Received 08/07/14

FURTHER INEQUALITIES FOR SEQUENCES AND POWER SERIES OF OPERATORS IN HILBERT SPACES VIA HERMITIAN FORMS

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ABSTRACT. By the use of some inequalities for nonnegative Hermitian forms some new inequalities for sequences and power series of bounded linear operators in complex Hilbert spaces are established. Applications for some fundamental functions of interest are also given.

1. INTRODUCTION

Let $\mathbb{K}$ be the field of real or complex numbers, i.e., $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and X be a linear space over $\mathbb{K}$.

Definition 1. *A functional* $(\cdot,\cdot) : X \times X \to \mathbb{K}$ *is said to be a Hermitian form on* X *if*

(H1) $(ax + by, z) = a(x, z) + b(y, z)$ *for* $a, b \in \mathbb{K}$ *and* $x, y, z \in X$;
(H2) $(x, y) = \overline{(y, x)}$ *for all* $x, y \in X$.

The functional $(\cdot,\cdot)$ is said to be *positive semi-definite* on a subspace Y of X if

(H3) $(y, y) \geq 0$ for every $y \in Y$,

and *positive definite* on Y if it is positive semi-definite on Y and

(H4) $(y, y) = 0$, $y \in Y$ implies $y = 0$.

The functional $(\cdot,\cdot)$ is said to be *definite* on Y provided that either $(\cdot,\cdot)$ or $-(\cdot,\cdot)$ is positive semi-definite on Y.

When a Hermitian functional $(\cdot,\cdot)$ is positive-definite on the whole space X, then, as usual, we will call it an *inner product* on X and will denote it by $\langle\cdot,\cdot\rangle$.

We use the following notations related to a given Hermitian form $(\cdot,\cdot)$ on X :

$$X_0 := \{x \in X | (x, x) = 0\}, \ K := \{x \in X | (x, x) < 0\}$$

and, for a given $z \in X$,

$$X^{(z)} := \{x \in X | (x, z) = 0\} \quad \text{and} \quad L(z) := \{az | a \in \mathbb{K}\}.$$

The following fundamental facts concerning Hermitian forms hold:

Theorem 1 (Kurepa, 1968 [28]). *Let* X *and* $(\cdot,\cdot)$ *be as above.*

(1) *If* $e \in X$ *is such that* $(e, e) \neq 0$, *then we have the decomposition*

$$X = L(e) \bigoplus X^{(e)}, \tag{1.1}$$

where $\bigoplus$ *denotes the direct sum of the linear subspaces* $X^{(e)}$ *and* $L(e)$;

1991 *Mathematics Subject Classification.* 47A63; 47A99.

(2) *If the functional* $(\cdot,\cdot)$ *is positive semi-definite on* $X^{(e)}$ *for at least one* $e\in K$, *then* $(\cdot,\cdot)$ *is positive semi-definite on* $X^{(f)}$ *for each* $f\in K$;
(3) *The functional* $(\cdot,\cdot)$ *is positive semi-definite on* $X^{(e)}$ *with* $e\in K$ *if and only if the inequality*

$$|(x,y)|^2\geq (x,x)(y,y) \tag{1.2}$$

holds for all $x\in K$ *and all* $y\in X$;
(4) *The functional* $(\cdot,\cdot)$ *is semi-definite on* X *if and only if the Schwarz's inequality*

$$|(x,y)|^2\leq (x,x)(y,y) \tag{1.3}$$

holds for all $x,y\in X$;
(5) *The case of equality holds in (1.3) for* $x,y\in X$ *and in (1.2), for* $x\in K$, $y\in X$, *respectively; if and only if there exists a scalar* $a\in\mathbb{K}$ *such that*

$$y-ax\in X_0^{(x)}:=X_0\cap X^{(x)}.$$

Let X be a linear space over the real or complex number field $\mathbb{K}$ and let us denote by $\mathcal{H}(X)$ the class of all positive semi-definite Hermitian forms on X, or, for simplicity, *nonnegative* forms on X.

If $(\cdot,\cdot)\in\mathcal{H}(X)$, then the functional $\|\cdot\|=(\cdot,\cdot)^{\frac{1}{2}}$ is a *semi-norm* on X and the following equivalent versions of Schwarz's inequality hold:

$$\|x\|^2\|y\|^2\geq|(x,y)|^2 \quad \text{or} \quad \|x\|\|y\|\geq|(x,y)| \tag{1.4}$$

for any $x,y\in X$.

Now, let us observe that $\mathcal{H}(X)$ is a *convex cone* in the linear space of all mappings defined on X^2 with values in $\mathbb{K}$, i.e.,

(e) $(\cdot,\cdot)_1,(\cdot,\cdot)_2\in\mathcal{H}(X)$ implies that $(\cdot,\cdot)_1+(\cdot,\cdot)_2\in\mathcal{H}(X)$;
(ee) $\alpha\geq 0$ and $(\cdot,\cdot)\in\mathcal{H}(X)$ implies that $\alpha(\cdot,\cdot)\in\mathcal{H}(X)$.

We can introduce on $\mathcal{H}(X)$ the following binary relation [23]:

$$(\cdot,\cdot)_2\geq(\cdot,\cdot)_1 \quad \text{if and only if} \quad \|x\|_2\geq\|x\|_1 \quad \text{for all } x\in X. \tag{1.5}$$

We observe that the following properties hold:

(b) $(\cdot,\cdot)_2\geq(\cdot,\cdot)_1$ for all $(\cdot,\cdot)\in\mathcal{H}(X)$;
(bb) $(\cdot,\cdot)_3\geq(\cdot,\cdot)_2$ and $(\cdot,\cdot)_2\geq(\cdot,\cdot)_1$ implies that $(\cdot,\cdot)_3\geq(\cdot,\cdot)_1$;
(bbb) $(\cdot,\cdot)_2\geq(\cdot,\cdot)_1$ and $(\cdot,\cdot)_1\geq(\cdot,\cdot)_2$ implies that $(\cdot,\cdot)_2=(\cdot,\cdot)_1$;

i.e., the binary relation defined by (1.5) is *an order relation* on $\mathcal{H}(X)$.

While (b) and (bb) are obvious from the definition, we should remark, for (bbb), that if $(\cdot,\cdot)_2\geq(\cdot,\cdot)_1$ and $(\cdot,\cdot)_1\geq(\cdot,\cdot)_2$, then obviously $\|x\|_2=\|x\|_1$ for all $x\in X$, which implies, by the following well known identity:

$$(x,y)_k:=\frac{1}{4}\left[\|x+y\|_k^2-\|x-y\|_k^2+i\left(\|x+iy\|_k^2-\|x-iy\|_k^2\right)\right] \tag{1.6}$$

with $x,y\in X$ and $k\in\{1,2\}$, that $(x,y)_2=(x,y)_1$ for all $x,y\in X$.

2. Inequalities for Hermitian Forms

The following result is of interest in itself as well:

Lemma 1. *Let X be a linear space over the real or complex number field $\mathbb{K}$ and $(\cdot,\cdot)$ a nonnegative Hermitian form on X. If $y \in X$ is such that $(y,y) \neq 0$, then*

$$p_y : H \times H \to \mathbb{K},\ \ p_y(x,z) = (x,z)\|y\|^2 - (x,y)(y,z) \tag{2.1}$$

is also a nonnegative Hermitian form on X.

We have the inequalities

$$\begin{aligned}&\left(\|x\|^2\|y\|^2 - |(x,y)|^2\right)\left(\|y\|^2\|z\|^2 - |(y,z)|^2\right) \\ &\geq \left|(x,z)\|y\|^2 - (x,y)(y,z)\right|^2\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}&\left(\|x+z\|^2\|y\|^2 - |(x+z,y)|^2\right)^{\frac{1}{2}} \\ &\leq \left(\|x\|^2\|y\|^2 - |(x,y)|^2\right)^{\frac{1}{2}} + \left(\|y\|^2\|z\|^2 - |(y,z)|^2\right)^{\frac{1}{2}}\end{aligned} \tag{2.3}$$

for any $x,y,z \in X$.

Proof. By Schwarz's inequality for the nonnegative Hermitian form $(\cdot,\cdot)$ we have

$$\begin{aligned}p_y(x,x) &= (x,x)\|y\|^2 - (x,y)(y,x) \\ &= \|x\|^2\|y\|^2 - |(x,y)|^2 \geq 0\end{aligned}$$

for any $x \in X$.

We have

$$\begin{aligned}p_y(\alpha x + \beta u, z) &= (\alpha x + \beta u, z)\|y\|^2 - (\alpha x + \beta u, y)(y,z) \\ &= \alpha(x,z)\|y\|^2 - \alpha(x,y)(y,z) \\ &+ \beta(u,z)\|y\|^2 - \beta(u,y)(y,z) \\ &= \alpha\left[(x,z)\|y\|^2 - (x,y)(y,z)\right] \\ &+ \beta\left[(u,z)\|y\|^2 - (u,y)(y,z)\right] \\ &= \alpha p_y(x,z) + \beta p_y(u,z)\end{aligned}$$

for any $\alpha, \beta \in \mathbb{K}$ and $x, u \in X$.

Also, we have

$$\begin{aligned}\overline{p_y(z,x)} &= \overline{(z,x)\|y\|^2 - (z,y)(y,x)} \\ &= \overline{(z,x)}\|y\|^2 - \overline{(z,y)(y,x)} \\ &= (x,z)\|y\|^2 - (x,y)(y,z) = p_y(x,z)\end{aligned}$$

for any $x, z \in X$.

If $y \in X$ is such that $(y,y) = 0$, then the inequalities (2.2) and (2.3) are obviously true.

If $y \in X$ is such that $(y,y) \neq 0$, then by Schwarz's inequality for $p_y(\cdot,\cdot)$ we have

$$|p_y(x,z)|^2 \leq p_y(x,x)\,p_y(z,z)$$

for any $x, z \in X$, which is equivalent to (2.2).

The inequality (2.3) follows by the triangle inequality for the nonnegative form $p_y(\cdot,\cdot)$. □

Remark 1. *The case when $(\cdot,\cdot)$ is an inner product in Lemma 1 was obtained in 1985 by S. S. Dragomir,* [3].

Remark 2. *Putting $z=\lambda y$ in (2.3), we get:*

$$0\le \|x+\lambda y\|^2\|y\|^2-|(x+\lambda y,y)|^2\le \|x\|^2\|y\|^2-|(x,y)|^2 \tag{2.4}$$

and, in particular,

$$0\le \|x\pm y\|^2\|y\|^2-|(x\pm y,y)|^2\le \|x\|^2\|y\|^2-|(x,y)|^2 \tag{2.5}$$

for every $x,y\in H$.

We note here that the inequality (2.4) is in fact equivalent to the following statement

$$\sup_{\lambda\in\mathbb{K}}\left[\|x+\lambda y\|^2\|y\|^2-|(x+\lambda y,y)|^2\right]=\|x\|^2\|y\|^2-|(x,y)|^2 \tag{2.6}$$

for each $x,y\in H$.

The following result holds:

Theorem 2. *Let X be a linear space over the real or complex number field $\mathbb{K}$ and $(\cdot,\cdot)$ a nonnegative Hermitian form on X. For any $x,y,z\in X$, the following refinement of the Schwarz inequality holds:*

$$\begin{aligned}\|x\|\|z\|\|y\|^2&\ge\left|(x,z)\|y\|^2-(x,y)(y,z)\right|+|(x,y)(y,z)|\\&\ge|(x,z)|\|y\|^2.\end{aligned} \tag{2.7}$$

Proof. Applying the inequality (2.2), we can state that

$$\begin{aligned}&\left(\|x\|^2\|y\|^2-|(x,y)|^2\right)\left(\|y\|^2\|z\|^2-|(y,z)|^2\right)\\&\ge\left|(x,z)\|y\|^2-(x,y)(y,z)\right|^2\end{aligned} \tag{2.8}$$

any $x,y,z\in X$.

Utilising the elementary inequality for real numbers

$$\left(m^2-n^2\right)\left(p^2-q^2\right)\le(mp-nq)^2, \tag{2.9}$$

we can easily see that

$$\begin{aligned}&\left(\|x\|^2\|y\|^2-|(x,y)|^2\right)\left(\|y\|^2\|z\|^2-|(y,z)|^2\right)\\&\le\left(\|x\|\|y\|^2\|z\|-|(x,y)(y,z)|\right)^2\end{aligned} \tag{2.10}$$

for any $x,y,z\in X$.

Since, by Schwarz's inequality we have

$$\|x\|\|y\|\ge|(x,y)| \text{ and } \|y\|\|z\|\ge|(y,z)|, \tag{2.11}$$

then

$$\|x\|\|y\|^2\|z\|-|(x,y)(y,z)|\ge0$$

for any $x,y,z\in X$.

Therefore, by (2.8) and (2.10) we deduce

$$\|x\|\|y\|^2\|z\|-|(x,y)(y,z)|\ge\left|(x,z)\|y\|^2-(x,y)(y,z)\right|,$$

which proves the first inequality in (2.7). □

Corollary 1. *For any $x, y, z \in X$ we have*

$$\frac{1}{2}\left[\|x\|\,\|z\| + |(x,z)|\right]\|y\|^2 \geq |(x,y)\,(y,z)|\,. \tag{2.12}$$

Proof. By the modulus property we have

$$\left|(x,z)\,\|y\|^2 - (x,y)\,(y,z)\right| \geq |(x,y)\,(y,z)| - |(x,z)|\,\|y\|^2$$

and by the first inequality in (2.7) we have

$$\begin{aligned}\|x\|\,\|z\|\,\|y\|^2 &\geq \left|(x,z)\,\|y\|^2 - (x,y)\,(y,z)\right| + |(x,y)\,(y,z)| \\ &\geq |(x,y)\,(y,z)| - |(x,z)|\,\|y\|^2 + |(x,y)\,(y,z)|\end{aligned}$$

for any $x, y, z \in X$, which is equivalent to (2.12). □

Remark 3. *We observe that if $(\cdot,\cdot)$ is an inner product, then (2.12) reduces to Buzano's inequality obtained in 1974* [2] *in a different way.*

3. Vector Inequalities for n-Tuple of Operators

Let $\mathbf{T} = (T_1, ..., T_n) \in \mathcal{B}(H) \times ... \times \mathcal{B}(H) := \mathcal{B}^{(n)}(H)$ be an n-tuple of bounded linear operators on the Hilbert space $(H; \langle\cdot,\cdot\rangle)$ and $\mathbf{p} = (p_1, ..., p_n) \in \mathbb{R}_+^{*n}$ an n-tuple of nonnegative weights not all of them equal to zero. For an $x \in H$, $x \neq 0$ we define

$$\langle \mathbf{T}, \mathbf{V}\rangle_{\mathbf{p},x} := \sum_{j=1}^{n} p_j \langle T_j x, V_j x\rangle = \left\langle \left(\sum_{j=1}^{n} p_j V_j^* T_j\right) x, x\right\rangle \tag{3.1}$$

where $\mathbf{T} = (T_1, ..., T_n)\,, \mathbf{V} = (V_1, ..., V_n) \in \mathcal{B}^{(n)}(H)\,.$

We need the following result:

Lemma 2. *For any $x \in H$, $x \neq 0$ and $\mathbf{p} = (p_1, ..., p_n) \in \mathbb{R}_+^{*n}$ we have that $\langle\cdot,\cdot\rangle_{\mathbf{p},x}$ is a nonnegative Hermitian form on $\mathcal{B}^{(n)}(H)$.*

Proof. We have that

$$\langle \mathbf{T}, \mathbf{T}\rangle_{\mathbf{p},x} = \left\langle \left(\sum_{j=1}^{n} p_j T_j^* T_j\right) x, x\right\rangle = \left\langle \left(\sum_{j=1}^{n} p_j |T_j|^2\right) x, x\right\rangle \geq 0, \tag{3.2}$$

for any $\mathbf{T} = (T_1, ..., T_n) \in \mathcal{B}^{(n)}(H)\,,$ where the *operator modulus* is defined by $|A|^2 = A^*A$, $A \in \mathcal{B}(H)\,.$

The functional $\langle\cdot,\cdot\rangle_{p,x}$ is linear in the first variable and

$$\begin{aligned}\overline{\langle \mathbf{V}, \mathbf{T}\rangle}_{\mathbf{p},x} &= \overline{\left\langle \left(\sum_{j=1}^{n} p_j T_j^* V_j\right) x, x\right\rangle} = \left\langle x, \left(\sum_{j=1}^{n} p_j T_j^* V_j\right) x\right\rangle \\ &= \left\langle \left(\sum_{j=1}^{n} p_j T_j^* V_j\right)^* x, x\right\rangle = \left\langle \left(\sum_{j=1}^{n} p_j V_j^* T_j\right) x, x\right\rangle = \langle \mathbf{T}, \mathbf{V}\rangle_{\mathbf{p},x}\end{aligned} \tag{3.3}$$

for any $\mathbf{T} = (T_1, ..., T_n)\,, \mathbf{V} = (V_1, ..., V_n) \in \mathcal{B}^{(n)}(H)\,.$ □

We have the following result for n-tuples of operators:

Theorem 3. *For* $\mathbf{T}=(T_1,...,T_n)$, $\mathbf{U}=(U_1,...,U_n)$, $\mathbf{V}=(V_1,...,V_n)\in\mathcal{B}^{(n)}(H)\setminus\{\mathbf{0}\}$, $\mathbf{p}=(p_1,...,p_n)\in\mathbb{R}_+^{*n}$ *and* $x\in H$, *we have*

$$\begin{aligned}&\left[\langle\mathbf{T},\mathbf{T}\rangle_{\mathbf{p},x}\langle\mathbf{U},\mathbf{U}\rangle_{\mathbf{p},x}-\left|\langle\mathbf{T},\mathbf{U}\rangle_{\mathbf{p},x}\right|^2\right]\\&\times\left[\langle\mathbf{U},\mathbf{U}\rangle_{\mathbf{p},x}\langle\mathbf{V},\mathbf{V}\rangle_{\mathbf{p},x}-\left|\langle\mathbf{U},\mathbf{V}\rangle_{\mathbf{p},x}\right|^2\right]\\&\geq\left|\langle\mathbf{T},\mathbf{V}\rangle_{\mathbf{p},x}\langle\mathbf{U},\mathbf{U}\rangle_{\mathbf{p},x}-\langle\mathbf{T},\mathbf{U}\rangle_{\mathbf{p},x}\langle\mathbf{U},\mathbf{V}\rangle_{\mathbf{p},x}\right|^2,\end{aligned}\tag{3.4}$$

$$\begin{aligned}&\left[\langle\mathbf{T}+\mathbf{V},\mathbf{T}+\mathbf{V}\rangle_{\mathbf{p},x}\langle\mathbf{U},\mathbf{U}\rangle_{\mathbf{p},x}-\left|\langle\mathbf{T}+\mathbf{V},\mathbf{U}\rangle_{\mathbf{p},x}\right|^2\right]^{1/2}\\&\leq\left[\langle\mathbf{T},\mathbf{T}\rangle_{\mathbf{p},x}\langle\mathbf{U},\mathbf{U}\rangle_{\mathbf{p},x}-\left|\langle\mathbf{T},\mathbf{U}\rangle_{\mathbf{p},x}\right|^2\right]^{1/2}\\&+\left[\langle\mathbf{V},\mathbf{V}\rangle_{\mathbf{p},x}\langle\mathbf{U},\mathbf{U}\rangle_{\mathbf{p},x}-\left|\langle\mathbf{V},\mathbf{U}\rangle_{\mathbf{p},x}\right|^2\right]^{1/2}\end{aligned}\tag{3.5}$$

$$\begin{aligned}&\langle\mathbf{T},\mathbf{T}\rangle_{\mathbf{p},x}^{1/2}\langle\mathbf{V},\mathbf{V}\rangle_{\mathbf{p},x}^{1/2}\langle\mathbf{U},\mathbf{U}\rangle_{\mathbf{p},x}\\&\geq\left|\langle\mathbf{T},\mathbf{V}\rangle_{\mathbf{p},x}\langle\mathbf{U},\mathbf{U}\rangle_{\mathbf{p},x}-\langle\mathbf{T},\mathbf{U}\rangle_{\mathbf{p},x}\langle\mathbf{U},\mathbf{V}\rangle_{\mathbf{p},x}\right|+\left|\langle\mathbf{T},\mathbf{U}\rangle_{\mathbf{p},x}\langle\mathbf{U},\mathbf{V}\rangle_{\mathbf{p},x}\right|\\&\geq\left|\langle\mathbf{T},\mathbf{V}\rangle_{\mathbf{p},x}\right|\langle\mathbf{U},\mathbf{U}\rangle_{\mathbf{p},x}\end{aligned}\tag{3.6}$$

and

$$\frac{1}{2}\left[\langle\mathbf{T},\mathbf{T}\rangle_{\mathbf{p},x}^{1/2}\langle\mathbf{V},\mathbf{V}\rangle_{\mathbf{p},x}^{1/2}+\left|\langle\mathbf{T},\mathbf{V}\rangle_{\mathbf{p},x}\right|\right]\langle\mathbf{U},\mathbf{U}\rangle_{\mathbf{p},x}\geq\left|\langle\mathbf{T},\mathbf{U}\rangle_{\mathbf{p},x}\langle\mathbf{U},\mathbf{V}\rangle_{\mathbf{p},x}\right|.\tag{3.7}$$

The proof follows from the corresponding inequalities above, namely (2.2), (2.3), (2.7) and (2.12) applied for the nonnegative Hermitian form $\langle\cdot,\cdot\rangle_{\mathbf{p},x}$, $x\in H$, $x\neq 0$. The details are omitted.

Remark 4. *The inequality (3.7) can be written as*

$$\begin{aligned}&\frac{1}{2}\left[\left\langle\left(\sum_{j=1}^n p_j\left|T_j\right|^2\right)x,x\right\rangle^{1/2}\left\langle\left(\sum_{j=1}^n p_j\left|V_j\right|^2\right)x,x\right\rangle^{1/2}\right.\\&\left.+\left|\left\langle\left(\sum_{j=1}^n p_jV_j^*T_j\right)x,x\right\rangle\right|\right]\left\langle\left(\sum_{j=1}^n p_j\left|U_j\right|^2\right)x,x\right\rangle\\&\geq\left|\left\langle\left(\sum_{j=1}^n p_jU_j^*T_j\right)x,x\right\rangle\left\langle\left(\sum_{j=1}^n p_jV_j^*U_j\right)x,x\right\rangle\right|,\end{aligned}\tag{3.8}$$

that holds for $(T_1,...,T_n)$, $(U_1,...,U_n)$, $(V_1,...,V_n)\in\mathcal{B}^{(n)}(H)\setminus\{\mathbf{0}\}$, $\mathbf{p}=(p_1,...,p_n)\in\mathbb{R}_+^{*n}$ *and* $x\in H$.

If we take $V_j = T_j^*$ for $j \in \{1, ..., n\}$ in (3.8), then we have

$$\frac{1}{2}\left[\left\langle\left(\sum_{j=1}^n p_j |T_j|^2\right)x, x\right\rangle^{1/2}\left\langle\left(\sum_{j=1}^n p_j \left|T_j^*\right|^2\right)x, x\right\rangle^{1/2} + \left|\left\langle\left(\sum_{j=1}^n p_j T_j^2\right)x, x\right\rangle\right|\right]\left\langle\left(\sum_{j=1}^n p_j |U_j|^2\right)x, x\right\rangle \geq \left|\left\langle\left(\sum_{j=1}^n p_j U_j^* T_j\right)x, x\right\rangle\left\langle\left(\sum_{j=1}^n p_j T_j U_j\right)x, x\right\rangle\right|, \tag{3.9}$$

and since

$$\left\langle\left(\sum_{j=1}^n p_j T_j U_j\right)x, x\right\rangle = \left\langle x, \left(\sum_{j=1}^n p_j T_j U_j\right)^* x\right\rangle = \left\langle x, \left(\sum_{j=1}^n p_j U_j^* T_j^*\right) x\right\rangle = \overline{\left\langle\left(\sum_{j=1}^n p_j U_j^* T_j^*\right)x, x\right\rangle},$$

then the inequality (3.9) can also be written as

$$\frac{1}{2}\left[\left\langle\left(\sum_{j=1}^n p_j |T_j|^2\right)x, x\right\rangle^{1/2}\left\langle\left(\sum_{j=1}^n p_j \left|T_j^*\right|^2\right)x, x\right\rangle^{1/2} + \left|\left\langle\left(\sum_{j=1}^n p_j T_j^2\right)x, x\right\rangle\right|\right]\left\langle\left(\sum_{j=1}^n p_j |U_j|^2\right)x, x\right\rangle \geq \left|\left\langle\left(\sum_{j=1}^n p_j U_j^* T_j\right)x, x\right\rangle\right|\left|\left\langle\left(\sum_{j=1}^n p_j U_j^* T_j^*\right)x, x\right\rangle\right|. \tag{3.10}$$

If T_j are normal operators for any $j \in \{1, ..., n\}$, then we get from (3.10) that

$$\frac{1}{2}\left[\left\langle\left(\sum_{j=1}^n p_j |T_j|^2\right)x, x\right\rangle + \left|\left\langle\left(\sum_{j=1}^n p_j T_j^2\right)x, x\right\rangle\right|\right] \times \left\langle\left(\sum_{j=1}^n p_j |U_j|^2\right)x, x\right\rangle \geq \left|\left\langle\left(\sum_{j=1}^n p_j U_j^* T_j\right)x, x\right\rangle\right|\left|\left\langle\left(\sum_{j=1}^n p_j U_j^* T_j^*\right)x, x\right\rangle\right|, \tag{3.11}$$

for any $(U_1, ..., U_n) \in \mathcal{B}^{(n)}(H) \setminus \{\mathbf{0}\}$ and $x \in H$.

If U_j are selfadjoint operators for any $j \in \{1, ..., n\}$, then we get from (3.9) that

$$\frac{1}{2}\left[\left\langle\left(\sum_{j=1}^{n} p_j |T_j|^2\right)x, x\right\rangle^{1/2}\left\langle\left(\sum_{j=1}^{n} p_j |T_j^*|^2\right)x, x\right\rangle^{1/2} + \left|\left\langle\left(\sum_{j=1}^{n} p_j T_j^2\right)x, x\right\rangle\right|\right]\left\langle\left(\sum_{j=1}^{n} p_j U_j^2\right)x, x\right\rangle \geq \left|\left\langle\left(\sum_{j=1}^{n} p_j U_j T_j\right)x, x\right\rangle\left\langle\left(\sum_{j=1}^{n} p_j T_j U_j\right)x, x\right\rangle\right|, \tag{3.12}$$

for any $(T_1, ..., T_n) \in \mathcal{B}^{(n)}(H) \setminus \{\mathbf{0}\}$ and $x \in H$.

Moreover, if $U_j T_j = T_j U_j$ for any $j \in \{1, ..., n\}$, then we get from (3.12) the inequality

$$\frac{1}{2}\left[\left\langle\left(\sum_{j=1}^{n} p_j |T_j|^2\right)x, x\right\rangle^{1/2}\left\langle\left(\sum_{j=1}^{n} p_j |T_j^*|^2\right)x, x\right\rangle^{1/2} + \left|\left\langle\left(\sum_{j=1}^{n} p_j T_j^2\right)x, x\right\rangle\right|\right]\left\langle\left(\sum_{j=1}^{n} p_j U_j^2\right)x, x\right\rangle \geq \left|\left\langle\left(\sum_{j=1}^{n} p_j U_j T_j\right)x, x\right\rangle\right|^2, \tag{3.13}$$

for any $(T_1, ..., T_n) \in \mathcal{B}^{(n)}(H) \setminus \{\mathbf{0}\}$, $(U_1, ..., U_n)$ selfadjoint operators and $x \in H$.

In particular, if $(T_1, ..., T_n)$ are normal operators and $(U_1, ..., U_n)$ are selfadjoint operators such that $U_j T_j = T_j U_j$ for any $j \in \{1, ..., n\}$, then we get from (3.13) the simpler inequality

$$\frac{1}{2}\left[\left\langle\left(\sum_{j=1}^{n} p_j |T_j|^2\right)x, x\right\rangle + \left|\left\langle\left(\sum_{j=1}^{n} p_j T_j^2\right)x, x\right\rangle\right|\right]\left\langle\left(\sum_{j=1}^{n} p_j U_j^2\right)x, x\right\rangle \geq \left|\left\langle\left(\sum_{j=1}^{n} p_j U_j T_j\right)x, x\right\rangle\right|^2, \tag{3.14}$$

for any $x \in H$.

Remark 5. *We notice that (3.14) is an operator version of de Bruijn inequality obtained in 1960 in* [1], *which provides the following refinement of the Cauchy-Bunyakovsky-Schwarz inequality:*

$$\left|\sum_{i=1}^{n} a_i z_i\right|^2 \leq \frac{1}{2}\sum_{i=1}^{n} a_i^2\left[\sum_{i=1}^{n} |z_i|^2 + \left|\sum_{i=1}^{n} z_i^2\right|\right], \tag{3.15}$$

provided that a_i are real numbers while z_i are complex for each $i \in \{1, ..., n\}$.

For some inequalities in inner product spaces and operators on Hilbert spaces see [4]-[26] and the references therein.

4. Applications for Functions of Normal Operators

Some important examples of power series with nonnegative coefficients are

$$\frac{1}{1-\lambda}=\sum_{n=0}^{\infty}\lambda^{n},\ \ \lambda\in D(0,1); \tag{4.1}$$
$$\ln\frac{1}{1-\lambda}=\sum_{n=1}^{\infty}\frac{1}{n}\lambda^{n},\ \ \lambda\in D(0,1);$$
$$\exp(\lambda)=\sum_{n=0}^{\infty}\frac{1}{n!}\lambda^{n},\ \ \lambda\in\mathbb{C};$$
$$\sinh\lambda=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\lambda^{2n+1},\ \ \lambda\in\mathbb{C};$$
$$\cosh\lambda=\sum_{n=0}^{\infty}\frac{1}{(2n)!}\lambda^{2n},\ \ \lambda\in\mathbb{C}.$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1},\ \ \lambda\in D(0,1); \tag{4.2}$$
$$\sin^{-1}(\lambda)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,(2n+1)\,n!}\lambda^{2n+1},\ \ \lambda\in D(0,1);$$
$$\tanh^{-1}(\lambda)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1},\ \ \lambda\in D(0,1);$$
$${}_2F_1(\alpha,\beta,\gamma,\lambda)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\,\Gamma(n+\beta)\,\Gamma(\gamma)}{n!\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(n+\gamma)}\lambda^{n},\alpha,\beta,\gamma>0,$$
$$\lambda\in D(0,1)$$

where Γ is *Gamma function*.

We have the following result:

Theorem 4. *Let $f(z):=\sum_{j=0}^{\infty}p_{j}z^{j}$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$, $R>0$. If T, U and V are normal operators and $V^{*}T=TV^{*}$, $U^{*}T=TU^{*}$, $V^{*}U=UV^{*}$ with $\|T\|^{2},\|U\|^{2},\|V\|^{2}<R$, then we have the inequalities*

$$\begin{aligned}&\frac{1}{2}\left[\left\langle f\left(|T|^{2}\right)x,x\right\rangle^{1/2}\left\langle f\left(|V|^{2}\right)x,x\right\rangle^{1/2}+|\langle f(V^{*}T)x,x\rangle|\right]\\&\times\left\langle f\left(|U|^{2}\right)x,x\right\rangle\\&\geq|\langle f(U^{*}T)x,x\rangle\langle f(V^{*}U)x,x\rangle|\end{aligned} \tag{4.3}$$

for any $x\in H$.

Proof. If we use the inequality (3.8) for powers of operators we have

$$\begin{aligned} (4.4) \quad & \frac{1}{2}\left[\left\langle\left(\sum_{j=0}^{m} p_j \left|T^j\right|^2\right) x, x\right\rangle^{1/2} \left\langle\left(\sum_{j=0}^{m} p_j \left|V^j\right|^2\right) x, x\right\rangle^{1/2} \right. \\ & \left. + \left|\left\langle\left(\sum_{j=0}^{m} p_j \left(V^j\right)^* T^j\right) x, x\right\rangle\right|\right] \left\langle\left(\sum_{j=0}^{m} p_j \left|U^j\right|^2\right) x, x\right\rangle \\ & \geq \left|\left\langle\left(\sum_{j=0}^{m} p_j \left(U^j\right)^* T^j\right) x, x\right\rangle \left\langle\left(\sum_{j=0}^{m} p_j \left(V^j\right)^* U^j\right) x, x\right\rangle\right|, \end{aligned}$$

for any $m \geq 1$ and $x \in H$.

Since T, U and V are normal operators and $V^*T = TV^*$, $U^*T = TU^*$, $V^*U = UV^*$, then from (4.4) we have

$$\begin{aligned} (4.5) \quad & \frac{1}{2}\left[\left\langle\left(\sum_{j=0}^{m} p_j \left|T\right|^{2j}\right) x, x\right\rangle^{1/2} \left\langle\left(\sum_{j=0}^{m} p_j \left|V\right|^{2j}\right) x, x\right\rangle^{1/2} \right. \\ & \left. + \left|\left\langle\left(\sum_{j=0}^{m} p_j \left(V^*T\right)^j\right) x, x\right\rangle\right|\right] \left\langle\left(\sum_{j=0}^{m} p_j \left|U\right|^{2j}\right) x, x\right\rangle \\ & \geq \left|\left\langle\left(\sum_{j=0}^{m} p_j \left(U^*T\right)^j\right) x, x\right\rangle \left\langle\left(\sum_{j=0}^{m} p_j \left(V^*U\right)^j\right) x, x\right\rangle\right|, \end{aligned}$$

for any $m \geq 1$ and $x \in H$.

Since all the series whose partial sums are involved in the inequality (4.5) are convergent then by letting $m \to \infty$ in (4.5) we get (4.3). □

Corollary 2. *Let $f(z) := \sum_{j=0}^{\infty} p_j z^j$ be a power series with nonnegative coefficients and convergent on the open disk $D(0,R)$, $R > 0$. If T and U are normal operators and $U^*T = TU^*$, $TU = UT$ with $\|T\|^2, \|U\|^2 < R$, then we have the inequalities*

$$\begin{aligned} (4.6) \quad & \frac{1}{2}\left[\left\langle f\left(|T|^2\right) x, x\right\rangle + \left|\left\langle f\left(T^2\right) x, x\right\rangle\right|\right] \left\langle f\left(|U|^2\right) x, x\right\rangle \\ & \geq \left|\left\langle f\left(U^*T\right) x, x\right\rangle \left\langle f\left(TU\right) x, x\right\rangle\right| \end{aligned}$$

for any $x \in H$, $x \neq 0$.

In particular, if T is normal and U is selfadjoint with $TU = UT$ and $\|T\|^2, \|U\|^2 < R$, then

$$(4.7) \quad \frac{1}{2}\left[\left\langle f\left(|T|^2\right) x, x\right\rangle + \left|\left\langle f\left(T^2\right) x, x\right\rangle\right|\right] \left\langle f\left(U^2\right) x, x\right\rangle \geq \left|\left\langle f\left(TU\right) x, x\right\rangle\right|^2$$

for any $x \in H$.

In order to provide various examples of interesting inequalities we use (4.7) for some fundamental functions.

If T is normal and U is selfadjoint with $TU = UT$ with $\|T\|, \|U\| < 1$, then

$$\frac{1}{2}\left[\left\langle\left(1_H - |T|^2\right)^{-1} x, x\right\rangle + \left|\left\langle\left(1_H - T^2\right)^{-1} x, x\right\rangle\right|\right]\left\langle\left(1_H - U^2\right)^{-1} x, x\right\rangle \geq \left|\left\langle\left(1_H - TU\right)^{-1} x, x\right\rangle\right|^2 \tag{4.8}$$

and

$$\frac{1}{2}\left[\ln\left\langle\left(1_H - |T|^2\right)^{-1} x, x\right\rangle + \left|\left\langle\ln\left(1_H - T^2\right)^{-1} x, x\right\rangle\right|\right] \times \left\langle\ln\left(1_H - U^2\right)^{-1} x, x\right\rangle \geq \left|\left\langle\ln\left(1_H - TU\right)^{-1} x, x\right\rangle\right|^2 \tag{4.9}$$

for any $x \in H$.

If T is normal and U is selfadjoint with $TU = UT$, then

$$\frac{1}{2}\left[\left\langle\exp\left(|T|^2\right) x, x\right\rangle + \left|\left\langle\exp\left(T^2\right) x, x\right\rangle\right|\right]\left\langle\exp\left(U^2\right) x, x\right\rangle \geq \left|\left\langle\exp\left(TU\right) x, x\right\rangle\right|^2, \tag{4.10}$$

$$\frac{1}{2}\left[\left\langle\sinh\left(|T|^2\right) x, x\right\rangle + \left|\left\langle\sinh\left(T^2\right) x, x\right\rangle\right|\right]\left\langle\sinh\left(U^2\right) x, x\right\rangle \geq \left|\left\langle\sinh\left(TU\right) x, x\right\rangle\right|^2, \tag{4.11}$$

and

$$\frac{1}{2}\left[\left\langle\cosh\left(|T|^2\right) x, x\right\rangle + \left|\left\langle\cosh\left(T^2\right) x, x\right\rangle\right|\right]\left\langle\cosh\left(U^2\right) x, x\right\rangle \geq \left|\left\langle\cosh\left(TU\right) x, x\right\rangle\right|^2 \tag{4.12}$$

for any $x \in H$.

5. Norm and Numerical Radius Inequalities

The *numerical radius* $w(T)$ of an operator T on H is given by [27, p. 8]:

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{5.1}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $\mathcal{B}(H)$ of all bounded linear operators $T : H \to H$. This norm is equivalent with the operator norm. In fact, the following more precise result holds [27, p. 9]:

Theorem 5 (Equivalent norm)**.** *For any $T \in \mathcal{B}(H)$ one has*

$$w(T) \leq \|T\| \leq 2w(T). \tag{5.2}$$

We recall also that if T is normal operator, then $w(T) = \|T\|$.

For a survey of recent inequalities for numerical radius, see [21] and the references therein.

Theorem 6. *Let* $(T_1,...,T_n) \in \mathcal{B}^{(n)}(H) \backslash \{\mathbf{0}\}$, $\mathbf{p} = (p_1,...,p_n) \in \mathbb{R}_+^{*n}$ *and* $(U_1,...,U_n)$ *be selfadjoint operators such that* $U_jT_j = T_jU_j$ *for any* $j \in \{1,...,n\}$. *Then we have the inequality*

$$w^2\left(\sum_{j=1}^n p_jU_jT_j\right) \le \frac{1}{2}\left\|\sum_{j=1}^n p_jU_j^2\right\| \times\left[\left\|\sum_{j=1}^n p_j\left|T_j\right|^2\right\|^{1/2}\left\|\sum_{j=1}^n p_j\left|T_j^*\right|^2\right\|^{1/2}+\left\|\sum_{j=1}^n p_jT_j^2\right\|\right]. \tag{5.3}$$

Proof. Taking the supremum over $\|x\| = 1$ in the inequality (3.13) and using its properties we have

$$\begin{aligned}\sup_{\|x\|=1}&\left|\left\langle\left(\sum_{j=1}^n p_jU_jT_j\right)x,x\right\rangle\right|^2\\ &\le\frac{1}{2}\sup_{\|x\|=1}\left\{\left[\left\langle\left(\sum_{j=1}^n p_j\left|T_j\right|^2\right)x,x\right\rangle^{1/2}\left\langle\left(\sum_{j=1}^n p_j\left|T_j^*\right|^2\right)x,x\right\rangle^{1/2}\right.\right.\\ &\quad\left.\left.+\left|\left\langle\left(\sum_{j=1}^n p_jT_j^2\right)x,x\right\rangle\right|\right]\left\langle\left(\sum_{j=1}^n p_jU_j^2\right)x,x\right\rangle\right\}\\ &\le\sup_{\|x\|=1}\left[\left\langle\left(\sum_{j=1}^n p_j\left|T_j\right|^2\right)x,x\right\rangle^{1/2}\left\langle\left(\sum_{j=1}^n p_j\left|T_j^*\right|^2\right)x,x\right\rangle^{1/2}\right.\\ &\quad\left.+\left|\left\langle\left(\sum_{j=1}^n p_jT_j^2\right)x,x\right\rangle\right|\right]\sup_{\|x\|=1}\left\langle\left(\sum_{j=1}^n p_jU_j^2\right)x,x\right\rangle.\end{aligned}\tag{5.4}$$

However, we have

$$\begin{aligned}\sup_{\|x\|=1}&\left[\left\langle\left(\sum_{j=1}^n p_j\left|T_j\right|^2\right)x,x\right\rangle^{1/2}\left\langle\left(\sum_{j=1}^n p_j\left|T_j^*\right|^2\right)x,x\right\rangle^{1/2}\right.\\ &\quad\left.+\left|\left\langle\left(\sum_{j=1}^n p_jT_j^2\right)x,x\right\rangle\right|\right]\\ &\le\sup_{\|x\|=1}\left[\left\langle\left(\sum_{j=1}^n p_j\left|T_j\right|^2\right)x,x\right\rangle^{1/2}\left\langle\left(\sum_{j=1}^n p_j\left|T_j^*\right|^2\right)x,x\right\rangle^{1/2}\right]\end{aligned}\tag{5.5}$$

$$\leq \sup_{\|x\|=1}\left[\left\langle\left(\sum_{j=1}^{n}p_j\left|T_j\right|^2\right)x,x\right\rangle^{1/2}\left\langle\left(\sum_{j=1}^{n}p_j\left|T_j^*\right|^2\right)x,x\right\rangle^{1/2}\right]$$
$$+\sup_{\|x\|=1}\left|\left\langle\left(\sum_{j=1}^{n}p_jT_j^2\right)x,x\right\rangle\right|$$
$$\leq \sup_{\|x\|=1}\left\langle\left(\sum_{j=1}^{n}p_j\left|T_j\right|^2\right)x,x\right\rangle^{1/2}\sup_{\|x\|=1}\left\langle\left(\sum_{j=1}^{n}p_j\left|T_j^*\right|^2\right)x,x\right\rangle^{1/2}$$
$$+\sup_{\|x\|=1}\left|\left\langle\left(\sum_{j=1}^{n}p_jT_j^2\right)x,x\right\rangle\right|.$$

Since

$$\sup_{\|x\|=1}\left\langle\left(\sum_{j=1}^{n}p_jU_j^2\right)x,x\right\rangle=\left\|\sum_{j=1}^{n}p_jU_j^2\right\|,$$
$$\sup_{\|x\|=1}\left\langle\left(\sum_{j=1}^{n}p_j\left|T_j\right|^2\right)x,x\right\rangle^{1/2}=\left\|\sum_{j=1}^{n}p_j\left|T_j\right|^2\right\|^{1/2},$$
$$\sup_{\|x\|=1}\left\langle\left(\sum_{j=1}^{n}p_j\left|T_j^*\right|^2\right)x,x\right\rangle^{1/2}=\left\|\sum_{j=1}^{n}p_j\left|T_j^*\right|^2\right\|^{1/2}$$

and

$$\sup_{\|x\|=1}\left|\left\langle\left(\sum_{j=1}^{n}p_jT_j^2\right)x,x\right\rangle\right|=\left\|\sum_{j=1}^{n}p_jT_j^2\right\|,$$

then by (5.4) and (5.5) we get the desired result (5.3). □

Remark 6. *If we take $U_j=a_j1_H$ with $j\in\{1,...,n\}$ where $a_j\in\mathbb{R}$, $j\in\{1,...,n\}$, then we get from (5.3)*

$$w^2\left(\sum_{j=1}^{n}p_ja_jT_j\right)\leq\frac{1}{2}\left\|\sum_{j=1}^{n}p_ja_j^2\right\|\times\left[\left\|\sum_{j=1}^{n}p_j\left|T_j\right|^2\right\|^{1/2}\left\|\sum_{j=1}^{n}p_j\left|T_j^*\right|^2\right\|^{1/2}+\left\|\sum_{j=1}^{n}p_jT_j^2\right\|\right]\tag{5.6}$$

for any $(T_1,...,T_n)\in\mathcal{B}^{(n)}(H)\setminus\{\mathbf{0}\}$ and $\mathbf{p}=(p_1,...,p_n)\in\mathbb{R}_+^{*n}$ and, in particular,

$$w^2\left(\sum_{j=1}^{n}a_jT_j\right)\leq\frac{1}{2}\left\|\sum_{j=1}^{n}a_j^2\right\|\times\left[\left\|\sum_{j=1}^{n}\left|T_j\right|^2\right\|^{1/2}\left\|\sum_{j=1}^{n}\left|T_j^*\right|^2\right\|^{1/2}+\left\|\sum_{j=1}^{n}T_j^2\right\|\right].\tag{5.7}$$

Moreover, if T_j are normal operators for any $j \in \{1, ..., n\}$, then we have

$$w^2\left(\sum_{j=1}^n a_j T_j\right) \le \frac{1}{2}\left\|\sum_{j=1}^n a_j^2\right\|\left[\left\|\sum_{j=1}^n |T_j|^2\right\| + \left\|\sum_{j=1}^n T_j^2\right\|\right]. \tag{5.8}$$

6. The Case for One and Two Operators

If we write the inequality (3.8) for $p_j = 1$, $j \in \{1, ..., n\}$, then we get

$$\begin{aligned} &\frac{1}{2}\left[\left\langle\left(\sum_{j=1}^n |T_j|^2\right)x, x\right\rangle^{1/2}\left\langle\left(\sum_{j=1}^n |V_j|^2\right)x, x\right\rangle^{1/2}\right. \\ &\left.+\left|\left\langle\left(\sum_{j=1}^n V_j^* T_j\right)x, x\right\rangle\right|\right]\left\langle\left(\sum_{j=1}^n |U_j|^2\right)x, x\right\rangle \\ &\ge \left|\left\langle\left(\sum_{j=1}^n U_j^* T_j\right)x, x\right\rangle\left\langle\left(\sum_{j=1}^n V_j^* U_j\right)x, x\right\rangle\right|, \end{aligned} \tag{6.1}$$

that holds for $(T_1, ..., T_n), (U_1, ..., U_n), (V_1, ..., V_n) \in \mathcal{B}^{(n)}(H) \setminus \{\mathbf{0}\}$ and $x \in H$.

If we write this inequality for $n = 1$ we get

$$\begin{aligned} &\frac{1}{2}\left[\left\langle|T|^2 x, x\right\rangle^{1/2}\left\langle|V|^2 x, x\right\rangle^{1/2} + |\langle V^* T x, x\rangle|\right]\left\langle|U|^2 x, x\right\rangle \\ &\ge |\langle(U^* T) x, x\rangle\langle(V^* U) x, x\rangle|, \end{aligned} \tag{6.2}$$

that holds for any $T, U, V \in \mathcal{B}(H)$ and $x \in H$.

If we take $V = T^*$ in (6.2), then we get

$$\begin{aligned} &\frac{1}{2}\left[\left\langle|T|^2 x, x\right\rangle^{1/2}\left\langle|T^*|^2 x, x\right\rangle^{1/2} + \left|\left\langle T^2 x, x\right\rangle\right|\right]\left\langle|U|^2 x, x\right\rangle \\ &\ge |\langle(U^* T) x, x\rangle\langle(TU) x, x\rangle|, \end{aligned} \tag{6.3}$$

that holds for any $T, U \in \mathcal{B}(H)$ and $x \in H$.

In particular, if T is normal, then from (6.3) we have

$$\frac{1}{2}\left[\left\langle|T|^2 x, x\right\rangle + \left|\left\langle T^2 x, x\right\rangle\right|\right]\left\langle|U|^2 x, x\right\rangle \ge |\langle(U^* T) x, x\rangle\langle(TU) x, x\rangle|, \tag{6.4}$$

for any $U \in \mathcal{B}(H)$ and $x \in H$.

Also, if U is selfadjoint, then from (6.3) we have

$$\begin{aligned} &\frac{1}{2}\left[\left\langle|T|^2 x, x\right\rangle^{1/2}\left\langle|T^*|^2 x, x\right\rangle^{1/2} + \left|\left\langle T^2 x, x\right\rangle\right|\right]\left\langle U^2 x, x\right\rangle \\ &\ge |\langle(UT) x, x\rangle\langle(TU) x, x\rangle|, \end{aligned} \tag{6.5}$$

for any $T \in \mathcal{B}(H)$ and $x \in H$.

Moreover, if U is selfadjoint and commuting with $T \in \mathcal{B}(H)$, then we have from (6.5)

$$\frac{1}{2}\left[\left\langle|T|^2 x, x\right\rangle^{1/2}\left\langle|T^*|^2 x, x\right\rangle^{1/2} + \left|\left\langle T^2 x, x\right\rangle\right|\right]\left\langle U^2 x, x\right\rangle \ge |\langle(TU) x, x\rangle|^2, \tag{6.6}$$

for any $x \in H$.

If we take the supremum over $\|x\|=1$ in (6.6), then we get

$$\begin{aligned} w^2(TU) &= \sup_{\|x\|=1} |\langle (TU)x,x\rangle|^2 \\ &\le \frac{1}{2}\sup_{\|x\|=1}\left\{\left[\left\langle |T|^2 x,x\right\rangle^{1/2}\left\langle |T^*|^2 x,x\right\rangle^{1/2} + \left|\left\langle T^2x,x\right\rangle\right|\right]\left\langle U^2x,x\right\rangle\right\} \\ &\le \frac{1}{2}\left[\sup_{\|x\|=1}\left\langle |T|^2 x,x\right\rangle^{1/2}\sup_{\|x\|=1}\left\langle |T^*|^2 x,x\right\rangle^{1/2} + \sup_{\|x\|=1}\left|\left\langle T^2x,x\right\rangle\right|\right] \\ &\times \sup_{\|x\|=1}\left\langle U^2x,x\right\rangle \\ &= \frac{1}{2}\left[\|T\|^2 + w\left(T^2\right)\right]\|U\|^2 \end{aligned}$$

since

$$\sup_{\|x\|=1}\left\langle |T|^2 x,x\right\rangle^{1/2} = \left[w\left(|T|^2\right)\right]^{1/2} = \left\||T|^2\right\|^{1/2} = \|T\|,$$

$$\sup_{\|x\|=1}\left\langle |T^*|^2 x,x\right\rangle^{1/2} = \left[w\left(|T^*|^2\right)\right]^{1/2} = \left\||T^*|^2\right\|^{1/2} = \|T^*\| = \|T\|$$

and

$$\sup_{\|x\|=1}\left\langle U^2x,x\right\rangle = \left\|U^2\right\| = \|U\|^2.$$

Therefore we get

$$w^2(TU) \le \frac{1}{2}\left[\|T\|^2 + w\left(T^2\right)\right]\|U\|^2, \tag{6.7}$$

for any $T \in \mathcal{B}(H)$ and U a selfadjoint operator that commutes with T.

If we take $U = I$ in (6.7), then we get the sharp inequality

$$w^2(T) \le \frac{1}{2}\left[\|T\|^2 + w\left(T^2\right)\right] \tag{6.8}$$

that has been firstly obtained in 2007 in [13].

If we write the inequality (6.1) for $n = 2$ we get

$$\begin{aligned} &\frac{1}{2}\left[\left\langle\left(|T_1|^2 + |T_2|^2\right)x,x\right\rangle^{1/2}\left\langle\left(|V_1|^2 + |V_2|^2\right)x,x\right\rangle^{1/2}\right. \\ &\left. + \ |\langle (V_1^*T_1 + V_2^*T_2)x,x\rangle|\right]\left\langle\left(|U_1|^2 + |U_2|^2\right)x,x\right\rangle \\ &\ge |\langle (U_1^*T_1 + U_2^*T_2)x,x\rangle\langle (V_1^*U_1 + V_2^*U_2)x,x\rangle|, \end{aligned} \tag{6.9}$$

for any $(T_1,T_2), (U_1,U_2), (V_1,V_2) \in \mathcal{B}^{(2)}(H)$ and $x \in H$.

If we take $T = (A,B)$ and $V = (B^*, \pm A^*)$ in (6.9), where $A, B \in \mathcal{B}(H)$, then we have

$$\begin{aligned} &\frac{1}{2}\left[\left\langle\left(|A|^2 + |B|^2\right)x,x\right\rangle^{1/2}\left\langle\left(|B^*|^2 + |A^*|^2\right)x,x\right\rangle^{1/2}\right. \\ &\left. + |\langle (BA \pm AB)x,x\rangle|\right]\left\langle\left(|U_1|^2 + |U_2|^2\right)x,x\right\rangle \\ &\ge |\langle (U_1^*A + U_2^*B)x,x\rangle\langle (BU_1 \pm AU_2)x,x\rangle|, \end{aligned} \tag{6.10}$$

for any $(U_1,U_2) \in \mathcal{B}^{(2)}(H)$ and $x \in H$.

If we take in this inequality $U_1 = B$ and $U_2 = A$, then we get

$$\frac{1}{2}\left[\left\langle\left(|A|^2+|B|^2\right)x,x\right\rangle^{1/2}\left\langle\left(|B^*|^2+|A^*|^2\right)x,x\right\rangle^{1/2} + |\langle(BA\pm AB)\,x,x\rangle|\right]\left\langle\left(|A|^2+|B|^2\right)x,x\right\rangle \geq \left|\langle(B^*A+A^*B)\,x,x\rangle\left\langle\left(B^2\pm A^2\right)x,x\right\rangle\right|, \tag{6.11}$$

for any $A, B \in \mathcal{B}(H)$. Moreover, if we take in this inequality $B = A^*$, then we get

$$\frac{1}{2}\left[\left\langle\left(|A|^2+|A^*|^2\right)x,x\right\rangle + |\langle(A^*A-AA^*)\,x,x\rangle|\right]\left\langle\left(|A|^2+|A^*|^2\right)x,x\right\rangle \geq \left|\left\langle\left(A^2+(A^*)^2\right)x,x\right\rangle\left\langle\left((A^*)^2-A^2\right)x,x\right\rangle\right|, \tag{6.12}$$

for any $A \in \mathcal{B}(H)$.

If we take $V_2 = T_1$ and $V_1 = T_2$ then we get from (6.9) that

$$\frac{1}{2}\left[\left\langle\left(|T_1|^2+|T_2|^2\right)x,x\right\rangle + |\langle(T_2^*T_1+T_1^*T_2)\,x,x\rangle|\right]\left\langle\left(|U_1|^2+|U_2|^2\right)x,x\right\rangle \geq |\langle(U_1^*T_1+U_2^*T_2)\,x,x\rangle\langle(T_2^*U_1+T_1^*U_2)\,x,x\rangle|, \tag{6.13}$$

for any $(T_1, T_2), (U_1, U_2) \in \mathcal{B}^{(2)}(H)$ and $x \in H$.

If we take $V_2 = T_1^*$ and $V_1 = T_2^*$ then we get from (6.9) that

$$\frac{1}{2}\left[\left\langle\left(|T_1|^2+|T_2|^2\right)x,x\right\rangle^{1/2}\left\langle\left(|T_1^*|^2+|T_2^*|^2\right)x,x\right\rangle^{1/2} + \left|\left\langle\left(T_1^2+T_2^2\right)x,x\right\rangle\right|\right]\left\langle\left(|U_1|^2+|U_2|^2\right)x,x\right\rangle \geq |\langle(U_1^*T_1+U_2^*T_2)\,x,x\rangle\langle(T_2U_1+T_1U_2)\,x,x\rangle|, \tag{6.14}$$

for any $(T_1, T_2), (U_1, U_2) \in \mathcal{B}^{(2)}(H)$ and $x \in H$.

One can state other particular inequalities by taking specific values for (T_1, T_2), (U_1, U_2). The details are however omitted.

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