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ČEBYSEV TYPE INEQUALITIES FOR CO-ORDINATED CONVEX FUNCTIONS

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Abstract. In this paper, we astablish new same Čebysev type inequalities for functions whose derivatives are co-ordinated convex in absolute value.

1. Introduction

In [3], P. L. Čebysev proved the following important integral inequality

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty \tag{1.1}$$

where $f,g:[a,b]\to\mathbb{R}$ are absolutely continuous functions whose derivatives $f',g'\in L_\infty[a,b]$ and

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \tag{1.2}$$

which is called the Čebysev functional, provided the integrals in (1.2) exist. In recent years many researchers have given the generalization of Čebysev type inequalities, we can mention the works [1, 2, 4, 5, 6, 7, 8, 9, 11].

Let us consider now a bidemensional interval $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$, amapping $f:\Delta\to\mathbb{R}$ is said to be convexs on Δ if the inequality

$$f(\lambda x+(1-\lambda)z,\lambda y+(1-\lambda)w)\leq \lambda f(x,y)+(1-\lambda)f(z,w),$$

holds for all $(x,y),(z,w)\in\Delta$ and $\lambda\in[0,1]$. The mapping f is said to be concave on the co-ordinates on if the above inequality holds in reversed direction, for all $(x,y),(z,w)\in\Delta$ and $\lambda\in[0,1]$.

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1. *A function $f:\Delta\to\mathbb{R}$ will be co-ordinated convex on Δ, for all $t,s\in[0,1]$ and $(x,y),(u,v)\in\Delta$, if the following inequality holds:*

$$\begin{aligned} & f(tx+(1-t)y,su+(1-s)v) \\ \leq\ & tsf(x,u)+s(1-t)f(y,u)+t(1-s)f(x,v)+(1-t)(1-s)f(y,v). \end{aligned}$$

In [5], the authours proved the following inequalities:

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Theorem 1. *Let $f,g:\Delta\to\mathbb{R}$ be a partial differentiable functions such that their second derivatives $f_{ts}=\frac{\partial^2 f(t,s)}{\partial t\partial s}$ and $g_{ts}=\frac{\partial^2 g(t,s)}{\partial t\partial s}$ are integrable on Δ, then*

$$|T(f,g)|\leq\frac{49}{3600}(b-a)^2(d-c)^2\|f_{ts}\|_\infty\|g_{ts}\|_\infty \tag{1.3}$$

where

$$\begin{aligned}T(f,g) &= \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)\,dydx\\ &-\frac{1}{(b-a)^2(d-c)}\int_a^b\int_c^d g(x,y)\int_a^b f(t,y)dtdydx\\ &-\frac{1}{(b-a)(d-c)^2}\int_a^b\int_c^d g(x,y)\int_c^d f(x,s)dsdydx\\ &+\frac{1}{(b-a)^2(d-c)^2}\left(\int_a^b\int_c^d f(x,s)dsdx\right)\left(\int_a^b\int_c^d g(t,y)dydt\right).\end{aligned} \tag{1.4}$$

In this paper, we shall improve the second inequality in (1.3) for functions whose partial derivatives are co-ordinated convex in absolute value.

2. Main Results

In order to prove our main theorems we need the following lemma (see [10]):

Lemma 1. *Let $f:\Delta\to\mathbb{R}$ be an absolutely continuous function on Δ. Then for any $(x,y)\in\Delta$, we have the equality:*

$$\begin{aligned}f(x,y) &= \frac{1}{d-c}\int_c^d f(x,s)ds+\frac{1}{b-a}\int_a^b f(t,y)dt-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt\\ &+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x-t)(y-s)\\ &\times\left[\int_0^1\int_0^1 f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)d\alpha d\lambda\right]dsdt\end{aligned} \tag{2.1}$$

Theorem 2. *Let $f,g:\Delta\to\mathbb{R}$ be a partial differentiable functions such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on Δ. If $f_{\lambda\alpha},g_{\lambda\alpha}\in L_\infty(\Delta)$ and $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are co-ordinated convex function, then*

$$|T(f,g)|\leq\frac{49}{57600}(b-a)^2(d-c)^2MN \tag{2.2}$$

and

$$\begin{aligned}|T(f,g)| &\leq \frac{1}{32(b-a)^2(d-c)^2}\int_a^b\int_c^d[M|g(x,y)|+N|f(x,y)|]\\ &\times\left[\left((x-a)^2+(b-x)^2\right)\left((y-c)^2+(d-y)^2\right)\right]dydx\end{aligned} \tag{2.3}$$

where $T(f,g)$ is defined in (1.4),

$$M := ess \sup_{\substack{x,t\in[a,b]\\ y,s\in[c,d]}} \left[|f_{\lambda\alpha}(x,y)|+|f_{\lambda\alpha}(x,s)|+|f_{\lambda\alpha}(t,y)|+|f_{\lambda\alpha}(t,s)|\right]$$

and

$$N := ess \sup_{\substack{x,t\in[a,b]\\ y,s\in[c,d]}} \left[|g_{\lambda\alpha}(x,y)|+|g_{\lambda\alpha}(x,s)|+|g_{\lambda\alpha}(t,y)|+|g_{\lambda\alpha}(t,s)|\right].$$

Proof. Let F, G, $\widetilde{F}$ and $\widetilde{G}$ be defined as fallows

$$F = f(x,y) - \frac{1}{d-c}\int_c^d f(x,s)ds - \frac{1}{b-a}\int_a^b f(t,y)dt + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)dsdt,$$

$$G = g(x,y) - \frac{1}{d-c}\int_c^d g(x,s)ds - \frac{1}{b-a}\int_a^b g(t,y)dt + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d g(t,s)dsdt,$$

$$\widetilde{F} = \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x-t)(y-s)\int_0^1\int_0^1 f_{\lambda\alpha}(\lambda x+(1-\lambda)t, \alpha y+(1-\alpha)s)d\alpha d\lambda dsdt$$

and

$$\widetilde{G} = \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x-t)(y-s)\int_0^1\int_0^1 g_{\lambda\alpha}(\lambda x+(1-\lambda)t, \alpha y+(1-\alpha)s)d\alpha d\lambda dsdt.$$

Since

$$FG = \widetilde{F}\widetilde{G},$$

by integrating FG over Δ, multiplying the resultant equality by $\frac{1}{(b-a)(d-c)}$ it yields

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)\,dydx - \frac{1}{(b-a)^2(d-c)}\int_a^b\int_c^d g(x,y)\int_a^b f(t,y)dtdydx\\
&-\frac{1}{(b-a)(d-c)^2}\int_a^b\int_c^d g(x,y)\int_c^d f(x,s)dsdydx\\
&+\frac{1}{(b-a)^2(d-c)^2}\left(\int_a^b\int_c^d f(x,s)dsdx\right)\left(\int_a^b\int_c^d g(t,y)dydt\right)\\
= \ &\frac{1}{(b-a)^3(d-c)^3}\int_a^b\int_c^d\left(\int_a^b\int_c^d (x-t)(y-s)\int_0^1\int_0^1 f_{\lambda\alpha}(\lambda x+(1-\lambda)t, \alpha y+(1-\alpha)s)d\alpha d\lambda dsdt\right)\\
&\times\left(\int_a^b\int_c^d (x-t)(y-s)\int_0^1\int_0^1 g_{\lambda\alpha}(\lambda x+(1-\lambda)t, \alpha y+(1-\alpha)s)d\alpha d\lambda dsdt\right)dydx.
\end{aligned}$$

Using co-ordinated convexity of $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$, we have

$$\begin{aligned}
|T(f,g)| &\leq \frac{1}{(b-a)^3(d-c)^3} \\
&\quad \times \int_a^b\int_c^d \left(\int_a^b\int_c^d |x-t|\,|y-s| \int_0^1\int_0^1 |f_{\lambda\alpha}(\lambda x+(1-\lambda)t, \alpha y+(1-\alpha)s)|\, d\alpha d\lambda ds dt\right) \\
&\quad \times \left(\int_a^b\int_c^d |x-t|\,|y-s| \int_0^1\int_0^1 |g_{\lambda\alpha}(\lambda x+(1-\lambda)t, \alpha y+(1-\alpha)s)|\, d\alpha d\lambda ds dt\right) dydx \\
&\leq \frac{1}{16(b-a)^3(d-c)^3} \\
&\quad \times \int_a^b\int_c^d \left(\int_a^b\int_c^d |x-t|\,|y-s| \left[|f_{\lambda\alpha}(x,y)|+|f_{\lambda\alpha}(x,s)|+|f_{\lambda\alpha}(t,y)|+|f_{\lambda\alpha}(t,s)|\right] dsdt\right) \\
&\quad \times \left(\int_a^b\int_c^d |x-t|\,|y-s| \left[|g_{\lambda\alpha}(x,y)|+|g_{\lambda\alpha}(x,s)|+|g_{\lambda\alpha}(t,y)|+|g_{\lambda\alpha}(t,s)|\right] dsdt\right) dydx \\
&\leq \frac{MN}{16(b-a)^3(d-c)^3}\int_a^b\int_c^d \left[\frac{(x-a)^2+(b-x)^2}{2}\right]^2 \left[\frac{(y-c)^2+(d-y)^2}{2}\right]^2 dydx \\
&= \frac{49}{57600}(b-a)^2(d-c)^2 MN.
\end{aligned}$$

Here, it is easy to observe that

$$\int_a^b \left[\frac{(x-a)^2+(b-x)^2}{2}\right]^2 dx = \frac{7}{60}(b-a)^5$$

and

$$\int_c^d \left[\frac{(y-c)^2+(d-y)^2}{2}\right]^2 dy = \frac{7}{60}(d-c)^5 .$$

Thus, the fist inequality (2.2) is proved.

Now, we prove the second inequality (2.3). Therefore, by applying (2.1) to the function g, we have

$$\begin{aligned}
g(x,y) &= \frac{1}{d-c}g(x,s)ds + \frac{1}{b-a}\int_a^b g(t,y)dt - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d g(t,s)dsdt \\
&\quad + \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (x-t)(y-s) \qquad (2.4)\\
&\quad \times \left[\int_0^1\int_0^1 g_{\lambda\alpha}(\lambda x+(1-\lambda)t, \alpha y+(1-\alpha)s)d\alpha d\lambda\right] dsdt.
\end{aligned}$$

Multiplying (2.1) by $\frac{1}{(b-a)(d-c)}g(x,y)$, (2.4) by $\frac{1}{(b-a)(d-c)}f(x,y)$, summing the resultant equalities, then integrating on Δ, we have

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)\,dydx-\frac{1}{(b-a)^2(d-c)}\int_a^b\int_c^d g(x,y)\int_a^b f(t,y)dtdydx \\
&-\frac{1}{(b-a)(d-c)^2}\int_a^b\int_c^d g(x,y)\int_c^d f(x,s)dsdydx \\
&+\frac{1}{(b-a)^2(d-c)^2}\left(\int_a^b\int_c^d f(x,s)dsdx\right)\left(\int_a^b\int_c^d g(t,y)dydt\right) \\
=\ &\frac{1}{2(b-a)(d-c)} \\
&\times\left[\int_a^b\int_c^d g(x,y)\left(\int_a^b\int_c^d (x-t)(y-s)\int_0^1\int_0^1 f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)d\alpha d\lambda dsdt\right)dydx\right. \\
&\left.+\int_a^b\int_c^d f(x,y)\left(\int_a^b\int_c^d (x-t)(y-s)\int_0^1\int_0^1 g_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)d\alpha d\lambda dsdt\right)dydx\right].
\end{aligned}$$

Thus, by definiton of $T(f,g)$ in (1.4), and using co-ordinated convexity of $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$, we obtain

$$\begin{aligned}
&|T(f,g)|\le\frac{1}{2(b-a)(d-c)} \\
&\times\left[\int_a^b\int_c^d |g(x,y)|\left(\int_a^b\int_c^d |x-t|\,|y-s|\int_0^1\int_0^1 |f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)|\,d\alpha d\lambda dsdt\right)dydx\right. \\
&\left.+\int_a^b\int_c^d |f(x,y)|\left(\int_a^b\int_c^d |x-t|\,|y-s|\int_0^1\int_0^1 |g_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)|\,d\alpha d\lambda dsdt\right)dydx\right] \\
\le\ &\frac{1}{8(b-a)(d-c)} \\
&\times\int_a^b\int_c^d\left[|g(x,y)|\left(\int_a^b\int_c^d |x-t|\,|y-s|\,[|f_{\lambda\alpha}(x,y)|+|f_{\lambda\alpha}(x,s)|+|f_{\lambda\alpha}(t,y)|+|f_{\lambda\alpha}(t,s)|]\,dsdt\right)\right. \\
&\left.+|f(x,y)|\left(\int_a^b\int_c^d |x-t|\,|y-s|\,[|g_{\lambda\alpha}(x,y)|+|g_{\lambda\alpha}(x,s)|+|g_{\lambda\alpha}(t,y)|+|g_{\lambda\alpha}(t,s)|]\,dsdt\right)\right]dydx \\
\le\ &\frac{1}{32(b-a)^2(d-c)^2}\int_a^b\int_c^d [M+N\,|f(x,y)|] \\
&\times\left[\left((x-a)^2+(b-x)^2\right)\left((y-c)^2+(d-y)^2\right)\right]dydx.
\end{aligned}$$

This completes the proof of the Theorem. □

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