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Univariate error function based neural network approximation

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Abstract

Here we research the univariate quantitative approximation of real and complex valued continuous functions on a compact interval or all the real line by quasi-interpolation, Baskakov type and quadrature type neural network operators. We perform also the related fractional approximation. These approximations are derived by establishing Jackson type inequalities involving the modulus of continuity of the engaged function or its high order derivative or fractional derivatives. Our operators are defined by using a density function induced by the error function. The approximations are pointwise and with respect to the uniform norm. The related feed-forward neural networks are with one hidden layer.

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1 Introduction

The author in [2] and [3], see Chapters 2-5, was the first to establish neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliagnet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support. Also in [3] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see Chapters 4-5 there.

The author inspired by [15], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [7], [9], [10], [11], [12], by treating both the univariate and multivariate cases. He did also the corresponding fractional case [13].

The author here performs univariate error function based neural network approximations to continuous functions over compact intervals of the real line or over the whole $\mathbb{R}$, the he extends his results to complex valued functions. Finally he treats completely the related fractional approximation. All convergences here are with rates expressed via the modulus of continuity of the involved function or its high order derivative, or fractional derivatives and given by very tight Jackson type inequalities.

The author comes up with the "right" precisely defined quasi-interpolation, Baskakov type and quadrature neural networks operators, associated with the error function and related to a compact interval or real line. Our compact intervals are not necessarily symmetric to the origin. Some of our upper bounds to error quantity are very flexible and general. In preparation to prove our results we establish important properties of the basic density function defining our operators.

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle a_j \cdot x\rangle + b_j\right), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x\rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental neural network models, the activation function is the error. About neural networks in general read [19], [20], [21].

2 Basics

We consider here the (Gauss) error special function ([1], [14])

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \quad x \in \mathbb{R}, \tag{1}$$

which is a sigmoidal type function and a strictly increasing function.

It has the basic properties

$$\operatorname{erf}(0) = 0, \quad \operatorname{erf}(-x) = -\operatorname{erf}(x), \quad \operatorname{erf}(+\infty) = 1, \quad \operatorname{erf}(-\infty) = -1, \tag{2}$$

and

$$(\operatorname{erf}(x))' = \frac{2}{\sqrt{\pi}} e^{-x^2}, \quad x \in \mathbb{R}, \tag{3}$$

$$\int \operatorname{erf}(x)\, dx = x \operatorname{erf}(x) + \frac{e^{-x^2}}{\sqrt{\pi}} + C, \tag{4}$$

where C is a constant.

The error function is related to the cumulative probability distribution function of the standard normal distribution

$$\Phi(x) = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right).$$

We consider the activation function

$$\chi(x) = \frac{1}{4}\left(\operatorname{erf}(x+1) - \operatorname{erf}(x-1)\right), \quad x \in \mathbb{R}, \tag{5}$$

and we notice that

$$\chi(-x) = \frac{1}{4}\left(\operatorname{erf}(-x+1) - \operatorname{erf}(-x-1)\right) =$$

$$\frac{1}{4}\left(\operatorname{erf}(-(x-1)) - \operatorname{erf}(-(x+1))\right) = \frac{1}{4}\left(-\operatorname{erf}(x-1) + \operatorname{erf}(x+1)\right) = \chi(x), \tag{6}$$

thus χ is an even function.

Since $x+1 > x-1$, then $\operatorname{erf}(x+1) > \operatorname{erf}(x-1)$, and $\chi(x) > 0$, all $x \in \mathbb{R}$.

We see that

$$\chi(0) = \frac{\operatorname{erf}(1)}{2} \simeq \frac{0.843}{2} = 0.4215. \tag{7}$$

Let $x > 0$, we have

$$\chi'(x) = \frac{1}{4}\left(\frac{2}{\sqrt{\pi}} e^{-(x+1)^2} - \frac{2}{\sqrt{\pi}} e^{-(x-1)^2}\right) =$$

$$\frac{1}{2\sqrt{\pi}}\left(\frac{1}{e^{(x+1)^2}} - \frac{1}{e^{(x-1)^2}}\right) = \frac{1}{2\sqrt{\pi}}\left(\frac{e^{(x-1)^2} - e^{(x+1)^2}}{e^{(x+1)^2} e^{(x-1)^2}}\right) < 0, \tag{8}$$

proving $\chi'(x) < 0$, for $x > 0$.

That is χ is strictly decreasing on $[0, \infty)$ and is strictly increasing on $(-\infty, 0]$, and $\chi'(0) = 0$.

Clearly the x-axis is the horizontal asymptote on χ.

Conclusion, χ is a bell symmetric function with maximum $\chi(0) \simeq 0.4215$.

We further present

Theorem 1 *We have that*

$$\sum_{i=-\infty}^{\infty} \chi(x-i) = 1, \quad all\ x \in \mathbb{R}. \tag{9}$$

Proof. We notice

$$\sum_{i=-\infty}^{\infty} \operatorname{erf}(x-i) - \operatorname{erf}(x-1-i) =$$

$$\sum_{i=0}^{\infty} (\operatorname{erf}(x-i) - \operatorname{erf}(x-1-i)) + \sum_{i=-\infty}^{-1} (\operatorname{erf}(x-i) - \operatorname{erf}(x-1-i)). \tag{10}$$

Furthermore ($\lambda \in \mathbb{Z}^+$) (telescoping sum)

$$\sum_{i=0}^{\infty} (\operatorname{erf}(x-i) - \operatorname{erf}(x-1-i)) =$$

$$\lim_{\lambda\to\infty} \sum_{i=0}^{\lambda} (\operatorname{erf}(x-i) - \operatorname{erf}(x-1-i)) =$$

$$\operatorname{erf}(x) - \lim_{\lambda\to\infty} \operatorname{erf}(x-1-\lambda) = 1 + \operatorname{erf}(x). \tag{11}$$

Similarly we get

$$\sum_{i=-\infty}^{-1} (\operatorname{erf}(x-i) - \operatorname{erf}(x-1-i)) =$$

$$\lim_{\lambda\to\infty} \sum_{i=-\lambda}^{-1} (\operatorname{erf}(x-i) - \operatorname{erf}(x-1-i)) =$$

$$\lim_{\lambda\to\infty} (\operatorname{erf}(x+\lambda) - \operatorname{erf}(x)) = 1 - \operatorname{erf}(x). \tag{12}$$

Adding (11) and (12), we get

$$\sum_{i=-\infty}^{\infty} (\operatorname{erf}(x-i) - \operatorname{erf}(x-1-i)) = 2, \quad \text{for any } x \in \mathbb{R}. \tag{13}$$

Hence (13) is true for $(x+1)$, giving us

$$\sum_{i=-\infty}^{\infty} (\operatorname{erf}(x+1-i) - \operatorname{erf}(x-i)) = 2, \quad \text{for any } x \in \mathbb{R}. \tag{14}$$

Adding (13) and (14) we obtain

$$\sum_{i=-\infty}^{\infty} (\operatorname{erf}(x+1-i) - \operatorname{erf}(x-1-i)) = 4, \quad \text{for any } x \in \mathbb{R}, \tag{15}$$

proving (9). ■

Thus

$$\sum_{i=-\infty}^{\infty} \chi(nx - i) = 1, \quad \forall\, n \in \mathbb{N},\ \forall\, x \in \mathbb{R}. \tag{16}$$

Furthermore we get:

Since χ is even it holds $\sum_{i=-\infty}^{\infty} \chi(i - x) = 1$, for any $x \in \mathbb{R}$.

Hence $\sum_{i=-\infty}^{\infty} \chi(i + x) = 1,\ \forall\, x \in \mathbb{R}$, and $\sum_{i=-\infty}^{\infty} \chi(x + i) = 1,\ \forall\, x \in \mathbb{R}$.

Theorem 2 *It holds*

$$\int_{-\infty}^{\infty} \chi(x)\, dx = 1. \tag{17}$$

Proof. We notice that

$$\int_{-\infty}^{\infty} \chi(x)\, dx = \sum_{j=-\infty}^{\infty} \int_{j}^{j+1} \chi(x)\, dx = \sum_{j=-\infty}^{\infty} \int_{0}^{1} \chi(x + j)\, dx =$$

$$\int_{0}^{1} \left(\sum_{j=-\infty}^{\infty} \chi(x + j) \right) dx = \int_{0}^{1} 1 dx = 1.$$

■

So $\chi(x)$ is a density function on $\mathbb{R}$.

Theorem 3 *Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} \geq 3$. It holds*

$$\sum_{\begin{cases} k = -\infty \\ : |nx - k| \geq n^{1-\alpha} \end{cases}}^{\infty} \chi(nx - k) < \frac{1}{2\sqrt{\pi}\left(n^{1-\alpha} - 2\right) e^{\left(n^{1-\alpha} - 2\right)^2}}. \tag{18}$$

Proof. Let $x \geq 1$. That is $0 \leq x - 1 < x + 1$. Applying the mean value theorem we get

$$\chi(x) = \frac{1}{4}\left(\operatorname{erf}(x + 1) - \operatorname{erf}(x - 1)\right) = \frac{1}{\sqrt{\pi}} e^{-\xi^2}, \tag{19}$$

where $x - 1 < \xi < x + 1$.

Hence

$$\chi(x) < \frac{e^{-(x-1)^2}}{\sqrt{\pi}}, \quad x \geq 1. \tag{20}$$

Thus we have

$$\sum_{\begin{cases} k = -\infty \\ : |nx - k| \geq n^{1-\alpha} \end{cases}}^{\infty} \chi(nx - k) = \sum_{\begin{cases} k = -\infty \\ : |nx - k| \geq n^{1-\alpha} \end{cases}}^{\infty} \chi(|nx - k|) <$$

$$\frac{1}{\sqrt{\pi}} \sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} e^{-(|nx-k|-1)^2} \leq \frac{1}{\sqrt{\pi}} \int_{(n^{1-\alpha}-1)}^{\infty} e^{-(x-1)^2} dx \tag{21}$$

$$= \frac{1}{\sqrt{\pi}} \int_{n^{1-\alpha}-2}^{\infty} e^{-z^2} dz$$

(see section 3.7.3 of [22])

$$= \frac{1}{2\sqrt{\pi}} \left(\min \left(\sqrt{\pi}, \frac{1}{(n^{1-\alpha}-2)} \right) \right) e^{-\left(n^{1-\alpha}-2\right)^2}$$

(by $n^{1-\alpha} - 2 \geq 1$, hence $\frac{1}{n^{1-\alpha}-2} \leq 1 < \sqrt{\pi}$)

$$< \frac{1}{2\sqrt{\pi}\left(n^{1-\alpha}-2\right) e^{(n^{1-\alpha}-2)^2}}, \tag{22}$$

proving the claim. ■

Denote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.

Theorem 4 *Let $x \in [a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k)} < \frac{1}{\chi(1)} \simeq 4.019, \quad \forall\, x \in [a, b]. \tag{23}$$

Proof. Let $x \in [a, b]$. We see that

$$1 = \sum_{k=-\infty}^{\infty} \chi(nx-k) > \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k) = \tag{24}$$

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(|nx-k|) > \chi(|nx-k_0|),$$

$\forall\, k_0 \in [\lceil na \rceil, \lfloor nb \rfloor] \cap \mathbb{Z}$.

We can choose $k_0 \in [\lceil na \rceil, \lfloor nb \rfloor] \cap \mathbb{Z}$ such that $|nx - k_0| < 1$.

Therefore

$$\chi(|nx-k_0|) > \chi(1) = \frac{1}{4}(\text{erf}(2) - \text{erf}(0)) =$$

$$\frac{\text{erf}(2)}{4} = \frac{0.99533}{4} = 0.2488325. \tag{25}$$

Consequently we get

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(|nx-k|) > \chi(1) \simeq 0.2488325, \tag{26}$$

and

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\chi\left(|nx-k|\right)}<\frac{1}{\chi\left(1\right)}\simeq 4.019, \tag{27}$$

proving the claim. ■

Remark 5 *We also notice that*

$$1-\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\chi\left(nb-k\right)=\sum_{k=-\infty}^{\lceil na\rceil-1}\chi\left(nb-k\right)+\sum_{k=\lfloor nb\rfloor+1}^{\infty}\chi\left(nb-k\right)$$

$$>\chi\left(nb-\lfloor nb\rfloor-1\right)$$

(call $\varepsilon:=nb-\lfloor nb\rfloor$, $0\leq\varepsilon<1$*)*

$$=\chi\left(\varepsilon-1\right)=\chi\left(1-\varepsilon\right)\geq\chi\left(1\right)>0. \tag{28}$$

Therefore

$$\lim_{n\to\infty}\left(1-\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\chi\left(nb-k\right)\right)>0.$$

Similarly,

$$1-\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\chi\left(na-k\right)=\sum_{k=-\infty}^{\lceil na\rceil-1}\chi\left(na-k\right)+\sum_{k=\lfloor nb\rfloor+1}^{\infty}\chi\left(na-k\right)$$

$$>\chi\left(na-\lceil na\rceil+1\right)$$

(call $\eta:=\lceil na\rceil-na,\ \ 0\leq\eta<1$*)*

$$=\chi\left(1-\eta\right)\geq\chi\left(1\right)>0.$$

Therefore again

$$\lim_{n\to\infty}\left(1-\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\chi\left(na-k\right)\right)>0. \tag{29}$$

Hence we derive that

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\chi\left(nx-k\right)\neq 1, \tag{30}$$

for at least some $x\in[a,b]$*.*

Note 6 *For large enough* n *we always obtain* $\lceil na\rceil\leq\lfloor nb\rfloor$*. Also* $a\leq\frac{k}{n}\leq b$*, iff* $\lceil na\rceil\leq k\leq\lfloor nb\rfloor$*. In general it holds (by* (16)*) that*

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\chi\left(nx-k\right)\leq 1. \tag{31}$$

We give

Definition 7 *Let* $f \in C([a,b])$ $n \in \mathbb{N}$. *We set*

$$A_n(f,x) = \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \chi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k)}, \quad \forall\ x \in [a.b], \tag{32}$$

A_n *is a neural network operator.*

Definition 8 *Let* $f \in C_B(\mathbb{R})$, *(continuous and bounded functions on* $\mathbb{R}$*),* $n \in \mathbb{N}$. *We introduce the quasi-interpolation operator*

$$B_n(f,x) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \chi(nx-k), \quad \forall\ x \in \mathbb{R}, \tag{33}$$

and the Kantorovich type operator

$$C_n(f,x) = \sum_{k=-\infty}^{\infty} \left(n \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt \right) \chi(nx-k), \quad \forall\ x \in \mathbb{R}. \tag{34}$$

B_n, C_n *are neural network operators.*

Also we give

Definition 9 *Let* $f \in C_B(\mathbb{R})$, $n \in \mathbb{N}$. *Let* $\theta \in \mathbb{N}$, $w_r \geq 0$, $\sum_{r=0}^{\theta} w_r = 1$, $k \in \mathbb{Z}$, *and*

$$\delta_{nk}(f) = \sum_{r=0}^{\theta} w_r f\left(\frac{k}{n} + \frac{r}{n\theta}\right). \tag{35}$$

We put

$$D_n(f,x) = \sum_{k=-\infty}^{\infty} \delta_{nk}(f) \chi(nx-k), \quad \forall\ x \in \mathbb{R}. \tag{36}$$

D_n *is a neural network operator of quadrature type.*

We need

Definition 10 *For* $f \in C([a,b])$, *the first modulus of continuity is given by*

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} |f(x) - f(y)|, \quad \delta > 0. \tag{37}$$

We have that $\lim_{\delta \to 0} \omega_1(f,\delta) = 0$.

Similarly $\omega_1(f,\delta)$ *is defined for* $f \in C_B(\mathbb{R})$.

We know that, f is uniformly continuous on $\mathbb{R}$ iff $\lim_{\delta \to 0} \omega_1 (f, \delta) = 0$.

We make

Remark 11 *We notice the following, that*

$$A_n (f, x) - f (x) \overset{(32)}{=} \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f \left(\frac{k}{n} \right) \chi (nx - k) - f (x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi (nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi (nx - k)}, \tag{38}$$

using (23) we get,

$$|A_n (f, x) - f (x)| \leq (4.019) \left| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f \left(\frac{k}{n} \right) \chi (nx - k) - f (x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi (nx - k) \right|. \tag{39}$$

Again here $0 < \alpha < 1$ and $n \in \mathbb{N}$ with $n^{1-\alpha} \geq 3$. Let the fixed $K, L > 0$; for the linear combination $\frac{K}{n^\alpha} + \frac{L}{(n^{1-\alpha}-2) e^{(n^{1-\alpha}-2)^2}}$, the dominant rate of convergence to zero, as $n \to \infty$, is $n^{-\alpha}$. The closer α is to 1, we get faster and better rate of convergence to zero.

In this article we study basic approximation properties of A_n, B_n, C_n, D_n neural network operators. That is, the quantitative pointwise and uniform convergence of these operators to the unit operator I.

3 Real Neural Network Approximations

Here we present a series of neural network approximations to a function given with rates.

We give

Theorem 12 *Let $f \in C([a, b])$, $0 < \alpha < 1$, $x \in [a, b]$, $n \in \mathbb{N}$ with $n^{1-\alpha} \geq 3$, $\|\cdot\|_\infty$ is the supremum norm. Then*

1)

$$|A_n (f, x) - f (x)| \leq (4.019) \left[\omega_1 \left(f, \frac{1}{n^\alpha} \right) + \frac{\|f\|_\infty}{\sqrt{\pi} (n^{1-\alpha} - 2) e^{(n^{1-\alpha}-2)^2}} \right] =: \mu_{1n}, \tag{40}$$

2)

$$\|A_n (f) - f\|_\infty \leq \mu_{1n}. \tag{41}$$

We notice that $\lim_{n \to \infty} A_n (f) = f$, pointwise and uniformly.

Proof. Using (39) we get

$$|A_n (f, x) - f (x)| \leq (4.019) \left[\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left| f \left(\frac{k}{n} \right) - f (x) \right| \chi (nx - k) \right] \leq$$

$$(4.019)\left[\sum_{\left\{\begin{array}{c} k=\lceil na\rceil \\ \left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}\end{array}\right.}^{\lfloor nb\rfloor}\left|f\left(\frac{k}{n}\right)-f(x)\right|\chi(nx-k)+\right.$$

$$\left.\sum_{\left\{\begin{array}{c} k=\lceil na\rceil \\ \left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}}\end{array}\right.}^{\lfloor nb\rfloor}\left|f\left(\frac{k}{n}\right)-f(x)\right|\chi(nx-k)\right]\leq \tag{42}$$

$$(4.019)\left[\omega_1\left(f,\frac{1}{n^{\alpha}}\right)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\chi(nx-k)\right)+\right.$$

$$\left.2\|f\|_{\infty}\left(\sum_{\left\{\begin{array}{c} k=\lceil na\rceil \\ |nx-k|\geq n^{1-\alpha}\end{array}\right.}^{\lfloor nb\rfloor}\chi(nx-k)\right)\right]\overset{\text{(by (18), (31))}}{\leq} \tag{43}$$

$$(4.019)\left[\omega_1\left(f,\frac{1}{n^{\alpha}}\right)+\frac{\|f\|_{\infty}}{\sqrt{\pi}\left(n^{1-\alpha}-2\right)e^{\left(n^{1-\alpha}-2\right)^2}}\right],$$

proving the claim. ■

We continue with

Theorem 13 *Let $f\in C_B(\mathbb{R})$, $0<\alpha<1$, $x\in\mathbb{R}$, $n\in\mathbb{N}$ with $n^{1-\alpha}\geq 3$. Then*

1)

$$|B_n(f,x)-f(x)|\leq\omega_1\left(f,\frac{1}{n^{\alpha}}\right)+\frac{\|f\|_{\infty}}{\sqrt{\pi}\left(n^{1-\alpha}-2\right)e^{\left(n^{1-\alpha}-2\right)^2}}=:\mu_{2n}, \tag{44}$$

2)

$$\|B_n(f)-f\|_{\infty}\leq\mu_{2n}. \tag{45}$$

For $f\in(C_B(\mathbb{R})\cap C_u(\mathbb{R}))$ ($C_u(\mathbb{R})$ uniformly continuous functions on $\mathbb{R}$) we get $\lim_{n\to\infty}B_n(f)=f$, pointwise and uniformly.

Proof. We see that

$$|B_n(f,x)-f(x)|\overset{\text{(by (16), (33))}}{=}\left|\sum_{k=-\infty}^{\infty}\left(f\left(\frac{k}{n}\right)-f(x)\right)\chi(nx-k)\right|\leq$$

$$\sum_{k=-\infty}^{\infty} \left| f\left(\frac{k}{n}\right) - f(x) \right| \chi(nx-k) \leq$$

$$\sum_{\substack{k=-\infty \\ \left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\alpha}}}}^{\infty} \left| f\left(\frac{k}{n}\right) - f(x) \right| \chi(nx-k) +$$

$$\sum_{\substack{k=-\infty \\ \left|\frac{k}{n}-x\right| \geq \frac{1}{n^{\alpha}}}}^{\infty} \left| f\left(\frac{k}{n}\right) - f(x) \right| \chi(nx-k) \leq \tag{46}$$

$$\omega_1\left(f, \frac{1}{n^{\alpha}}\right) \left(\sum_{\substack{k=-\infty \\ \left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\alpha}}}}^{\infty} \chi(nx-k) \right) +$$

$$2\|f\|_{\infty} \left(\sum_{\substack{k=-\infty \\ |nx-k| \geq n^{1-\alpha}}}^{\infty} \chi(nx-k) \right) \overset{\text{(by (16), (18))}}{\leq} \tag{47}$$

$$\omega_1\left(f, \frac{1}{n^{\alpha}}\right) + \frac{\|f\|_{\infty}}{\sqrt{\pi}\left(n^{1-\alpha}-2\right) e^{\left(n^{1-\alpha}-2\right)^2}}.$$

■

We continue with

Theorem 14 *Let $f \in C_B(\mathbb{R})$, $0 < \alpha < 1$, $x \in \mathbb{R}$, $n \in \mathbb{N}$ with $n^{1-\alpha} \geq 3$. Then*

1)

$$|C_n(f,x) - f(x)| \leq \omega_1\left(f, \frac{1}{n} + \frac{1}{n^{\alpha}}\right) + \frac{\|f\|_{\infty}}{\sqrt{\pi}\left(n^{1-\alpha}-2\right) e^{\left(n^{1-\alpha}-2\right)^2}} =: \mu_{3n}, \tag{48}$$

2)

$$\|C_n(f) - f\|_{\infty} \leq \mu_{3n}. \tag{49}$$

For $f \in (C_B(\mathbb{R}) \cap C_u(\mathbb{R}))$ we get $\lim_{n\to\infty} C_n(f) = f$, pointwise and uniformly.

Proof. We notice that

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt = \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt. \tag{50}$$

Hence we can write

$$C_n(f,x) = \sum_{k=-\infty}^{\infty} \left(n \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt \right) \chi(nx - k). \tag{51}$$

We observe that

$$|C_n(f,x) - f(x)| = \left| \sum_{k=-\infty}^{\infty} \left(\left(n \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt \right) - f(x) \right) \chi(nx-k) \right| =$$

$$\left| \sum_{k=-\infty}^{\infty} \left(n \int_0^{\frac{1}{n}} \left(f\left(t + \frac{k}{n}\right) - f(x) \right) dt \right) \chi(nx-k) \right| \leq \tag{52}$$

$$\sum_{k=-\infty}^{\infty} \left(n \int_0^{\frac{1}{n}} \left| f\left(t + \frac{k}{n}\right) - f(x) \right| dt \right) \chi(nx-k) \leq$$

$$\sum_{\begin{cases} k=-\infty \\ \left|x - \frac{k}{n}\right| \leq \frac{1}{n^\alpha} \end{cases}}^{\infty} \left(n \int_0^{\frac{1}{n}} \left| f\left(t + \frac{k}{n}\right) - f(x) \right| dt \right) \chi(nx-k) + \tag{53}$$

$$\sum_{\begin{cases} k=-\infty \\ \left|x - \frac{k}{n}\right| \geq \frac{1}{n^\alpha} \end{cases}}^{\infty} \left(n \int_0^{\frac{1}{n}} \left| f\left(t + \frac{k}{n}\right) - f(x) \right| dt \right) \chi(nx-k) \leq$$

$$\sum_{\begin{cases} k=-\infty \\ \left|x - \frac{k}{n}\right| \leq \frac{1}{n^\alpha} \end{cases}}^{\infty} \left(n \int_0^{\frac{1}{n}} \omega_1\left(f, \left|t + \frac{k}{n} - x\right|\right) dt \right) \chi(nx-k) +$$

$$2\|f\|_\infty \left(\sum_{\begin{cases} k=-\infty \\ |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \chi(|nx-k|) \right) \leq \tag{54}$$

$$\sum_{\begin{cases} k=-\infty \\ |nx-k| \leq n^{1-\alpha} \end{cases}}^{\infty} \left(n \int_0^{\frac{1}{n}} \omega_1\left(f, |t| + \frac{1}{n^\alpha}\right) dt \right) \chi(nx-k)$$

$$+ \frac{\|f\|_\infty}{\sqrt{\pi}\,(n^{1-\alpha} - 2)\, e^{(n^{1-\alpha}-2)^2}} \leq$$

$$\omega_1\left(f,\frac{1}{n}+\frac{1}{n^{\alpha}}\right)\left(\sum_{\left\{\begin{array}{c}k=-\infty\\ |nx-k|\leq n^{1-\alpha}\end{array}\right.}^{\infty}\chi(nx-k)\right) \tag{55}$$

$$+\frac{\|f\|_{\infty}}{\sqrt{\pi}\left(n^{1-\alpha}-2\right)e^{\left(n^{1-\alpha}-2\right)^2}}\leq$$

$$\omega_1\left(f,\frac{1}{n}+\frac{1}{n^{\alpha}}\right)+\frac{\|f\|_{\infty}}{\sqrt{\pi}\left(n^{1-\alpha}-2\right)e^{\left(n^{1-\alpha}-2\right)^2}},$$

proving the claim. ■

We give next

Theorem 15 *Let* $f\in C_B(\mathbb{R})$, $0<\alpha<1$, $x\in\mathbb{R}$, $n\in\mathbb{N}$ *with* $n^{1-\alpha}\geq 3$. *Then*

1)

$$|D_n(f,x)-f(x)|\leq\mu_{3n}, \tag{56}$$

and

2)

$$\|D_n(f)-f\|_{\infty}\leq\mu_{3n}, \tag{57}$$

where μ_{3n} *as in (48).*

For $f\in(C_B(\mathbb{R})\cap C_u(\mathbb{R}))$ *we get* $\lim_{n\to\infty}D_n(f)=f$*, pointwise and uniformly.*

Proof. We see that

$$|D_n(f,x)-f(x)|\overset{\text{(by (35), (36))}}{=}$$

$$\left|\sum_{k=-\infty}^{\infty}\left(\left(\sum_{r=0}^{\theta}w_r f\left(\frac{k}{n}+\frac{r}{n\theta}\right)\right)-f(x)\right)\chi(nx-k)\right|=$$

$$\left|\sum_{k=-\infty}^{\infty}\left(\sum_{r=0}^{\theta}w_r\left(f\left(\frac{k}{n}+\frac{r}{n\theta}\right)-f(x)\right)\right)\chi(nx-k)\right|\leq \tag{58}$$

$$\sum_{k=-\infty}^{\infty}\left(\sum_{r=0}^{\theta}w_r\left|f\left(\frac{k}{n}+\frac{r}{n\theta}\right)-f(x)\right|\right)\chi(nx-k)\leq$$

$$\sum_{\left\{\begin{array}{c}k=-\infty\\ \left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}\end{array}\right.}^{\infty}\left(\sum_{r=0}^{\theta}w_r\left|f\left(\frac{k}{n}+\frac{r}{n\theta}\right)-f(x)\right|\right)\chi(nx-k)+ \tag{59}$$

$$2\|f\|_\infty \sum_{\substack{k=-\infty \\ |nx-k|\geq n^{1-\alpha}}}^{\infty} \chi(|nx-k|) \leq$$

(see that $\frac{r}{n\theta} \leq \frac{1}{n}$)

$$\sum_{\substack{k=-\infty \\ \left|\frac{k}{n}-x\right|\leq \frac{1}{n^\alpha}}}^{\infty} \left(\sum_{r=0}^{\theta} w_r \omega_1\left(f, \frac{1}{n^\alpha}+\frac{1}{n}\right)\right) \chi(nx-k) +$$

$$\frac{\|f\|_\infty}{\sqrt{\pi}\left(n^{1-\alpha}-2\right) e^{\left(n^{1-\alpha}-2\right)^2}} \leq \tag{60}$$

$$\omega_1\left(f, \frac{1}{n^\alpha}+\frac{1}{n}\right)+\frac{\|f\|_\infty}{\sqrt{\pi}\left(n^{1-\alpha}-2\right) e^{\left(n^{1-\alpha}-2\right)^2}}=\mu_{3n},$$

proving the claim. ■

In the next we discuss high order of approximation by using the smoothness of f.

Theorem 16 *Let $f \in C^N([a,b])$, $n, N \in \mathbb{N}$, $n^{1-\alpha} \geq 3$, $0<\alpha<1$, $x \in [a,b]$. Then*

i)

$$|A_n(f,x)-f(x)| \leq (4.019) \cdot \tag{61}$$

$$\left\{\sum_{j=1}^{N} \frac{\left|f^{(j)}(x)\right|}{j!}\left[\frac{1}{n^{\alpha j}}+\frac{(b-a)^j}{2\sqrt{\pi}\left(n^{1-\alpha}-2\right) e^{\left(n^{1-\alpha}-2\right)^2}}\right]+\right.$$

$$\left.\left[\omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N} N!}+\frac{\left\|f^{(N)}\right\|_\infty (b-a)^N}{N!\sqrt{\pi}\left(n^{1-\alpha}-2\right) e^{\left(n^{1-\alpha}-2\right)^2}}\right]\right\},$$

ii) assume further $f^{(j)}(x_0)=0$, $j=1,\ldots,N$, for some $x_0 \in [a,b]$, it holds

$$|A_n(f,x_0)-f(x_0)| \leq (4.019) \cdot \tag{62}$$

$$\left[\omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N} N!}+\frac{\left\|f^{(N)}\right\|_\infty (b-a)^N}{N!\sqrt{\pi}\left(n^{1-\alpha}-2\right) e^{\left(n^{1-\alpha}-2\right)^2}}\right],$$

notice here the extremely high rate of convergence at $n^{-(N+1)\alpha}$,

iii)

$$\|A_n(f)-f\|_\infty \leq (4.019) \cdot \tag{63}$$

$$\left\{\sum_{j=1}^{N} \frac{\left\|f^{(j)}\right\|_\infty}{j!}\left[\frac{1}{n^{\alpha j}}+\frac{(b-a)^j}{2\sqrt{\pi}\left(n^{1-\alpha}-2\right) e^{\left(n^{1-\alpha}-2\right)^2}}\right]+\right.$$

$$\left[\omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N} N!} + \frac{\left\|f^{(N)}\right\|_\infty (b-a)^N}{N!\sqrt{\pi}\left(n^{1-\alpha}-2\right) e^{\left(n^{1-\alpha}-2\right)^2}}\right]\right\}.$$

Proof. We use (39).

Call

$$A_n^*(f,x) := \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)\chi(nx-k),$$

that is

$$A_n(f,x) = \frac{A_n^*(f,x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\chi(nx-k)}.$$

Next we apply Taylor's formula with integral remainder.

We have (here $\frac{k}{n}, x \in [a,b]$)

$$f\left(\frac{k}{n}\right) = \sum_{j=0}^{N}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^j + \int_x^{\frac{k}{n}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt.$$

Then

$$f\left(\frac{k}{n}\right)\chi(nx-k) = \sum_{j=0}^{N}\frac{f^{(j)}(x)}{j!}\chi(nx-k)\left(\frac{k}{n}-x\right)^j +$$

$$\chi(nx-k)\int_x^{\frac{k}{n}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt.$$

Hence

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)\chi(nx-k) - f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\chi(nx-k) =$$

$$\sum_{j=1}^{N}\frac{f^{(j)}(x)}{j!}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\chi(nx-k)\left(\frac{k}{n}-x\right)^j +$$

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\chi(nx-k)\int_x^{\frac{k}{n}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt.$$

Thus

$$A_n^*(f,x) - f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\chi(nx-k)\right) = \sum_{j=1}^{N}\frac{f^{(j)}(x)}{j!}A_n^*\left((\cdot - x)^j\right) + \Lambda_n(x), \tag{64}$$

where

$$\Lambda_n(x) := \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\chi(nx-k)\int_x^{\frac{k}{n}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt. \tag{65}$$

We assume that $b-a>\frac{1}{n^{\alpha}}$, which is always the case for large enough $n\in\mathbb{N}$, that is when $n>\left\lceil (b-a)^{-\frac{1}{\alpha}}\right\rceil$.

Thus $\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}$ or $\left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}}$.

As in [3], pp. 72-73 for

$$\gamma:=\int_{x}^{\frac{k}{n}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt, \tag{66}$$

in case of $\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}$, we find that

$$|\gamma|\leq\omega_{1}\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}$$

(for $x\leq\frac{k}{n}$ or $x\geq\frac{k}{n}$).

Notice also for $x\leq\frac{k}{n}$ that

$$\left|\int_{x}^{\frac{k}{n}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt\right|\leq$$

$$\int_{x}^{\frac{k}{n}}\left|f^{(N)}(t)-f^{(N)}(x)\right|\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt\leq$$

$$2\left\|f^{(N)}\right\|_{\infty}\int_{x}^{\frac{k}{n}}\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt=2\left\|f^{(N)}\right\|_{\infty}\frac{\left(\frac{k}{n}-x\right)^{N}}{N!}\leq2\left\|f^{(N)}\right\|_{\infty}\frac{(b-a)^{N}}{N!}.$$

Next assume $\frac{k}{n}\leq x$, then

$$\left|\int_{x}^{\frac{k}{n}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt\right|=$$

$$\left|\int_{\frac{k}{n}}^{x}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt\right|\leq$$

$$\int_{\frac{k}{n}}^{x}\left|f^{(N)}(t)-f^{(N)}(x)\right|\frac{\left(t-\frac{k}{n}\right)^{N-1}}{(N-1)!}dt\leq$$

$$2\left\|f^{(N)}\right\|_{\infty}\int_{\frac{k}{n}}^{x}\frac{\left(t-\frac{k}{n}\right)^{N-1}}{(N-1)!}dt=2\left\|f^{(N)}\right\|_{\infty}\frac{\left(x-\frac{k}{n}\right)^{N}}{N!}\leq2\left\|f^{(N)}\right\|_{\infty}\frac{(b-a)^{N}}{N!}.$$

Thus

$$|\gamma|\leq2\left\|f^{(N)}\right\|_{\infty}\frac{(b-a)^{N}}{N!}, \tag{67}$$

in all two cases.

Therefore

$$\Lambda_n(x) = \sum_{\substack{k=\lceil na \rceil \\ \left|\frac{k}{n}-x\right| \le \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \chi(nx-k)\,\gamma + \sum_{\substack{k=\lceil na \rceil \\ \left|\frac{k}{n}-x\right| > \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \chi(nx-k)\,\gamma.$$

Hence

$$|\Lambda_n(x)| \le \sum_{\substack{k=\lceil na \rceil \\ \left|\frac{k}{n}-x\right| \le \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \chi(nx-k)\left(\omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{N!n^{N\alpha}}\right) +$$

$$\left(\sum_{\substack{k=\lceil na \rceil \\ \left|\frac{k}{n}-x\right| > \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \chi(nx-k)\right) 2\left\|f^{(N)}\right\|_\infty \frac{(b-a)^N}{N!} \le$$

$$\omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{N!n^{N\alpha}} + \left\|f^{(N)}\right\|_\infty \frac{(b-a)^N}{N!}\frac{1}{\sqrt{\pi}\left(n^{1-\alpha}-2\right)e^{\left(n^{1-\alpha}-2\right)^2}}.$$

Consequently we have

$$|\Lambda_n(x)| \le \omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N}N!} + \frac{\left\|f^{(N)}\right\|_\infty (b-a)^N}{N!\sqrt{\pi}\left(n^{1-\alpha}-2\right)e^{\left(n^{1-\alpha}-2\right)^2}}. \tag{68}$$

We further see that

$$A_n^*\left((\cdot - x)^j\right) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k)\left(\frac{k}{n}-x\right)^j.$$

Therefore

$$\left|A_n^*\left((\cdot - x)^j\right)\right| \le \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k)\left|\frac{k}{n}-x\right|^j =$$

$$\sum_{\begin{cases} k=\lceil na \rceil \\ \left|\frac{k}{n}-x\right| \le \frac{1}{n^\alpha} \end{cases}}^{\lfloor nb \rfloor} \chi(nx-k)\left|\frac{k}{n}-x\right|^j + \sum_{\begin{cases} k=\lceil na \rceil \\ \left|\frac{k}{n}-x\right| > \frac{1}{n^\alpha} \end{cases}}^{\lfloor nb \rfloor} \chi(nx-k)\left|\frac{k}{n}-x\right|^j \le$$

$$\frac{1}{n^{\alpha j}} \sum_{\begin{cases} k=\lceil na \rceil \\ \left|\frac{k}{n}-x\right| \le \frac{1}{n^\alpha} \end{cases}}^{\lfloor nb \rfloor} \chi(nx-k) + (b-a)^j \cdot \sum_{\begin{cases} k=\lceil na \rceil \\ |k-nx| > n^{1-\alpha} \end{cases}}^{\lfloor nb \rfloor} \chi(nx-k)$$

$$\le \frac{1}{n^{\alpha j}} + (b-a)^j \cdot \frac{1}{2\sqrt{\pi}\left(n^{1-\alpha}-2\right)e^{\left(n^{1-\alpha}-2\right)^2}}.$$

Hence

$$\left|A_n^*\left((\cdot - x)^j\right)\right| \leq \frac{1}{n^{\alpha j}} + \frac{(b-a)^j}{2\sqrt{\pi}\left(n^{1-\alpha}-2\right)e^{\left(n^{1-\alpha}-2\right)^2}}, \tag{69}$$

for $j = 1, ..., N$.

Putting things together we have proved

$$\left|A_n^*(f,x) - f(x)\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k)\right| \leq$$

$$\sum_{j=1}^{N} \frac{\left|f^{(j)}(x)\right|}{j!}\left[\frac{1}{n^{\alpha j}} + \frac{(b-a)^j}{2\sqrt{\pi}\left(n^{1-\alpha}-2\right)e^{\left(n^{1-\alpha}-2\right)^2}}\right] + \tag{70}$$

$$\left[\omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N} N!} + \frac{\left\|f^{(N)}\right\|_\infty (b-a)^N}{N!\sqrt{\pi}\left(n^{1-\alpha}-2\right)e^{\left(n^{1-\alpha}-2\right)^2}}\right],$$

that is establishing theorem. ■

4 Fractional Neural Network Approximation

We need

Definition 17 *Let $\nu \geq 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f \in AC^n([a,b])$ (space of functions f with $f^{(n-1)} \in AC([a,b])$, absolutely continuous functions). We call left Caputo fractional derivative (see [16], pp. 49-52, [18], [23]) the function*

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)}\int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\,dt, \tag{71}$$

$\forall\ x \in [a,b]$, where Γ is the gamma function $\Gamma(\nu) = \int_0^\infty e^{-t}t^{\nu-1}dt$, $\nu > 0$.

Notice $D_{*a}^{\nu} f \in L_1([a,b])$ and $D_{*a}^{\nu} f$ exists a.e.on $[a,b]$.

We set $D_{*a}^{0} f(x) = f(x)$, $\forall\ x \in [a,b]$.

Lemma 18 *([6]) Let $\nu > 0$, $\nu \notin \mathbb{N}$, $n = \lceil \nu \rceil$, $f \in C^{n-1}([a,b])$ and $f^{(n)} \in L_\infty([a,b])$. Then $D_{*a}^{\nu} f(a) = 0$.*

Definition 19 *(see also [4], [17], [18]). Let $f \in AC^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. The right Caputo fractional derivative of order $\alpha > 0$ is given by*

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)}\int_x^b (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta)\,d\zeta, \tag{72}$$

$\forall\ x \in [a,b]$. We set $D_{b-}^{0} f(x) = f(x)$. Notice $D_{b-}^{\alpha} f \in L_1([a,b])$ and $D_{b-}^{\alpha} f$ exists a.e.on $[a,b]$.

Lemma 20 *([6]) Let* $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. *Then* $D_{b-}^{\alpha} f(b) = 0$.

Convention 21 *We assume that*

$$D_{*x_0}^{\alpha} f(x) = 0, \text{ for } x < x_0, \tag{73}$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0, \tag{74}$$

for all $x, x_0 \in (a,b]$.

We mention

Proposition 22 *([6]) Let* $f \in C^n([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$. *Then* $D_{*a}^{\nu} f(x)$ *is continuous in* $x \in [a,b]$.

Also we have

Proposition 23 *([6]) Let* $f \in C^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. *Then* $D_{b-}^{\alpha} f(x)$ *is continuous in* $x \in [a,b]$.

We further mention

Proposition 24 *([6]) Let* $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$ *and*

$$D_{*x_0}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{75}$$

for all $x, x_0 \in [a,b] : x \geq x_0$.

Then $D_{*x_0}^{\alpha} f(x)$ *is continuous in* x_0.

Proposition 25 *([6]) Let* $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$ *and*

$$D_{x_0-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_{x}^{x_0} (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta)\, d\zeta, \tag{76}$$

for all $x, x_0 \in [a,b] : x \leq x_0$.

Then $D_{x_0-}^{\alpha} f(x)$ *is continuous in* x_0.

Proposition 26 *([6]) Let* $f \in C^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $x, x_0 \in [a,b]$. *Then* $D_{*x_0}^{\alpha} f(x)$, $D_{x_0-}^{\alpha} f(x)$ *are jointly continuous functions in* (x, x_0) *from* $[a,b]^2$ *into* $\mathbb{R}$.

We recall

Theorem 27 *([6]) Let $f : [a,b]^2 \to \mathbb{R}$ be jointly continuous. Consider*

$$G(x) = \omega_1(f(\cdot, x), \delta, [x,b]), \tag{77}$$

$\delta > 0,\ x \in [a,b]$.

Then G is continuous in $x \in [a,b]$.

Also it holds

Theorem 28 *([6]) Let $f : [a,b]^2 \to \mathbb{R}$ be jointly continuous. Then*

$$H(x) = \omega_1(f(\cdot, x), \delta, [a,x]), \tag{78}$$

$x \in [a,b]$, is continuous in $x \in [a,b]$, $\delta > 0$.

We need

Remark 29 *([6]) Let $f \in C^{n-1}([a,b])$, $f^{(n)} \in L_\infty([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$, $\nu \notin \mathbb{N}$. Then we have*

$$|D_{*a}^{\nu} f(x)| \leq \frac{\left\| f^{(n)} \right\|_\infty}{\Gamma(n-\nu+1)} (x-a)^{n-\nu}, \ \ \forall\, x \in [a,b]. \tag{79}$$

Thus we observe

$$\omega_1(D_{*a}^{\nu} f, \delta) = \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} |D_{*a}^{\nu} f(x) - D_{*a}^{\nu} f(y)|$$

$$\leq \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} \left(\frac{\left\| f^{(n)} \right\|_\infty}{\Gamma(n-\nu+1)} (x-a)^{n-\nu} + \frac{\left\| f^{(n)} \right\|_\infty}{\Gamma(n-\nu+1)} (y-a)^{n-\nu} \right)$$

$$\leq \frac{2\left\| f^{(n)} \right\|_\infty}{\Gamma(n-\nu+1)} (b-a)^{n-\nu}.$$

Consequently

$$\omega_1(D_{*a}^{\nu} f, \delta) \leq \frac{2\left\| f^{(n)} \right\|_\infty}{\Gamma(n-\nu+1)} (b-a)^{n-\nu}. \tag{80}$$

Similarly, let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, then

$$\omega_1\left(D_{b-}^{\alpha} f, \delta\right) \leq \frac{2\left\| f^{(m)} \right\|_\infty}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}. \tag{81}$$

So for $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, we find

$$\sup_{x_0 \in [a,b]} \omega_1\left(D_{*x_0}^{\alpha} f, \delta\right)_{[x_0,b]} \leq \frac{2\left\| f^{(m)} \right\|_\infty}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}, \tag{82}$$

and

$$\sup_{x_0\in[a,b]}\omega_1\left(D^{\alpha}_{x_0-}f,\delta\right)_{[a,x_0]}\le\frac{2\left\|f^{(m)}\right\|_{\infty}}{\Gamma(m-\alpha+1)}(b-a)^{m-\alpha}. \tag{83}$$

By Proposition 15.114, p. 388 of [5], we get here that $D^{\alpha}_{*x_0}f\in C([x_0,b])$, *and by [8] we obtain that* $D^{\alpha}_{x_0-}f\in C([a,x_0])$.

Here comes our main fractional result

Theorem 30 *Let* $\alpha>0$, $N=\lceil\alpha\rceil$, $\alpha\notin\mathbb{N}$, $f\in AC^N([a,b])$, *with* $f^{(N)}\in L_{\infty}([a,b])$, $0<\beta<1$, $x\in[a,b]$, $n\in\mathbb{N}$, $n^{1-\beta}\ge3$. *Then*

i)

$$\left|A_n(f,x)-\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}A_n\left((\cdot-x)^j\right)(x)-f(x)\right|\le \tag{84}$$

$$\frac{(4.019)}{\Gamma(\alpha+1)}\cdot\left\{\frac{\left(\omega_1\left(D^{\alpha}_{x-}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D^{\alpha}_{*x}f,\frac{1}{n_{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}\cdot$$

$$\left.\left(\left\|D^{\alpha}_{x-}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D^{\alpha}_{*x}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\},$$

ii) if $f^{(j)}(x)=0$, *for* $j=1,...,N-1$, *we have*

$$|A_n(f,x)-f(x)|\le\frac{(4.019)}{\Gamma(\alpha+1)}\cdot \tag{85}$$

$$\left\{\frac{\left(\omega_1\left(D^{\alpha}_{x-}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D^{\alpha}_{*x}f,\frac{1}{n_{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}\cdot$$

$$\left.\left(\left\|D^{\alpha}_{x-}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D^{\alpha}_{*x}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\},$$

when $\alpha>1$ *notice here the extremely high rate of convergence at* $n^{-(\alpha+1)\beta}$,

iii)

$$|A_n(f,x)-f(x)|\le(4.019)\cdot \tag{86}$$

$$\left\{\sum_{j=1}^{N-1}\frac{\left|f^{(j)}(x)\right|}{j!}\left\{\frac{1}{n^{\beta j}}+(b-a)^j\frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n_{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}\cdot$$

$$\left.\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}\Bigg\},$$

$\forall\ x\in[a,b]$,

and

iv)

$$\|A_nf-f\|_{\infty}\leq(4.019)\cdot$$

$$\left\{\sum_{j=1}^{N-1}\frac{\left\|f^{(j)}\right\|_{\infty}}{j!}\left\{\frac{1}{n^{\beta j}}+(b-a)^j\frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n_{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right. \qquad (87)$$

$$\frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}\cdot$$

$$\left.(b-a)^{\alpha}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\right\}\Bigg\}.$$

Above, when $N=1$ the sum $\sum_{j=1}^{N-1}\cdot=0$.

As we see here we obtain fractionally type pointwise and uniform convergence with rates of $A_n\to I$ the unit operator, as $n\to\infty$.

Proof. Let $x\in[a,b]$. We have that $D_{x-}^{\alpha}f(x)=D_{*x}^{\alpha}f(x)=0$.

From [16], p. 54, we get by the left Caputo fractional Taylor formula that

$$f\left(\frac{k}{n}\right)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^j+ \qquad (88)$$

$$\frac{1}{\Gamma(\alpha)}\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right)dJ,$$

for all $x\leq\frac{k}{n}\leq b$.

Also from [4], using the right Caputo fractional Taylor formula we get

$$f\left(\frac{k}{n}\right)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^{j}+ \tag{89}$$

$$\frac{1}{\Gamma(\alpha)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J)-D_{x-}^{\alpha}f(x)\right)dJ,$$

for all $a\leq\frac{k}{n}\leq x$.

Hence we have

$$f\left(\frac{k}{n}\right)\chi(nx-k)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\chi(nx-k)\left(\frac{k}{n}-x\right)^{j}+ \tag{90}$$

$$\frac{\chi(nx-k)}{\Gamma(\alpha)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right)dJ,$$

for all $x\leq\frac{k}{n}\leq b$, iff $\lceil nx\rceil\leq k\leq\lfloor nb\rfloor$, and

$$f\left(\frac{k}{n}\right)\chi(nx-k)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\chi(nx-k)\left(\frac{k}{n}-x\right)^{j}+ \tag{91}$$

$$\frac{\chi(nx-k)}{\Gamma(\alpha)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J)-D_{x-}^{\alpha}f(x)\right)dJ,$$

for all $a\leq\frac{k}{n}\leq x$, iff $\lceil na\rceil\leq k\leq\lfloor nx\rfloor$.

We have that $\lceil nx\rceil\leq\lfloor nx\rfloor+1$.

Therefore it holds

$$\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\chi(nx-k)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\chi(nx-k)\left(\frac{k}{n}-x\right)^{j}+$$

$$\frac{1}{\Gamma(\alpha)}\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\chi(nx-k)\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right)dJ, \tag{92}$$

and

$$\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}f\left(\frac{k}{n}\right)\chi(nx-k)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\chi(nx-k)\left(\frac{k}{n}-x\right)^{j}+ \tag{93}$$

$$\frac{1}{\Gamma(\alpha)}\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\chi(nx-k)\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J)-D_{x-}^{\alpha}f(x)\right)dJ.$$

Adding the last two equalities (92) and (93) we obtain

$$A_n^*(f,x) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right)\chi(nx-k) = \tag{94}$$

$$\sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k)\left(\frac{k}{n}-x\right)^j +$$

$$\frac{1}{\Gamma(\alpha)}\left\{\sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \chi(nx-k)\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J)-D_{x-}^{\alpha}f(x)\right)dJ+\right.$$

$$\left.\sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} \chi(nx-k)\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right)dJ\right\}.$$

So we have derived

$$A_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k)\right) = \tag{95}$$

$$\sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n^*\left((\cdot - x)^j\right)(x) + \theta_n(x),$$

where

$$\theta_n(x) := \frac{1}{\Gamma(\alpha)}\left\{\sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \chi(nx-k)\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J)-D_{x-}^{\alpha}f(x)\right)dJ\right.$$

$$\left. + \sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} \chi(nx-k)\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right)dJ\right\}. \tag{96}$$

We set

$$\theta_{1n}(x) := \frac{1}{\Gamma(\alpha)}\sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \chi(nx-k)\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J)-D_{x-}^{\alpha}f(x)\right)dJ, \tag{97}$$

and

$$\theta_{2n} := \frac{1}{\Gamma(\alpha)}\sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} \chi(nx-k)\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right)dJ, \tag{98}$$

i.e.

$$\theta_n(x) = \theta_{1n}(x) + \theta_{2n}(x). \tag{99}$$

We assume $b-a>\frac{1}{n^{\beta}}$, $0<\beta<1$, which is always the case for large enough $n\in\mathbb{N}$, that is when $n>\left\lceil (b-a)^{-\frac{1}{\beta}}\right\rceil$. It is always true that either $\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\beta}}$ or $\left|\frac{k}{n}-x\right|>\frac{1}{n^{\beta}}$.

For $k=\lceil na\rceil,...,\lfloor nx\rfloor$, we consider

$$\gamma_{1k}:=\left|\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J)-D_{x-}^{\alpha}f(x)\right)dJ\right| \tag{100}$$

$$=\left|\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ\right|\leq\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left|D_{x-}^{\alpha}f(J)\right|dJ$$

$$\leq\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}\frac{\left(x-\frac{\kappa}{n}\right)^{\alpha}}{\alpha}\leq\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}\frac{(x-a)^{\alpha}}{\alpha}. \tag{101}$$

That is

$$\gamma_{1k}\leq\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}\frac{(x-a)^{\alpha}}{\alpha}, \tag{102}$$

for $k=\lceil na\rceil,...,\lfloor nx\rfloor$.

Also we have in case of $\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\beta}}$ that

$$\gamma_{1k}\leq\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left|D_{x-}^{\alpha}f(J)-D_{x-}^{\alpha}f(x)\right|dJ \tag{103}$$

$$\leq\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\omega_{1}\left(D_{x-}^{\alpha}f,|J-x|\right)_{[a,x]}dJ$$

$$\leq\omega_{1}\left(D_{x-}^{\alpha}f,\left|x-\frac{k}{n}\right|\right)_{[a,x]}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}dJ$$

$$\leq\omega_{1}\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}\frac{\left(x-\frac{k}{n}\right)^{\alpha}}{\alpha}\leq\omega_{1}\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}\frac{1}{\alpha n^{a\beta}}. \tag{104}$$

That is when $\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\beta}}$, then

$$\gamma_{1k}\leq\frac{\omega_{1}\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}}{\alpha n^{a\beta}}. \tag{105}$$

Consequently we obtain

$$|\theta_{1n}(x)|\leq\frac{1}{\Gamma(\alpha)}\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\chi(nx-k)\,\gamma_{1k}=$$

$$\frac{1}{\Gamma(\alpha)}\left\{\sum_{\left\{\begin{array}{c}k=\lceil na\rceil\\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\beta}}\end{array}\right.}^{\lfloor nx\rfloor}\chi(nx-k)\,\gamma_{1k}+\sum_{\left\{\begin{array}{c}k=\lceil na\rceil\\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\beta}}\end{array}\right.}^{\lfloor nx\rfloor}\chi(nx-k)\,\gamma_{1k}\right\}\leq$$

$$\frac{1}{\Gamma(\alpha)}\left\{\left(\sum_{\substack{\left\{\begin{array}{c} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\beta}}\end{array}\right.}}^{\lfloor nx\rfloor}\chi(nx-k)\right)\frac{\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}}{\alpha n^{\alpha\beta}}+\right.$$

$$\left.\left(\sum_{\substack{\left\{\begin{array}{c} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\beta}}\end{array}\right.}}^{\lfloor nx\rfloor}\chi(nx-k)\right)\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}\frac{(x-a)^{\alpha}}{\alpha}\right\}\leq \tag{106}$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}}{n^{\alpha\beta}}+\right.$$

$$\left.\left(\sum_{\substack{\left\{\begin{array}{c} k=-\infty \\ :|nx-k|>n^{1-\beta}\end{array}\right.}}^{\infty}\chi(nx-k)\right)\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}\right\}\overset{(18)}{\leq} \tag{107}$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}}{n^{\alpha\beta}}+\right.$$

$$\left.\frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^{2}}}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}\right\}.$$

So we have proved that

$$|\theta_{1n}(x)|\leq\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}}{n^{\alpha\beta}}+\right. \tag{108}$$

$$\left.\frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^{2}}}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}\right\}.$$

Next when $k=\lfloor nx\rfloor+1,...,\lfloor nb\rfloor$ we consider

$$\gamma_{2k}:=\left|\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right)dJ\right|\leq$$

$$\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left|D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right|dJ=$$

$$\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left|D_{*x}^{\alpha}f(J)\right|dJ\leq\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\frac{\left(\frac{k}{n}-x\right)^{\alpha}}{\alpha}\leq \tag{109}$$

$$\left\| D^{\alpha}_{*x} f \right\|_{\infty,[x,b]} \frac{(b-x)^{\alpha}}{\alpha}. \tag{110}$$

Therefore when $k = \lfloor nx \rfloor + 1, ..., \lfloor nb \rfloor$ we get that

$$\gamma_{2k} \leq \left\| D^{\alpha}_{*x} f \right\|_{\infty,[x,b]} \frac{(b-x)^{\alpha}}{\alpha}. \tag{111}$$

In case of $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}}$, we get

$$\gamma_{2k} \leq \int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - J \right)^{\alpha-1} \omega_1 \left(D^{\alpha}_{*x} f, |J - x| \right)_{[x,b]} dJ \leq \tag{112}$$

$$\omega_1 \left(D^{\alpha}_{*x} f, \left| \frac{k}{n} - x \right| \right)_{[x,b]} \int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - J \right)^{\alpha-1} dJ \leq$$

$$\omega_1 \left(D^{\alpha}_{*x} f, \frac{1}{n^{\beta}} \right)_{[x,b]} \frac{\left(\frac{k}{n} - x \right)^{\alpha}}{\alpha} \leq \omega_1 \left(D^{\alpha}_{*x} f, \frac{1}{n^{\beta}} \right)_{[x,b]} \frac{1}{\alpha n^{\alpha\beta}}. \tag{113}$$

So when $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}}$ we derived that

$$\gamma_{2k} \leq \frac{\omega_1 \left(D^{\alpha}_{*x} f, \frac{1}{n^{\beta}} \right)_{[x,b]}}{\alpha n^{\alpha\beta}}. \tag{114}$$

Similarly we have that

$$|\theta_{2n}(x)| \leq \frac{1}{\Gamma(\alpha)} \left(\sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} \chi(nx-k) \gamma_{2k} \right) = \tag{115}$$

$$\frac{1}{\Gamma(\alpha)} \left\{ \sum_{\begin{cases} k = \lfloor nx \rfloor + 1 \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}} \end{cases}}^{\lfloor nb \rfloor} \chi(nx-k) \gamma_{2k} + \sum_{\begin{cases} k = \lfloor nx \rfloor + 1 \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^{\beta}} \end{cases}}^{\lfloor nb \rfloor} \chi(nx-k) \gamma_{2k} \right\} \leq$$

$$\frac{1}{\Gamma(\alpha)} \left\{ \left(\sum_{\begin{cases} k = \lfloor nx \rfloor + 1 \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}} \end{cases}}^{\lfloor nb \rfloor} \chi(nx-k) \right) \frac{\omega_1 \left(D^{\alpha}_{*x} f, \frac{1}{n^{\beta}} \right)_{[x,b]}}{\alpha n^{\alpha\beta}} + \right.$$

$$\left. \left(\sum_{\begin{cases} k = \lfloor nx \rfloor + 1 \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^{\beta}} \end{cases}}^{\lfloor nb \rfloor} \chi(nx-k) \right) \left\| D^{\alpha}_{*x} f \right\|_{\infty,[x,b]} \frac{(b-x)^{\alpha}}{\alpha} \right\} \leq \tag{116}$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}}{n^{\alpha\beta}}+\right.$$

$$\left.\left(\sum_{\substack{k=-\infty \\ \left\{:\left|\frac{k}{n}-x\right|>\frac{1}{n^{\beta}}\right.}}^{\infty}\chi(nx-k)\right)\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right\}\overset{(18)}{\leq} \tag{117}$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}}{n^{\alpha\beta}}+\right.$$

$$\left.\frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^{2}}}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right\}.$$

So we have proved that

$$|\theta_{2n}(x)|\leq\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}}{n^{\alpha\beta}}+\right. \tag{118}$$

$$\left.\frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^{2}}}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right\}.$$

Therefore

$$|\theta_n(x)|\leq|\theta_{1n}(x)|+|\theta_{2n}(x)|\leq \tag{119}$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}}{n^{\alpha\beta}}+\right. \tag{120}$$

$$\left.\frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^{2}}}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}.$$

As in (69) we get that

$$\left|A_n^*\left((\cdot-x)^j\right)(x)\right|\leq\frac{1}{n^{\beta j}}+(b-a)^j\frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^{2}}}, \tag{121}$$

for $j=1,...,N-1$, $\forall\ x\in[a,b]$.

Putting things together, we have established

$$\left|A_n^*(f,x)-f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\chi(nx-k)\right)\right|\leq \tag{122}$$

$$\sum_{j=1}^{N-1}\frac{\left|f^{(j)}(x)\right|}{j!}\left[\frac{1}{n^{\beta j}}+(b-a)^j\frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^{2}}}\right]+$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}\cdot$$

$$\left.\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}=:T_n(x). \quad (123)$$

As a result, see (39), we derive

$$|A_n(f,x)-f(x)|\leq(4.019)\,T_n(x), \quad (124)$$

$\forall\ x\in[a,b]$.

We further have that

$$\|T_n\|_{\infty}\leq\sum_{j=1}^{N-1}\frac{\left\|f^{(j)}\right\|_{\infty}}{j!}\left[\frac{1}{n^{\beta j}}+(b-a)^j\frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}\right]+ \quad (125)$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left\{\sup_{x\in[a,b]}\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}\right)+\sup_{x\in[a,b]}\left(\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)\right\}}{n^{\alpha\beta}}\right.$$

$$+\frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}(b-a)^{\alpha}\cdot$$

$$\left.\left\{\left(\sup_{x\in[a,b]}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}\right)+\sup_{x\in[a,b]}\left(\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\right)\right\}\right\}=:E_n.$$

Hence it holds

$$\|A_nf-f\|_{\infty}\leq(4.019)\,E_n. \quad (126)$$

Since $f\in AC^N([a,b])$, $N=\lceil\alpha\rceil$, $\alpha>0$, $\alpha\notin\mathbb{N}$, $f^{(N)}\in L_{\infty}([a,b])$, $x\in[a,b]$, then we get that $f\in AC^N([a,x])$, $f^{(N)}\in L_{\infty}([a,x])$ and $f\in AC^N([x,b])$, $f^{(N)}\in L_{\infty}([x,b])$.

We have

$$\left(D_{x-}^{\alpha}f\right)(y)=\frac{(-1)^N}{\Gamma(N-\alpha)}\int_y^x(J-y)^{N-\alpha-1}f^{(N)}(J)\,dJ, \quad (127)$$

$\forall\ y\in[a,x]$ and

$$\left|\left(D_{x-}^{\alpha}f\right)(y)\right|\leq\frac{1}{\Gamma(N-\alpha)}\left(\int_y^x(J-y)^{N-\alpha-1}dJ\right)\left\|f^{(N)}\right\|_{\infty} \quad (128)$$

$$=\frac{1}{\Gamma(N-\alpha)}\frac{(x-y)^{N-\alpha}}{(N-\alpha)}\left\|f^{(N)}\right\|_{\infty}=$$

$$\frac{(x-y)^{N-\alpha}}{\Gamma(N-\alpha+1)}\left\|f^{(N)}\right\|_{\infty} \leq \frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)}\left\|f^{(N)}\right\|_{\infty}.$$

That is

$$\left\|D_{x-}^{\alpha} f\right\|_{\infty,[a,x]} \leq \frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)}\left\|f^{(N)}\right\|_{\infty}, \tag{129}$$

and

$$\sup_{x\in[a,b]}\left\|D_{x-}^{\alpha} f\right\|_{\infty,[a,x]} \leq \frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)}\left\|f^{(N)}\right\|_{\infty}. \tag{130}$$

Similarly we have

$$\left(D_{*x}^{\alpha} f\right)(y)=\frac{1}{\Gamma(N-\alpha)}\int_{x}^{y}(y-t)^{N-\alpha-1} f^{(N)}(t)\,dt, \tag{131}$$

$\forall\ y\in[x,b]$.

Thus we get

$$\left|\left(D_{*x}^{\alpha} f\right)(y)\right| \leq \frac{1}{\Gamma(N-\alpha)}\left(\int_{x}^{y}(y-t)^{N-\alpha-1}\,dt\right)\left\|f^{(N)}\right\|_{\infty} \leq$$

$$\frac{1}{\Gamma(N-\alpha)}\frac{(y-x)^{N-\alpha}}{(N-\alpha)}\left\|f^{(N)}\right\|_{\infty} \leq \frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)}\left\|f^{(N)}\right\|_{\infty}.$$

Hence

$$\left\|D_{*x}^{\alpha} f\right\|_{\infty,[x,b]} \leq \frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)}\left\|f^{(N)}\right\|_{\infty}, \tag{132}$$

and

$$\sup_{x\in[a,b]}\left\|D_{*x}^{\alpha} f\right\|_{\infty,[x,b]} \leq \frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)}\left\|f^{(N)}\right\|_{\infty}. \tag{133}$$

From (82) and (83) we get

$$\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\alpha} f,\frac{1}{n^{\beta}}\right)_{[a,x]} \leq \frac{2\left\|f^{(N)}\right\|_{\infty}}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}, \tag{134}$$

and

$$\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\alpha} f,\frac{1}{n^{\beta}}\right)_{[x,b]} \leq \frac{2\left\|f^{(N)}\right\|_{\infty}}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}. \tag{135}$$

So that $E_n<\infty$.

We finally notice that

$$A_n(f,x)-\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}A_n\left((\cdot-x)^{j}\right)(x)-f(x)=\frac{A_n^{*}(f,x)}{\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\chi(nx-k)\right)}$$

$$-\frac{1}{\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\chi(nx-k)\right)}\left(\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}A_n^{*}\left((\cdot-x)^{j}\right)(x)\right)-f(x)$$

$$= \frac{1}{\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \chi(nx-k)\right)} \cdot \tag{136}$$

$$\left[A_n^*(f,x) - \left(\sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n^* \left((\cdot - x)^j \right)(x) \right) - \left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \chi(nx-k) \right) f(x) \right].$$

Therefore we get

$$\left| A_n(f,x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n \left((\cdot - x)^j \right)(x) - f(x) \right| \leq (4.019) \cdot$$

$$\left| A_n^*(f,x) - \left(\sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n^* \left((\cdot - x)^j \right)(x) \right) - \left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \chi(nx-k) \right) f(x) \right|, \tag{137}$$

$\forall\, x \in [a,b]$.

The proof of the theorem is now complete. ■

Next we apply Theorem 30 for $N = 1$.

Corollary 31 *Let $0 < \alpha, \beta < 1$, $n^{1-\beta} \geq 3$, $f \in AC([a,b])$, $f' \in L_\infty([a,b])$, $n \in \mathbb{N}$. Then*

$$\|A_n f - f\|_\infty \leq \frac{(4.019)}{\Gamma(\alpha+1)} \cdot \tag{138}$$

$$\left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1 \left(D_{x-}^\alpha f, \frac{1}{n^\beta} \right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1 \left(D_{*x}^\alpha f, \frac{1}{n_\beta} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right) e^{\left(n^{1-\beta}-2\right)^2}} (b-a)^\alpha \cdot$$

$$\left. \left(\sup_{x\in[a,b]} \left\| D_{x-}^\alpha f \right\|_{\infty,[a,x]} + \sup_{x\in[a,b]} \left\| D_{*x}^\alpha f \right\|_{\infty,[x,b]} \right) \right\}.$$

Remark 32 *Let $0 < \alpha < 1$, then by (130), we get*

$$\sup_{x\in[a,b]} \left\| D_{x-}^\alpha f \right\|_{\infty,[a,x]} \leq \frac{(b-a)^{1-\alpha}}{\Gamma(2-\alpha)} \|f'\|_\infty, \tag{139}$$

and by (133), we obtain

$$\sup_{x\in[a,b]} \left\| D_{*x}^\alpha f \right\|_{\infty,[x,b]} \leq \frac{(b-a)^{1-\alpha}}{\Gamma(2-\alpha)} \|f'\|_\infty, \tag{140}$$

given that $f \in AC([a,b])$ and $f' \in L_\infty([a,b])$.

Next we specialize to $\alpha = \frac{1}{2}$.

Corollary 33 *Let $0 < \beta < 1$, $n^{1-\beta} \geq 3$, $f \in AC([a,b])$, $f' \in L_\infty([a,b])$, $n \in \mathbb{N}$. Then*

$$\|A_n f - f\|_\infty \leq \frac{(8.038)}{\sqrt{\pi}} \cdot$$

$$\left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^\beta} \right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\frac{\beta}{2}}} + \right.$$

$$\frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right) e^{\left(n^{1-\beta}-2\right)^2}} \sqrt{b-a} \cdot$$

$$\left. \left(\sup_{x\in[a,b]} \left\| D_{x-}^{\frac{1}{2}} f \right\|_{\infty,[a,x]} + \sup_{x\in[a,b]} \left\| D_{*x}^{\frac{1}{2}} f \right\|_{\infty,[x,b]} \right) \right\}, \tag{141}$$

Remark 34 *(to Corollary 33) Assume that*

$$\omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^\beta} \right)_{[a,x]} \leq \frac{K_1}{n^\beta}, \tag{142}$$

and

$$\omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n_\beta} \right)_{[x,b]} \leq \frac{K_2}{n^\beta}, \tag{143}$$

$\forall\, x \in [a,b]$, $\forall\, n \in \mathbb{N}$, where $K_1, K_2 > 0$.

Then for large enough $n \in \mathbb{N}$, by (141), we obtain

$$\|A_n f - f\|_\infty \leq \frac{M}{n^{\frac{3}{2}\beta}}, \tag{144}$$

for some $M > 0$.

The speed of convergence in (144) is much higher than the corresponding speeds achieved in (40), which were there $\frac{1}{n^\beta}$.

5 Complex Neural Network Approximations

We make

Remark 35 *Let $X := [a,b]$, $\mathbb{R}$ and $f : X \to \mathbb{C}$ with real and imaginary parts $f_1, f_2 : f = f_1 + i f_2$, $i = \sqrt{-1}$. Clearly f is continuous iff f_1 and f_2 are continuous.*

Also it holds

$$f^{(j)}(x) = f_1^{(j)}(x) + i f_2^{(j)}(x), \tag{145}$$

for all $j = 1, ..., N$, *given that* $f_1, f_2 \in C^N(X)$, $N \in \mathbb{N}$.

We denote by $C_B(\mathbb{R}, \mathbb{C})$ *the space of continuous and bounded functions* $f : \mathbb{R} \to \mathbb{C}$. *Clearly* f *is bounded, iff both* f_1, f_2 *are bounded from* $\mathbb{R}$ *into* $\mathbb{R}$, *where* $f = f_1 + if_2$.

Here we define

$$A_n(f, x) := A_n(f_1, x) + iA_n(f_2, x), \tag{146}$$

and

$$B_n(f, x) := B_n(f_1, x) + iB_n(f_2, x). \tag{147}$$

We observe here that

$$|A_n(f, x) - f(x)| \le |A_n(f_1, x) - f_1(x)| + |A_n(f_2, x) - f_2(x)|, \tag{148}$$

and

$$\|A_n(f) - f\|_\infty \le \|A_n(f_1) - f_1\|_\infty + \|A_n(f_2) - f_2\|_\infty. \tag{149}$$

Similarly we get

$$|B_n(f, x) - f(x)| \le |B_n(f_1, x) - f_1(x)| + |B_n(f_2, x) - f_2(x)|, \tag{150}$$

and

$$\|B_n(f) - f\|_\infty \le \|B_n(f_1) - f_1\|_\infty + \|B_n(f_2) - f_2\|_\infty. \tag{151}$$

We present

Theorem 36 *Let* $f \in C([a, b], \mathbb{C})$, $f = f_1 + if_2$, $0 < \alpha < 1$, $n \in \mathbb{N}$, $n^{1-\alpha} \ge 3$, $x \in [a, b]$. *Then*

i)

$$|A_n(f, x) - f(x)| \le (4.019) \cdot \tag{152}$$

$$\left[\left(\omega_1\left(f_1, \frac{1}{n^\alpha}\right) + \omega_1\left(f_2, \frac{1}{n^\alpha}\right)\right) + (\|f_1\|_\infty + \|f_2\|_\infty)\frac{1}{\sqrt{\pi}\left(n^{1-\alpha} - 2\right)e^{\left(n^{1-\alpha}-2\right)^2}}\right]$$

$$=: \psi_1,$$

and

ii)

$$\|A_n(f) - f\|_\infty \le \psi_1. \tag{153}$$

Proof. Based on Remark 35 and Theorem 12. ■

We give

Theorem 37 *Let* $f \in C_B(\mathbb{R}, \mathbb{C})$, $f = f_1 + if_2$, $0 < \alpha < 1$, $n \in \mathbb{N}$, $n^{1-\alpha} \ge 3$, $x \in \mathbb{R}$. *Then* *i)*

$$|B_n(f,x) - f(x)| \leq \left(\omega_1\left(f_1, \frac{1}{n^\alpha}\right) + \omega_1\left(f_2, \frac{1}{n^\alpha}\right)\right) + \tag{154}$$

$$(\|f_1\|_\infty + \|f_2\|_\infty) \frac{1}{\sqrt{\pi}(n^{1-\alpha}-2)e^{(n^{1-\alpha}-2)^2}} =: \psi_2,$$

ii)

$$\|B_n(f) - f\|_\infty \leq \psi_2. \tag{155}$$

Proof. Based on Remark 35 and Theorem 13. ■

Next we present a result of high order complex neural network approximation.

Theorem 38 *Let $f : [a,b] \to \mathbb{C}$, $[a,b] \subset \mathbb{R}$, such that $f = f_1 + if_2$. Assume $f_1, f_2 \in C^N([a,b])$, $n, N \in \mathbb{N}$, $n^{1-\alpha} \geq 3$, $0 < \alpha < 1$, $x \in [a,b]$. Then*

i)

$$|A_n(f,x) - f(x)| \leq (4.019) \cdot \tag{156}$$

$$\left\{\sum_{j=1}^{N} \frac{\left|f_1^{(j)}(x)\right| + \left|f_2^{(j)}(x)\right|}{j!}\left[\frac{1}{n^{\alpha j}} + \frac{(b-a)^j}{2\sqrt{\pi}(n^{1-\alpha}-2)e^{(n^{1-\alpha}-2)^2}}\right] + \right.$$

$$\left[\frac{\omega_1\left(f_1^{(N)}, \frac{1}{n^\alpha}\right) + \omega_1\left(f_2^{(N)}, \frac{1}{n^\alpha}\right)}{n^{\alpha N} N!} + \right.$$

$$\left.\left.\left(\frac{\left(\left\|f_1^{(N)}\right\|_\infty + \left\|f_2^{(N)}\right\|_\infty\right)(b-a)^N}{N!\sqrt{\pi}(n^{1-\alpha}-2)e^{(n^{1-\alpha}-2)^2}}\right)\right]\right\},$$

ii) assume further $f_1^{(j)}(x_0) = f_2^{(j)}(x_0) = 0$, $j = 1, ..., N$, for some $x_0 \in [a,b]$, it holds

$$|A_n(f,x_0) - f(x_0)| \leq (4.019) \cdot \tag{157}$$

$$\left[\frac{\omega_1\left(f_1^{(N)}, \frac{1}{n^\alpha}\right) + \omega_1\left(f_2^{(N)}, \frac{1}{n^\alpha}\right)}{n^{\alpha N} N!} + \right.$$

$$\left.\left(\frac{\left(\left\|f_1^{(N)}\right\|_\infty + \left\|f_2^{(N)}\right\|_\infty\right)(b-a)^N}{N!\sqrt{\pi}(n^{1-\alpha}-2)e^{(n^{1-\alpha}-2)^2}}\right)\right],$$

notice here the extremely high rate of convergence at $n^{-(N+1)\alpha}$,

iii)

$$\|A_n(f) - f\|_\infty \leq (4.019) \cdot \tag{158}$$

$$\left\{\sum_{j=1}^{N} \frac{\left(\left\|f_1^{(j)}\right\|_\infty + \left\|f_2^{(j)}\right\|_\infty\right)}{j!}\left[\frac{1}{n^{\alpha j}} + \frac{(b-a)^j}{2\sqrt{\pi}(n^{1-\alpha}-2)e^{(n^{1-\alpha}-2)^2}}\right] + \right.$$

$$\left[\frac{\left(\omega_1\left(f_1^{(N)},\frac{1}{n^\alpha}\right)+\omega_1\left(f_2^{(N)},\frac{1}{n^\alpha}\right)\right)}{n^{\alpha N}N!}+\right.$$

$$\left.\left.\frac{\left(\left\|f_1^{(N)}\right\|_\infty+\left\|f_2^{(N)}\right\|_\infty\right)(b-a)^N}{N!\sqrt{\pi}\left(n^{1-\alpha}-2\right)e^{\left(n^{1-\alpha}-2\right)^2}}\right]\right\}.$$

Proof. Based on Remark 35 and Theorem 16. ■

We continue with high order complex fractional neural network approximation.

Theorem 39 *Let $f:[a,b]\to\mathbb{C}$, $[a,b]\subset\mathbb{R}$, such that $f=f_1+if_2$; $\alpha>0$, $N=\lceil\alpha\rceil$, $\alpha\notin\mathbb{N}$, $0<\beta<1$, $x\in[a,b]$, $n\in\mathbb{N}$, $n^{1-\beta}\geq 3$. Assume $f_1,f_2\in AC^N([a,b])$, with $f_1^{(N)},f_2^{(N)}\in L_\infty([a,b])$. Then*

i) assume further $f_1^{(j)}(x)=f_2^{(j)}(x)=0$, $j=1,...,N-1$, we have

$$|A_n(f,x)-f(x)|\leq\frac{(4.019)}{\Gamma(\alpha+1)}\cdot$$

$$\left\{\frac{1}{n^{\alpha\beta}}\left[\left(\omega_1\left(D_{x-}^\alpha f_1,\frac{1}{n^\beta}\right)_{[a,x]}+\omega_1\left(D_{*x}^\alpha f_1,\frac{1}{n^\beta}\right)_{[x,b]}\right)+\right.\right.$$

$$\left.\left(\omega_1\left(D_{x-}^\alpha f_2,\frac{1}{n^\beta}\right)_{[a,x]}+\omega_1\left(D_{*x}^\alpha f_2,\frac{1}{n^\beta}\right)_{[x,b]}\right)\right]+$$

$$\frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}\cdot$$

$$\left[\left(\left\|D_{x-}^\alpha f_1\right\|_{\infty,[a,x]}(x-a)^\alpha+\left\|D_{*x}^\alpha f_1\right\|_{\infty,[x,b]}(b-x)^\alpha\right)+\right.$$

$$\left.\left.\left(\left\|D_{x-}^\alpha f_2\right\|_{\infty,[a,x]}(x-a)^\alpha+\left\|D_{*x}^\alpha f_2\right\|_{\infty,[x,b]}(b-x)^\alpha\right)\right]\right\},\tag{159}$$

when $\alpha>1$ notice here the extremely high rate of convergence at $n^{-(\alpha+1)\beta}$,

ii)

$$|A_n(f,x)-f(x)|\leq(4.019)\cdot\left\{\sum_{j=1}^{N-1}\frac{\left(\left|f_1^{(j)}(x)\right|+\left|f_2^{(j)}(x)\right|\right)}{j!}\right.$$

$$\left\{\frac{1}{n^{\beta j}}+\frac{(b-a)^j}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}\right\}+$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{1}{n^{\alpha\beta}}\left[\left(\omega_1\left(D_{x-}^\alpha f_1,\frac{1}{n^\beta}\right)_{[a,x]}+\omega_1\left(D_{*x}^\alpha f_1,\frac{1}{n^\beta}\right)_{[x,b]}\right)+\right.\right.$$

$$\left(\omega_1\left(D^{\alpha}_{x-}f_2,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D^{\alpha}_{*x}f_2,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)\Bigg]+$$

$$\frac{1}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}\cdot$$

$$\Big[\left(\left\|D^{\alpha}_{x-}f_1\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D^{\alpha}_{*x}f_1\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)+$$

$$\left(\left\|D^{\alpha}_{x-}f_2\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D^{\alpha}_{*x}f_2\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\Big]\Big\}, \tag{160}$$

and

iii)

$$\|A_n(f)-f\|_{\infty}\leq(4.019)\cdot \tag{158}$$

$$\left\{\sum_{j=1}^{N-1}\frac{\left(\left\|f_1^{(j)}\right\|_{\infty}+\left\|f_2^{(j)}\right\|_{\infty}\right)}{j!}\left\{\frac{1}{n^{\beta j}}+\frac{(b-a)^j}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{1}{n^{\alpha\beta}}\left\{\left[\sup_{x\in[a,b]}\omega_1\left(D^{\alpha}_{x-}f_1,\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D^{\alpha}_{*x}f_1,\frac{1}{n^{\beta}}\right)_{[x,b]}+\right.\right.\right.$$

$$\left.\left.\sup_{x\in[a,b]}\omega_1\left(D^{\alpha}_{x-}f_2,\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D^{\alpha}_{*x}f_2,\frac{1}{n^{\beta}}\right)_{[x,b]}\right]\right\}+$$

$$\frac{(b-a)^{\alpha}}{2\sqrt{\pi}\left(n^{1-\beta}-2\right)e^{\left(n^{1-\beta}-2\right)^2}}\cdot$$

$$\left[\left(\sup_{x\in[a,b]}\left\|D^{\alpha}_{x-}f_1\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D^{\alpha}_{*x}f_1\right\|_{\infty,[x,b]}\right)+\right.$$

$$\left.\left.\left(\sup_{x\in[a,b]}\left\|D^{\alpha}_{x-}f_2\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D^{\alpha}_{*x}f_2\right\|_{\infty,[x,b]}\right)\right]\right\}. \tag{161}$$

Above, when $N=1$ the sum $\sum_{j=1}^{N-1}\cdot=0$.

As we see here we obtain fractionally type pointwise and uniform convergence with rates of complex $A_n\to I$ the unit operator, as $n\to\infty$.

Proof. Using Theorem 30 and Remark 35. ■

We need

Definition 40 *Let $f\in C_B(\mathbb{R},\mathbb{C})$, with $f=f_1+if_2$. We define*

$$C_n(f,x):=C_n(f_1,x)+iC_n(f_2,x),$$

$$D_n(f,x):=D_n(f_1,x)+iD_n(f_2,x),\quad\forall\, x\in\mathbb{R},\ n\in\mathbb{N}. \tag{162}$$

We finish with

Theorem 41 *Let* $f \in C_B(\mathbb{R}, \mathbb{C})$, $f = f_1 + if_2$, $0 < \alpha < 1$, $n \in \mathbb{N}$, $n^{1-\alpha} \geq 3$, $x \in \mathbb{R}$. *Then*

i)

$$\left\{ \begin{array}{l} |C_n(f,x) - f(x)| \\ |D_n(f,x) - f(x)| \end{array} \right. \leq \left(\omega_1 \left(f_1, \frac{1}{n} + \frac{1}{n^\alpha} \right) + \omega_1 \left(f_2, \frac{1}{n} + \frac{1}{n^\alpha} \right) \right)$$

$$+ \frac{(\|f_1\|_\infty + \|f_2\|_\infty)}{\sqrt{\pi}(n^{1-\alpha} - 2) e^{(n^{1-\alpha}-2)^2}} =: \mu_{3n}(f_1, f_2), \tag{163}$$

and

ii)

$$\left\{ \begin{array}{l} \|C_n(f) - f\|_\infty \\ \|D_n(f) - f\|_\infty \end{array} \right. \leq \mu_{3n}(f_1, f_2). \tag{164}$$

Proof. By Theorems 14, 15, also see (162). ■

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