu_i}{n+\nu_i} \right) \right) - f(x) \right) b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)} = \tag{33}$$

$$\frac{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} \left(\sum_{i=1}^{r} w_i \left(f\left(\frac{k+\mu_i}{n+\nu_i} \right) - f(x) \right) \right) b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}.$$

Hence it holds

$$\left| \left(H_n^* (f) \right) (x) - f(x) \right| \leq$$

$$\frac{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} \left(\sum_{i=1}^{r} w_i \left| f\left(\frac{k+\mu_i}{n+\nu_i} \right) - f(x) \right| \right) b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)} \leq \tag{34}$$

$$\frac{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} \left(\sum_{i=1}^{r} w_i \omega_1 \left(f, \left\| \frac{k+\mu_i}{n+\nu_i} - x \right\|_\infty \right) \right) b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx+Tn^\alpha]} b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)} \overset{(22)}{\leq}$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{r} w_i\omega_1\left(f,\left(\frac{\|\nu_i\|_{\infty}\|x\|_{\infty}+\|\mu_i\|_{\infty}}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_{\infty}}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right)\right)\cdot}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)} \tag{35}$$

$$b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)=$$

$$\frac{\left[\sum_{i=1}^{r} w_i\omega_1\left(f,\left(\frac{\|\nu_i\|_{\infty}\|x\|_{\infty}+\|\mu_i\|_{\infty}}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_{\infty}}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right)\right]\cdot}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}$$

$$\left(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right)$$

$$=\sum_{i=1}^{r} w_i\omega_1\left(f,\left(\frac{\|\nu_i\|_{\infty}\|x\|_{\infty}+\|\mu_i\|_{\infty}}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_{\infty}}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right), \tag{36}$$

proving the claim. ■

Corollary 4 *(to Theorem 3) Let* $x \in \prod_{j=1}^{d}\left[-\gamma_j,\gamma_j\right] \subset \mathbb{R}^d$, $\gamma_j > 0$, $\gamma^* = \max\{\gamma_1,...,\gamma_d\}$ *and* $n \in \mathbb{N}$ *such that* $n \geq \max_{j\in\{1,...,d\}}\{T_j+\gamma_j, T_j^{-\frac{1}{\alpha}}\}$. *Then*

$$\|H_n^*(f)-f\|_{\infty,\prod_{j=1}^{d}\left[-\gamma_j,\gamma_j\right]} \leq$$

$$\sum_{i_1=1}^{r_1}\cdots\sum_{i_d=1}^{r_d} w_{i_1,...,i_d}\omega_1\left(f,\left(\frac{\|\nu_i\|_{\infty}\gamma^*+\|\mu_i\|_{\infty}}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_{\infty}}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right). \tag{37}$$

Proof. By (31). ■

We continue with

Theorem 5 *All assumptions as in Theorem 3. Then*

$$|(K_n^*(f))(x)-f(x)| \leq$$

$$\sum_{i=1}^{r} w_i\omega_1\left(f,\left(\frac{\rho_i\|x\|_{\infty}+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right)= \tag{38}$$

$$\sum_{i_1=1}^{r_1}\cdots\sum_{i_d=1}^{r_d} w_{i_1,...,i_d}\omega_1\left(f,\left(\frac{\rho_{i_1,...,i_d}\|x\|_{\infty}+\lambda_{i_1,...,i_d}+1}{n+\rho_{i_1,...,i_d}}\right)+\left(1+\frac{\rho_{i_1,...,i_d}}{n+\rho_{i_1,...,i_d}}\right)\frac{T^*}{n^{1-\alpha}}\right).$$

Proof. We observe the following

$$\left|\left(K_n^*(f)\right)(x) - f(x)\right| =$$

$$\left|\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{r} w_i (n+\rho_i)^d \int_0^{\frac{1}{n+\rho_i}} f\left(t+\frac{k+\lambda_i}{n+\rho_i}\right) dt\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} - f(x)\right| = \tag{39}$$

$$\left|\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{r} w_i (n+\rho_i)^d \int_0^{\frac{1}{n+\rho_i}} f\left(t+\frac{k+\lambda_i}{n+\rho_i}\right) dt\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) - f(x)V(x)}{V(x)}\right| =$$

$$\left|\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left[\left(\sum_{i=1}^{r} w_i (n+\rho_i)^d \int_0^{\frac{1}{n+\rho_i}} f\left(t+\frac{k+\lambda_i}{n+\rho_i}\right) dt\right) - f(x)\right] b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\right| =$$

$$\left|\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left[\sum_{i=1}^{r} w_i \left[(n+\rho_i)^d \int_0^{\frac{1}{n+\rho_i}} f\left(t+\frac{k+\lambda_i}{n+\rho_i}\right) dt - f(x)\right]\right] b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\right| \tag{40}$$

$$= \frac{\left|\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left[\sum_{i=1}^{r} w_i \left[(n+\rho_i)^d \int_0^{\frac{1}{n+\rho_i}} \left[f\left(t+\frac{k+\lambda_i}{n+\rho_i}\right) - f(x)\right] dt\right]\right] b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right|}{V(x)} \leq$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left[\sum_{i=1}^{r} w_i \left[(n+\rho_i)^d \int_0^{\frac{1}{n+\rho_i}} \left|f\left(t+\frac{k+\lambda_i}{n+\rho_i}\right) - f(x)\right| dt\right]\right] b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \leq$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left[\sum_{i=1}^{r} w_i \left[(n+\rho_i)^d \int_0^{\frac{1}{n+\rho_i}} \omega_1\left(f, \left\|t+\frac{k+\lambda_i}{n+\rho_i} - x\right\|_{\infty}\right) dt\right]\right] b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \ \text{(by (28))} \leq$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left[\sum_{i=1}^{r} w_i \left[(n+\rho_i)^d \int_0^{\frac{1}{n+\rho_i}} \omega_1\left(f, \left(\frac{\rho_i \|x\|_{\infty}+\lambda_i+1}{n+\rho_i}\right) + \left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right) dt\right]\right] b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} =$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left[\sum_{i=1}^{r} w_i \omega_1\left(f, \left(\frac{\rho_i \|x\|_{\infty}+\lambda_i+1}{n+\rho_i}\right) + \left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right)\right] b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \tag{41}$$

$$= \sum_{i=1}^{r} w_i \omega_1\left(f, \left(\frac{\rho_i \|x\|_{\infty}+\lambda_i+1}{n+\rho_i}\right) + \left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right), \tag{42}$$

proving the claim. ■

Corollary 6 *(to Theorem 5) All here as in Corollary 4. It holds*

$$\left\|K_n^*(f) - f\right\|_{\infty, \prod_{j=1}^{d}\left[-\gamma_j, \gamma_j\right]} \leq$$

$$\sum_{i=1}^{r} w_i \omega_1 \left(f, \left(\frac{\rho_i \gamma^* + \lambda_i + 1}{n + \rho_i} \right) + \left(1 + \frac{\rho_i}{n + \rho_i} \right) \frac{T^*}{n^{1-\alpha}} \right) = \tag{43}$$

$$\sum_{i_1=1}^{r_1} \dots \sum_{i_d=1}^{r_d} w_{i_1,\dots,i_d} \omega_1 \left(f, \left(\frac{\rho_{i_1,\dots,i_d} \gamma^* + \lambda_{i_1,\dots,i_d} + 1}{n + \rho_{i_1,\dots,i_d}} \right) + \left(1 + \frac{\rho_{i_1,\dots,i_d}}{n + \rho_{i_1,\dots,i_d}} \right) \frac{T^*}{n^{1-\alpha}} \right).$$

Proof. By (38). ■

We also present

Theorem 7 *All here as in Theorem 3. Then*

$$|M_n^* (f) (x) - f (x)| \leq \omega_1 \left(f, \frac{T^*}{n^{1-\alpha}} + \frac{1}{n} \right). \tag{44}$$

Proof. We observe that

$$|M_n^* (f) (x) - f (x)| =$$

$$\left| \frac{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx + Tn^\alpha]} \left(\sum_{i=1}^{r} w_i f \left(\frac{k}{n} + \frac{i}{nr} \right) \right) b \left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}{V (x)} - f (x) \right| = \tag{45}$$

$$\left| \frac{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx + Tn^\alpha]} \left(\sum_{i=1}^{r} w_i f \left(\frac{k}{n} + \frac{i}{nr} \right) \right) b \left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right) - f (x) V (x)}{V (x)} \right| =$$

$$\frac{\left| \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx + Tn^\alpha]} \left[\left(\sum_{i=1}^{r} w_i f \left(\frac{k}{n} + \frac{i}{nr} \right) \right) - f (x) \right] b \left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right) \right|}{V (x)} = \tag{46}$$

$$\frac{\left| \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx + Tn^\alpha]} \left[\sum_{i=1}^{r} w_i \left(f \left(\frac{k}{n} + \frac{i}{nr} \right) - f (x) \right) \right] b \left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right) \right|}{V (x)} \leq$$

$$\frac{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx + Tn^\alpha]} \left[\sum_{i=1}^{r} w_i \left| f \left(\frac{k}{n} + \frac{i}{nr} \right) - f (x) \right| \right] b \left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}{V (x)} \leq$$

$$\frac{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx + Tn^\alpha]} \left[\sum_{i=1}^{r} w_i \omega_1 \left(f, \left\| \frac{k}{n} + \frac{i}{nr} - x \right\|_\infty \right) \right] b \left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}{V (x)} \overset{(\text{by } (30))}{\leq} \tag{47}$$

$$\frac{\sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx + Tn^\alpha]} \left[\sum_{i=1}^{r} w_i \omega_1 \left(f, \frac{T^*}{n^{1-\alpha}} + \frac{1}{n} \right) \right] b \left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}{V (x)}$$

$$= \omega_1 \left(f, \frac{T^*}{n^{1-\alpha}} + \frac{1}{n} \right), \tag{48}$$

proving the claim. ■

Corollary 8 *(to Theorem 7) All here as in Corollary 4. It holds*

$$\left\| M_n^* (f) - f \right\|_{\infty, \prod_{j=1}^{d} [-\gamma_j, \gamma_j]} \leq \omega_1 \left(f, \frac{T^*}{n^{1-\alpha}} + \frac{1}{n} \right). \quad (49)$$

Proof. By (44). ■

Note 9 *Theorems 3, 5, 7 and Corollaries 4, 6, 8 given that f is uniformly continuous, produce the pointwise and uniform convergences with rates, at speed $\frac{1}{n^{1-\alpha}}$, of multivariate neural network operators H_n^*, K_n^*, M_n^* to the unit operator. Notice that the right hand sides of inequalities (31), (37), (38), (43), (44) and (49) do not depend on b.*

Next we present higher order of approximation results based on the high order differentiability of the approximated function.

Theorem 10 *Let $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$ such that $n \geq \max_{j \in \{1,\dots,d\}} \left(T_j + \left| x_j \right|, T_j^{-\frac{1}{\alpha}} \right)$, $T_j > 0$, $0 < \alpha < 1$. Let also $f \in C^N \left(\mathbb{R}^d \right)$, $N \in \mathbb{N}$, such that all of its partial derivatives $f_{\widetilde{\alpha}}$ of order N, $\widetilde{\alpha} : |\widetilde{\alpha}| = \sum_{j=1}^{d} \alpha_j = N$, are uniformly continuous or continuous and bounded. Then*

$$\left| \left(H_n^* (f) \right) (x) - f (x) \right| \leq \sum_{l=1}^{N} \frac{1}{l!} \left(\left(\sum_{j=1}^{d} \left| \frac{\partial}{\partial x_j} \right| \right)^l f (x) \right) \cdot$$

$$\left[\sum_{i=1}^{r} w_i \left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n + \nu_i^{\min}} \right) + \left(1 + \frac{\|\nu_i\|_\infty}{n + \nu_i^{\min}} \right) \frac{T^*}{n^{1-\alpha}} \right]^l \right] +$$

$$\frac{d^N}{N!} \sum_{i=1}^{r} w_i \left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n + \nu_i^{\min}} \right) + \left(1 + \frac{\|\nu_i\|_\infty}{n + \nu_i^{\min}} \right) \frac{T^*}{n^{1-\alpha}} \right]^N \cdot$$

$$\max_{\widetilde{\alpha} : |\widetilde{\alpha}| = N} \omega_1 \left(f_{\widetilde{\alpha}}, \left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n + \nu_i^{\min}} \right) + \left(1 + \frac{\|\nu_i\|_\infty}{n + \nu_i^{\min}} \right) \frac{T^*}{n^{1-\alpha}} \right] \right). \quad (50)$$

Inequality (50) implies the pointwise convergence with rates on $\left(H_n^* (f) \right) (x)$to $f (x)$, as $n \to \infty$, at speed $\frac{1}{n^{1-\alpha}}$.

Proof. Set

$$g_{\frac{k+\mu_i}{n+\nu_i}} (t) = f \left(x + t \left(\frac{k + \mu_i}{n + \nu_i} - x \right) \right), \quad 0 \leq t \leq 1. \quad (51)$$

Then we have

$$g_{\frac{k+\mu_i}{n+\nu_i}}^{(l)} (t) = \left[\left(\sum_{j=1}^{d} \left(\frac{k_j + \mu_{i_j}}{n + \nu_{i_j}} - x_j \right) \frac{\partial}{\partial x_j} \right)^l f \right] \quad (52)$$

$$\left(x_1+t\left(\frac{k_1+\mu_{i_1}}{n+\nu_{i_1}}-x_1\right),...,x_d+t\left(\frac{k_d+\mu_{i_d}}{n+\nu_{i_d}}-x_d\right)\right),$$

and

$$g_{\frac{k+\mu_i}{n+\nu_i}}(0)=f(x).$$

By Taylor's formula, we get

$$f\left(\frac{k_1+\mu_{i_1}}{n+\nu_{i_1}},...,\frac{k_d+\mu_{i_d}}{n+\nu_{i_d}}\right)=g_{\frac{k+\mu_i}{n+\nu_i}}(1)= \tag{53}$$

$$\sum_{l=0}^{N}\frac{g^{(l)}_{\frac{k+\mu_i}{n+\nu_i}}(0)}{l!}+R_N\left(\frac{k+\mu_i}{n+\nu_i},0\right),$$

where

$$R_N\left(\frac{k+\mu_i}{n+\nu_i},0\right)=\int_0^1\left(\int_0^{t_1}...\left(\int_0^{t_{N-1}}\left(g^{(N)}_{\frac{k+\mu_i}{n+\nu_i}}(t_N)-g^{(N)}_{\frac{k+\mu_i}{n+\nu_i}}(0)\right)dt_N\right)...\right)dt_1. \tag{54}$$

Here we denote by

$$f_{\widetilde{\alpha}}:=\frac{\partial^{\widetilde{\alpha}}f}{\partial x^{\widetilde{\alpha}}},\quad \widetilde{\alpha}:=(\alpha_1,...,\alpha_d),\ \ \alpha_j\in\mathbb{Z}^+,\ j=1,...,d,$$

such that $|\widetilde{\alpha}|=\sum_{j=1}^{d}\alpha_j=N$. Thus

$$\sum_{i=1}^{r}w_if\left(\frac{k+\mu_i}{n+\nu_i}\right)=\sum_{l=0}^{N}\sum_{i=1}^{r}w_i\frac{g^{(l)}_{\frac{k+\mu_i}{n+\nu_i}}(0)}{l!}+\sum_{i=1}^{r}w_iR_N\left(\frac{k+\mu_i}{n+\nu_i},0\right), \tag{55}$$

and

$$(H_n^*(f))(x)=\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{r}w_if\left(\frac{k+\mu_i}{n+\nu_i}\right)\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}=$$

$$\sum_{l=0}^{N}\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{r}w_i\frac{g^{(l)}_{\frac{k+\mu_i}{n+\nu_i}}(0)}{l!}\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}+ \tag{56}$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\left(\sum_{i=1}^{r}w_iR_N\left(\frac{k+\mu_i}{n+\nu_i},0\right)\right).$$

Therefore it holds

$$(H_n^*(f))(x)-f(x)= \tag{57}$$

$$\sum_{l=1}^{N} \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{r} w_i \frac{g^{(l)}_{\frac{k+\mu_i}{n+\nu_i}}(0)}{l!} \right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} + R^*,$$

where

$$R^* = \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \left(\sum_{i=1}^{r} w_i R_N \left(\frac{k+\mu_i}{n+\nu_i}, 0 \right) \right). \tag{58}$$

Consequently, we obtain

$$|(H_n^*(f))(x) - f(x)| \leq \tag{51}$$

$$\sum_{l=1}^{N} \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{r} w_i \frac{\left| g^{(l)}_{\frac{k+\mu_i}{n+\nu_i}}(0) \right|}{l!} \right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} + |R^*|. \tag{59}$$

Notice that

$$\left| g^{(l)}_{\frac{k+\mu_i}{n+\nu_i}}(0) \right| \overset{(21)}{\leq} \left(\left(\sum_{j=1}^{d} \left| \frac{\partial}{\partial x_j} \right| \right)^l f(x) \right)$$

$$\left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n+\nu_i^{\min}} \right) + \left(1 + \frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}} \right) \frac{T^*}{n^{1-\alpha}} \right]^l, \tag{60}$$

and so far we have

$$|(H_n^*(f))(x) - f(x)| \leq \sum_{l=1}^{N} \frac{1}{l!} \left(\left(\sum_{j=1}^{d} \left| \frac{\partial}{\partial x_j} \right| \right)^l f(x) \right)$$

$$\left[\sum_{i=1}^{r} w_i \left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n+\nu_i^{\min}} \right) + \left(1 + \frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}} \right) \frac{T^*}{n^{1-\alpha}} \right]^l \right] + |R^*|. \tag{61}$$

Next, we need to estimate $|R^*|$. For that, we observe $(0 \leq t_N \leq 1)$

$$\left| g^{(N)}_{\frac{k+\mu_i}{n+\nu_i}}(t_N) - g^{(N)}_{\frac{k+\mu_i}{n+\nu_i}}(0) \right| =$$

$$\left| \left[\left(\sum_{j=1}^{d} \left(\frac{k_j+\mu_{i_j}}{n+\nu_{i_j}} - x_j \right) \frac{\partial}{\partial x_j} \right)^N f \right] \left(x + t_N \left(\frac{k+\mu_i}{n+\nu_i} - x \right) \right) - \right. \tag{62}$$

$$\left[\left(\sum_{j=1}^{d}\left(\frac{k_j+\mu_{i_j}}{n+\nu_{i_j}}-x_j\right)\frac{\partial}{\partial x_j}\right)^N f\right](x)\,|\overset{\text{(by (21), (22))}}{\leq}$$

$$d^N\left[\left(\frac{\|\nu_i\|_\infty\|x\|_\infty+\|\mu_i\|_\infty}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]^N\cdot$$

$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\left[\left(\frac{\|\nu_i\|_\infty\|x\|_\infty+\|\mu_i\|_\infty}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]\right). \quad (63)$$

Thus we find

$$\left|R_N\left(\frac{k+\mu_i}{n+\nu_i},0\right)\right|\leq$$

$$\int_0^1\left(\int_0^{t_1}...\left(\int_0^{t_{N-1}}\left|g^{(N)}_{\frac{k+\mu_i}{n+\nu_i}}(t_N)-g^{(N)}_{\frac{k+\mu_i}{n+\nu_i}}(0)\right|dt_N\right)...\right)dt_1\leq$$

$$\frac{d^N}{N!}\left[\left(\frac{\|\nu_i\|_\infty\|x\|_\infty+\|\mu_i\|_\infty}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]^N\cdot$$

$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\left[\left(\frac{\|\nu_i\|_\infty\|x\|_\infty+\|\mu_i\|_\infty}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]\right). \quad (64)$$

Finally we obtain

$$|R^*|\leq\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\left(\sum_{i=1}^{r}w_i\left|R_N\left(\frac{k+\mu_i}{n+\nu_i},0\right)\right|\right)\leq$$

$$\sum_{i=1}^{r}w_i\frac{d^N}{N!}\left[\left(\frac{\|\nu_i\|_\infty\|x\|_\infty+\|\mu_i\|_\infty}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]^N\cdot$$

$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\left[\left(\frac{\|\nu_i\|_\infty\|x\|_\infty+\|\mu_i\|_\infty}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]\right). \quad (65)$$

Using (61) and (65) we derive (50). ■

Corollary 11 *(to Theorem 10). Let all as in Theorem 10. Additionally assume that all $f_{\widetilde{\alpha}}(x)=0$, $\widetilde{\alpha}:|\widetilde{\alpha}|=\rho$, $1\leq\rho\leq N$. Then*

$$|(H_n^*(f))(x)-f(x)|\leq$$

$$\frac{d^N}{N!}\sum_{i=1}^{r}w_i\left[\left(\frac{\|\nu_i\|_\infty\|x\|_\infty+\|\mu_i\|_\infty}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]^N\cdot$$

$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\left[\left(\frac{\|\nu_i\|_\infty\|x\|_\infty+\|\mu_i\|_\infty}{n+\nu_i^{\min}}\right)+\left(1+\frac{\|\nu_i\|_\infty}{n+\nu_i^{\min}}\right)\frac{T^*}{n^{1-\alpha}}\right]\right). \quad (66)$$

Inequality (66) implies the pointwise convergence with rates of $(H_n^*(f))(x)$ to $f(x)$, as $n\to\infty$, at the high speed $\frac{1}{n^{(1-\alpha)(N+1)}}$.

Proof. By (50). ■

The uniform convergence with rates follows from

Corollary 12 *(to Theorem 10) All as in Theorem 10, but now* $x \in G = \prod_{j=1}^{d} \left[-\gamma_j, \gamma_j\right] \subset \mathbb{R}^d$, $\gamma_j > 0$, $\gamma^* = \max\{\gamma_1, ..., \gamma_d\}$ *and* $n \in \mathbb{N}$:
$n \geq \max_{j \in \{1,...,d\}} \left(T_j + \gamma_j, T_j^{-\frac{1}{\alpha}}\right)$. *Then*

$$\|H_n^*(f) - f\|_{\infty, \prod_{j=1}^{d}\left[-\gamma_j, \gamma_j\right]} \leq \sum_{l=1}^{N} \frac{1}{l!} \left(\left(\sum_{j=1}^{d} \left| \frac{\partial}{\partial x_j} \right| \right)^l \|f\|_{\infty, \prod_{j=1}^{d}\left[-\gamma_j, \gamma_j\right]} \right)$$

$$\left[\sum_{i=1}^{r} w_i \left[\left(\frac{\|\nu_i\|_\infty \gamma^* + \|\mu_i\|_\infty}{n + \nu_i^{\min}} \right) + \left(1 + \frac{\|\nu_i\|_\infty}{n + \nu_i^{\min}} \right) \frac{T^*}{n^{1-\alpha}} \right]^l \right] +$$

$$\frac{d^N}{N!} \sum_{i=1}^{r} w_i \left[\left(\frac{\|\nu_i\|_\infty \gamma^* + \|\mu_i\|_\infty}{n + \nu_i^{\min}} \right) + \left(1 + \frac{\|\nu_i\|_\infty}{n + \nu_i^{\min}} \right) \frac{T^*}{n^{1-\alpha}} \right]^N \cdot$$

$$\max_{\widetilde{\alpha}: |\widetilde{\alpha}| = N} \omega_1 \left(f_{\widetilde{\alpha}}, \left[\left(\frac{\|\nu_i\|_\infty \gamma^* + \|\mu_i\|_\infty}{n + \nu_i^{\min}} \right) + \left(1 + \frac{\|\nu_i\|_\infty}{n + \nu_i^{\min}} \right) \frac{T^*}{n^{1-\alpha}} \right] \right). \quad (67)$$

Inequality (67) implies the uniform convergence with rates of $H_n^*(f)$ *to* f *on* G, *as* $n \to \infty$, *at speed* $\frac{1}{n^{1-\alpha}}$.

Proof. By (50). ■

Corollary 13 *(to Theorem 10) All as in Theorem 10 with* $N = 1$. *Then*

$$|(H_n^*(f))(x) - f(x)| \leq \left(\sum_{j=1}^{d} \left| \frac{\partial f(x)}{\partial x_j} \right| \right)$$

$$\left[\sum_{i=1}^{r} w_i \left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n + \nu_i^{\min}} \right) + \left(1 + \frac{\|\nu_i\|_\infty}{n + \nu_i^{\min}} \right) \frac{T^*}{n^{1-\alpha}} \right] \right] +$$

$$d \sum_{i=1}^{r} w_i \left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n + \nu_i^{\min}} \right) + \left(1 + \frac{\|\nu_i\|_\infty}{n + \nu_i^{\min}} \right) \frac{T^*}{n^{1-\alpha}} \right] \cdot$$

$$\max_{\widetilde{\alpha}: |\widetilde{\alpha}| = N} \omega_1 \left(f_{\widetilde{\alpha}}, \left[\left(\frac{\|\nu_i\|_\infty \|x\|_\infty + \|\mu_i\|_\infty}{n + \nu_i^{\min}} \right) + \left(1 + \frac{\|\nu_i\|_\infty}{n + \nu_i^{\min}} \right) \frac{T^*}{n^{1-\alpha}} \right] \right). \quad (68)$$

Inequality (68) implies the pointwise convergence with rates of $(H_n^*(f))(x)$ *to* $f(x)$, *as* $n \to \infty$, *at speed* $\frac{1}{n^{1-\alpha}}$.

Proof. By (50). ■

We continue with

Theorem 14 *All here as in Theorem 10. Then*

$$|(K_n^*(f))(x) - f(x)| \leq$$

$$\sum_{l=1}^{N} \frac{1}{l!} \left(\sum_{i=1}^{r} w_i \left[\left(\frac{\rho_i \|x\|_\infty + \lambda_i + 1}{n + \rho_i} \right) + \left(1 + \frac{\rho_i}{n + \rho_i} \right) \frac{T^*}{n^{1-\alpha}} \right]^l \right) \cdot$$

$$\left(\left(\sum_{j=1}^{d} \left| \frac{\partial}{\partial x_j} \right| \right)^l f(x) \right) + \tag{69}$$

$$\frac{d^N}{N!} \sum_{i=1}^{r} w_i \left[\left(\frac{\rho_i \|x\|_\infty + \lambda_i + 1}{n + \rho_i} \right) + \left(1 + \frac{\rho_i}{n + \rho_i} \right) \frac{T^*}{n^{1-\alpha}} \right]^N \cdot$$

$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1 \left(f_{\widetilde{\alpha}}, \left[\left(\frac{\rho_i \|x\|_\infty + \lambda_i + 1}{n + \rho_i} \right) + \left(1 + \frac{\rho_i}{n + \rho_i} \right) \frac{T^*}{n^{1-\alpha}} \right] \right).$$

Inequality (69) implies the pointwise convergence with rates of $(K_n^*(f))(x)$ *to* $f(x)$*, as* $n \to \infty$*, at speed* $\frac{1}{n^{1-\alpha}}$*.*

Proof. Set

$$g_{t+\frac{k+\lambda_i}{n+\rho_i}}(\lambda^*) = f\left(x + \lambda^* \left(t + \frac{k + \lambda_i}{n + \rho_i} - x \right) \right), \quad 0 \leq \lambda^* \leq 1. \tag{70}$$

Then we have

$$g^{(l)}_{t+\frac{k+\lambda_i}{n+\rho_i}}(\lambda^*) = \tag{71}$$

$$\left[\left(\sum_{j=1}^{d} \left(t_j + \frac{k_j + \lambda_i}{n + \rho_i} - x_j \right) \frac{\partial}{\partial x_j} \right)^l f \right] \left(x + \lambda^* \left(t + \frac{k + \lambda_i}{n + \rho_i} - x \right) \right),$$

and

$$g_{t+\frac{k+\lambda_i}{n+\rho_i}}(0) = f(x). \tag{72}$$

By Taylor's formula, we get

$$f\left(t + \frac{k + \lambda_i}{n + \rho_i} \right) = g_{t+\frac{k+\lambda_i}{n+\rho_i}}(1) = \tag{73}$$

$$\sum_{l=0}^{N} \frac{g^{(l)}_{t+\frac{k+\lambda_i}{n+\rho_i}}(0)}{l!} + R_N\left(t + \frac{k + \lambda_i}{n + \rho_i}, 0 \right),$$

where

$$R_N\left(t + \frac{k + \lambda_i}{n + \rho_i}, 0 \right) =$$

$$\int_0^1 \left(\int_0^{\lambda_1^*} ... \left(\int_0^{\lambda_{N-1}^*} \left(g_{t+\frac{k+\lambda_i}{n+\rho_i}}^{(N)} \left(\lambda_N^* \right) - g_{t+\frac{k+\lambda_i}{n+\rho_i}}^{(N)} (0) \right) d\lambda_N^* \right) ... \right) d\lambda_1^*. \tag{74}$$

Here we denote by

$$f_{\widetilde{\alpha}} := \frac{\partial^{\widetilde{\alpha}} f}{\partial x^{\widetilde{\alpha}}}, \quad \widetilde{\alpha} := (\alpha_1, ..., \alpha_d), \ \ \alpha_j \in \mathbb{Z}^+, \ j = 1, ..., d,$$

such that $|\widetilde{\alpha}| = \sum_{j=1}^{d} \alpha_j = N$. Thus

$$\sum_{i=1}^{r} w_i (n+\rho_i)^d \int_0^{\frac{1}{n+\rho_i}} f\left(t + \frac{k+\lambda_i}{n+\rho_i} \right) dt = \tag{75}$$

$$\sum_{l=0}^{N} \frac{\sum_{i=1}^{r} w_i (n+\rho_i)^d \int_0^{\frac{1}{n+\rho_i}} g_{t+\frac{k+\lambda_i}{n+\rho_i}}^{(l)} (0)\, dt}{l!} +$$

$$\sum_{i=1}^{r} w_i (n+\rho_i)^d \int_0^{\frac{1}{n+\rho_i}} R_N \left(t + \frac{k+\lambda_i}{n+\rho_i}, 0 \right) dt.$$

Hence it holds

$$\left(K_n^* (f) \right)(x) =$$

$$\frac{\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{r} w_i (n+\rho_i)^d \int_0^{\frac{1}{n+\rho_i}} f\left(t + \frac{k+\lambda_i}{n+\rho_i} \right) dt \right) b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}{V(x)} =$$

$$\sum_{l=0}^{N} \frac{1}{l!} \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}{V(x)} \left(\sum_{i=1}^{r} w_i (n+\rho_i)^d \int_0^{\frac{1}{n+\rho_i}} g_{t+\frac{k+\lambda_i}{n+\rho_i}}^{(l)} (0)\, dt \right) + \tag{76}$$

$$\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}{V(x)} \left(\sum_{i=1}^{r} w_i (n+\rho_i)^d \int_0^{\frac{1}{n+\rho_i}} R_N \left(t + \frac{k+\lambda_i}{n+\rho_i}, 0 \right) dt \right).$$

So we see that

$$\left(K_n^* (f) \right)(x) - f(x) = \tag{77}$$

$$\sum_{l=1}^{N} \frac{1}{l!} \sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}{V(x)} \left(\sum_{i=1}^{r} w_i (n+\rho_i)^d \int_0^{\frac{1}{n+\rho_i}} g_{t+\frac{k+\lambda_i}{n+\rho_i}}^{(l)} (0)\, dt \right) + R^*,$$

where

$$R^* =$$

$$\sum_{k=\lceil nx - Tn^{\alpha} \rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha} \left(x - \frac{k}{n} \right) \right)}{V(x)} \left(\sum_{i=1}^{r} w_i (n+\rho_i)^d \int_0^{\frac{1}{n+\rho_i}} R_N \left(t + \frac{k+\lambda_i}{n+\rho_i}, 0 \right) dt \right). \tag{78}$$

Consequently, we obtain

$$\left|\left(K_n^*(f)\right)(x)-f(x)\right| \leq$$

$$\sum_{l=1}^{N}\frac{1}{l!}\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\left(\sum_{i=1}^{r}w_i\left(n+\rho_i\right)^d\int_0^{\frac{1}{n+\rho_i}}\left|g_{t+\frac{k+\lambda_i}{n+\rho_i}}^{(l)}(0)\right|dt\right) \tag{79}$$

$$+\left|R^*\right|.$$

Notice that

$$\left|g_{t+\frac{k+\lambda_i}{n+\rho_i}}^{(l)}(0)\right|\overset{(27)}{\leq}\left(\left(\sum_{j=1}^{d}\left|\frac{\partial}{\partial x_j}\right|\right)^l f(x)\right)\cdot$$

$$\left[\left(\frac{\rho_i\|x\|_{\infty}+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]^l, \tag{80}$$

and so far we have

$$\left|\left(K_n^*(f)\right)(x)-f(x)\right| \leq$$

$$\sum_{l=1}^{N}\frac{1}{l!}\left(\sum_{i=1}^{r}w_i\left[\left(\frac{\rho_i\|x\|_{\infty}+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]^l\right) \tag{81}$$

$$\cdot\left(\left(\sum_{j=1}^{d}\left|\frac{\partial}{\partial x_j}\right|\right)^l f(x)\right)+\left|R^*\right|.$$

Next, we need to estimate $\left|R^*\right|$. For that, we observe $(0\leq\lambda_N^*\leq 1)$

$$\left|g_{t+\frac{k+\lambda_i}{n+\rho_i}}^{(N)}\left(\lambda_N^*\right)-g_{t+\frac{k+\lambda_i}{n+\rho_i}}^{(N)}(0)\right|=$$

$$\left|\left[\left(\sum_{j=1}^{d}\left(t_j+\frac{k_j+\lambda_i}{n+\rho_i}-x_j\right)\frac{\partial}{\partial x_j}\right)^N f\right]\left(x+\lambda_N^*\left(t+\frac{k+\lambda_i}{n+\rho_i}-x\right)\right)-\right. \tag{82}$$

$$\left.\left[\left(\sum_{j=1}^{d}\left(t_j+\frac{k_j+\lambda_i}{n+\rho_i}-x_j\right)\frac{\partial}{\partial x_j}\right)^N f\right](x)\right|\overset{(\text{by }(27),\,(28))}{\leq}$$

$$d^N\left[\left(\frac{\rho_i\|x\|_{\infty}+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]^N\cdot$$

$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\left[\left(\frac{\rho_i\|x\|_{\infty}+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]\right). \tag{83}$$

Thus we find

$$\left|R_N\left(t+\frac{k+\lambda_i}{n+\rho_i},0\right)\right| \leq$$

$$\int_0^1\left(\int_0^{\lambda_1^*}...\left(\int_0^{\lambda_{N-1}^*}\left|g_{t+\frac{k+\lambda_i}{n+\rho_i}}^{(N)}\left(\lambda_N^*\right)-g_{t+\frac{k+\lambda_i}{n+\rho_i}}^{(N)}(0)\right|d\lambda_N^*\right)...\right)d\lambda_1^* \leq \tag{84}$$

$$\frac{d^N}{N!}\left[\left(\frac{\rho_i\|x\|_\infty+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]^N \cdot \tag{85}$$

$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\left[\left(\frac{\rho_i\|x\|_\infty+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]\right).$$

Finally we obtain

$$|R^*| \leq \sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\cdot$$

$$\left(\sum_{i=1}^r w_i\left(n+\rho_i\right)^d\int_0^{\frac{1}{n+\rho_i}}\left|R_N\left(t+\frac{k+\lambda_i}{n+\rho_i},0\right)\right|dt\right) \leq$$

$$\sum_{i=1}^r w_i\frac{d^N}{N!}\left[\left(\frac{\rho_i\|x\|_\infty+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]^N \cdot$$

$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\left[\left(\frac{\rho_i\|x\|_\infty+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]\right). \tag{86}$$

Using (81) and (86) we derive (69). ■

Corollary 15 *(to Theorem 14) Let all as in Corollary 11. Then*

$$\left|\left(K_n^*(f)\right)(x)-f(x)\right| \leq$$

$$\frac{d^N}{N!}\sum_{i=1}^r w_i\left[\left(\frac{\rho_i\|x\|_\infty+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]^N \cdot \tag{87}$$

$$\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\left[\left(\frac{\rho_i\|x\|_\infty+\lambda_i+1}{n+\rho_i}\right)+\left(1+\frac{\rho_i}{n+\rho_i}\right)\frac{T^*}{n^{1-\alpha}}\right]\right).$$

Inequality (87) implies the pointwise convergence with rates of $\left(K_n^*(f)\right)(x)$ *to* $f(x)$*, as* $n\to\infty$*, at the high speed* $\frac{1}{n^{(1-\alpha)(N+1)}}$*.*

Proof. By (69). ■

The uniform convergence with rates follows from

Corollary 16 *(to Theorem 14) Let all as in Corollary 12. Then*

$$\left\| K_n^* (f) - f \right\|_{\infty, \prod_{j=1}^{d} [-\gamma_j, \gamma_j]} \leq \sum_{l=1}^{N} \frac{1}{l!} \left(\left(\sum_{j=1}^{d} \left| \frac{\partial}{\partial x_j} \right| \right)^l \|f\|_{\infty, \prod_{j=1}^{d} [-\gamma_j, \gamma_j]} \right)$$

$$\left[\sum_{i=1}^{r} w_i \left[\left(\frac{\rho_i \gamma^* + \lambda_i + 1}{n + \rho_i} \right) + \left(1 + \frac{\rho_i}{n + \rho_i} \right) \frac{T^*}{n^{1-\alpha}} \right]^l \right] + \tag{88}$$

$$\frac{d^N}{N!} \sum_{i=1}^{r} w_i \left[\left(\frac{\rho_i \gamma^* + \lambda_i + 1}{n + \rho_i} \right) + \left(1 + \frac{\rho_i}{n + \rho_i} \right) \frac{T^*}{n^{1-\alpha}} \right]^N \cdot$$

$$\max_{\widetilde{\alpha} : |\widetilde{\alpha}| = N} \omega_1 \left(f_{\widetilde{\alpha}}, \left[\left(\frac{\rho_i \gamma^* + \lambda_i + 1}{n + \rho_i} \right) + \left(1 + \frac{\rho_i}{n + \rho_i} \right) \frac{T^*}{n^{1-\alpha}} \right] \right).$$

Inequality (88) implies the uniform convergence with rates of $K_n^* (f)$ to f on G, as $n \to \infty$, at speed $\frac{1}{n^{1-\alpha}}$.

Proof. By (69). ∎

Corollary 17 *(to Theorem 14) All as in Theorem 14 with $N = 1$. Then*

$$\left| \left(K_n^* (f) \right) (x) - f (x) \right| \leq \left(\sum_{j=1}^{d} \left| \frac{\partial f (x)}{\partial x_j} \right| \right)$$

$$\left[\sum_{i=1}^{r} w_i \left[\left(\frac{\rho_i \|x\|_\infty + \lambda_i + 1}{n + \rho_i} \right) + \left(1 + \frac{\rho_i}{n + \rho_i} \right) \frac{T^*}{n^{1-\alpha}} \right] \right] + \tag{89}$$

$$d \sum_{i=1}^{r} w_i \left[\left(\frac{\rho_i \|x\|_\infty + \lambda_i + 1}{n + \rho_i} \right) + \left(1 + \frac{\rho_i}{n + \rho_i} \right) \frac{T^*}{n^{1-\alpha}} \right] \cdot$$

$$\max_{\widetilde{\alpha} : |\widetilde{\alpha}| = N} \omega_1 \left(f_{\widetilde{\alpha}}, \left[\left(\frac{\rho_i \|x\|_\infty + \lambda_i + 1}{n + \rho_i} \right) + \left(1 + \frac{\rho_i}{n + \rho_i} \right) \frac{T^*}{n^{1-\alpha}} \right] \right).$$

Inequality (89) implies the pointwise convergence with rates of $\left(K_n^* (f) \right) (x)$ to $f (x)$, as $n \to \infty$, at speed $\frac{1}{n^{1-\alpha}}$.

Proof. By (69). ∎

We also give

Theorem 18 *All here as in Theorem 10. Then*

$$\left| \left(M_n^* (f) \right) (x) - f (x) \right| \leq \sum_{l=1}^{N} \frac{1}{l!} \left(\left(\sum_{j=1}^{d} \left| \frac{\partial}{\partial x_j} \right| \right)^l f (x) \right) \cdot$$

$$\left(\frac{T^*}{n^{1-\alpha}}+\frac{1}{n}\right)^l+\frac{d^N}{N!}\left(\frac{T^*}{n^{1-\alpha}}+\frac{1}{n}\right)^N\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\frac{T^*}{n^{1-\alpha}}+\frac{1}{n}\right). \tag{90}$$

Inequality (90) implies the pointwise convergence with rates on $\left(M_n^*(f)\right)(x)$*to* $f(x)$*, as* $n\to\infty$*, at speed* $\frac{1}{n^{1-\alpha}}$*.*

Proof. Set

$$g_{\frac{k}{n}+\frac{i}{nr}}(t)=f\left(x+t\left(\frac{k}{n}+\frac{i}{nr}-x\right)\right),\quad 0\leq t\leq 1. \tag{91}$$

Then we have

$$g^{(l)}_{\frac{k}{n}+\frac{i}{nr}}(t)=\left[\left(\sum_{j=1}^{d}\left(\frac{k_j}{n}+\frac{i_j}{nr_j}-x_j\right)\frac{\partial}{\partial x_j}\right)^l f\right]\left(x+t\left(\frac{k}{n}+\frac{i}{nr}-x\right)\right), \tag{92}$$

and

$$g_{\frac{k}{n}+\frac{i}{nr}}(0)=f(x). \tag{93}$$

By Taylor's formula, we get

$$f\left(\frac{k}{n}+\frac{i}{nr}\right)=g_{\frac{k}{n}+\frac{i}{nr}}(1)= \tag{94}$$

$$\sum_{l=0}^{N}\frac{g^{(l)}_{\frac{k}{n}+\frac{i}{nr}}(0)}{l!}+R_N\left(\frac{k}{n}+\frac{i}{nr},0\right),$$

where

$$R_N\left(\frac{k}{n}+\frac{i}{nr},0\right)=$$

$$\int_0^1\left(\int_0^{t_1}...\left(\int_0^{t_{N-1}}\left(g^{(N)}_{\frac{k}{n}+\frac{i}{nr}}(t_N)-g^{(N)}_{\frac{k}{n}+\frac{i}{nr}}(0)\right)dt_N\right)...\right)dt_1. \tag{95}$$

Here we denote by

$$f_{\widetilde{\alpha}}:=\frac{\partial^{\widetilde{\alpha}}f}{\partial x^{\widetilde{\alpha}}},\quad \widetilde{\alpha}:=(\alpha_1,...,\alpha_d),\ \ \alpha_j\in\mathbb{Z}^+,\ j=1,...,d,$$

such that $|\widetilde{\alpha}|=\sum_{j=1}^{d}\alpha_j=N$. Thus

$$\left(M_n^*(f)\right)(x)=\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{r}w_i f\left(\frac{k}{n}+\frac{i}{nr}\right)\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}=$$

$$\frac{\sum_{l=0}^{N}\frac{1}{l!}\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{r}w_i g^{(l)}_{\frac{k}{n}+\frac{i}{nr}}(0)\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}+ \tag{96}$$

$$\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\left(\sum_{i=1}^{r} w_i R_N\left(\frac{k}{n}+\frac{i}{nr},0\right)\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}.$$

Therefore it holds

$$\left(M_n^*(f)\right)(x)-f(x)= \tag{97}$$

$$\sum_{l=1}^{N}\frac{1}{l!}\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\left(\sum_{i=1}^{r} w_i g_{\frac{k}{n}+\frac{i}{nr}}^{(l)}(0)\right)+R^*,$$

where

$$R^*=\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\left(\sum_{i=1}^{r} w_i R_N\left(\frac{k}{n}+\frac{i}{nr},0\right)\right). \tag{98}$$

Consequently, we obtain

$$\left|\left(M_n^*(f)\right)(x)-f(x)\right|\leq$$

$$\sum_{l=1}^{N}\frac{1}{l!}\frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\left(\sum_{i=1}^{r} w_i\left|g_{\frac{k}{n}+\frac{i}{nr}}^{(l)}(0)\right|\right)+\left|R^*\right|. \tag{99}$$

Notice that

$$\left|g_{\frac{k}{n}+\frac{i}{nr}}^{(l)}(0)\right|\overset{(29)}{\leq}\left(\left(\sum_{j=1}^{d}\left|\frac{\partial}{\partial x_j}\right|\right)^{l} f(x)\right)\left(\frac{T^*}{n^{1-\alpha}}+\frac{1}{n}\right)^{l}, \tag{100}$$

and so far we have

$$\left|\left(M_n^*(f)\right)(x)-f(x)\right|\leq\sum_{l=1}^{N}\frac{1}{l!}\left(\left(\sum_{j=1}^{d}\left|\frac{\partial}{\partial x_j}\right|\right)^{l} f(x)\right)\left(\frac{T^*}{n^{1-\alpha}}+\frac{1}{n}\right)^{l}+\left|R^*\right|. \tag{101}$$

Next, we need to estimate $|R^*|$. For that, we observe $(0\leq t_N\leq 1)$

$$\left|g_{\frac{k}{n}+\frac{i}{nr}}^{(N)}(t_N)-g_{\frac{k}{n}+\frac{i}{nr}}^{(N)}(0)\right|=$$

$$|\left[\left(\sum_{j=1}^{d}\left(\frac{k_j}{n}+\frac{i_j}{nr_j}-x_j\right)\frac{\partial}{\partial x_j}\right)^{N} f\right]\left(x+t_N\left(\frac{k}{n}+\frac{i}{nr}-x\right)\right)- \tag{102}$$

$$\left[\left(\sum_{j=1}^{d}\left(\frac{k_j}{n}+\frac{i_j}{nr_j}-x_j\right)\frac{\partial}{\partial x_j}\right)^{N} f\right](x)|\overset{(\text{by }(29),\,(30))}{\leq}$$

$$d^N\left(\frac{T^*}{n^{1-\alpha}}+\frac{1}{n}\right)^N \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\frac{T^*}{n^{1-\alpha}}+\frac{1}{n}\right). \tag{103}$$

Thus we find

$$\left|R_N\left(\frac{k}{n}+\frac{i}{nr},0\right)\right|\leq$$

$$\int_0^1\left(\int_0^{t_1}...\left(\int_0^{t_{N-1}}\left|g^{(N)}_{\frac{k}{n}+\frac{i}{nr}}(t_N)-g^{(N)}_{\frac{k}{n}+\frac{i}{nr}}(0)\right|dt_N\right)...\right)dt_1\leq$$

$$\frac{d^N}{N!}\left(\frac{T^*}{n^{1-\alpha}}+\frac{1}{n}\right)^N \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\frac{T^*}{n^{1-\alpha}}+\frac{1}{n}\right). \tag{104}$$

Hence we get

$$|R^*|\leq\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}\left(\sum_{i=1}^r w_i\left|R_N\left(\frac{k}{n}+\frac{i}{nr},0\right)\right|\right)\leq \tag{105}$$

$$\frac{d^N}{N!}\left(\frac{T^*}{n^{1-\alpha}}+\frac{1}{n}\right)^N \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\frac{T^*}{n^{1-\alpha}}+\frac{1}{n}\right), \tag{106}$$

proving the claim. ■

Corollary 19 *(to Theorem 18) Let all as in Corollary 11. Then*

$$|(M_n^*(f))(x)-f(x)|\leq\frac{d^N}{N!}\left(\frac{T^*}{n^{1-\alpha}}+\frac{1}{n}\right)^N \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\frac{T^*}{n^{1-\alpha}}+\frac{1}{n}\right). \tag{107}$$

Inequality (107) implies the pointwise convergence with rates of $(M_n^*(f))(x)$ *to* $f(x)$*, as* $n\to\infty$*, at the high speed* $\frac{1}{n^{(1-\alpha)(N+1)}}$*.*

Proof. By (90). ■

The uniform convergence with rates comes from

Corollary 20 *(to Theorem 18) Let all as in Corollary 12. Then*

$$\|M_n^*(f)-f\|_{\infty,\prod_{j=1}^d[-\gamma_j,\gamma_j]}\leq\sum_{l=1}^N\frac{1}{l!}\left(\left(\sum_{j=1}^d\left|\frac{\partial}{\partial x_j}\right|\right)^l\|f\|_{\infty,\prod_{j=1}^d[-\gamma_j,\gamma_j]}\right)$$

$$\left(\frac{T^*}{n^{1-\alpha}}+\frac{1}{n}\right)^l+\frac{d^N}{N!}\left(\frac{T^*}{n^{1-\alpha}}+\frac{1}{n}\right)^N \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\frac{T^*}{n^{1-\alpha}}+\frac{1}{n}\right). \tag{108}$$

Inequality (108) implies the uniform convergence with rates of $M_n^*(f)$ *to* f*, as* $n\to\infty$*, at the high speed* $\frac{1}{n^{1-\alpha}}$*.*

Proof. By (90). ■

Corollary 21 *(to Theorem 18) All as in Theorem 18 with $N = 1$. Then*

$$\left|\left(M_n^*\left(f\right)\right)\left(x\right) - f\left(x\right)\right| \leq \left[\sum_{j=1}^{d}\left|\frac{\partial f\left(x\right)}{\partial x_j}\right|\right.$$

$$\left.+d\max_{j\in\{1,\ldots,d\}}\omega_1\left(\frac{\partial f}{\partial x_j}, \frac{T^*}{n^{1-\alpha}}+\frac{1}{n}\right)\right]\left(\frac{T^*}{n^{1-\alpha}}+\frac{1}{n}\right). \tag{109}$$

Inequality (109) implies the pointwise convergence with rates of $\left(M_n^*\left(f\right)\right)\left(x\right)$ to $f\left(x\right)$, as $n \to \infty$, at speed $\frac{1}{n^{1-\alpha}}$.

Proof. By (90). ■

Note 22 *We also observe that all the right hand sides of convergence inequalities (50), (66), (67), (68), (69), (87), (88), (89), (90), (107), (108), (109), are independent of b.*

Note 23 *We observe that*

$$H_n^*\left(1\right) = K_n^*\left(1\right) = M_n^*\left(1\right) = 1, \tag{110}$$

thus unitary operators.

Also, given that f is bounded, we obtain

$$\left\|H_n^*\left(f\right)\right\|_{\infty,\mathbb{R}^d} \leq \left\|f\right\|_{\infty,\mathbb{R}^d}, \tag{111}$$

$$\left\|K_n^*\left(f\right)\right\|_{\infty,\mathbb{R}^d} \leq \left\|f\right\|_{\infty,\mathbb{R}^d}, \tag{112}$$

and

$$\left\|M_n^*\left(f\right)\right\|_{\infty,\mathbb{R}^d} \leq \left\|f\right\|_{\infty,\mathbb{R}^d}. \tag{113}$$

Operators H_n^*, K_n^*, M_n^* are positive linear operators, and of course bounded operators directly by (111)-(113).

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