

## AN INEQUALITY OF GRÜSS LIKE VIA VARIANT OF POMPEIU'S MEAN VALUE THEOREM

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ABSTRACT. The main of this paper is to establish an integral inequality of Grüss type by using a mean value theorem.

### 1. INTRODUCTION

In 1935, G. Grüss [4] proved the following inequality:

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \frac{1}{b-a} \int_a^b g(x)dx \right| \leq \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma),$$

provided that  $f$  and  $g$  are two integrable function on  $[a, b]$  satisfying the condition

$$\varphi \leq f(x) \leq \Phi \quad \text{and} \quad \gamma \leq g(x) \leq \Gamma \quad \text{for all } x \in [a, b].$$

The constant  $\frac{1}{4}$  is best possible.

In 1882, P. L. Čebyšev [2] gave the following inequality:

$$|T(f, g)| \leq \frac{1}{12}(b-a)^2 \|f'\|_{\infty} \|g'\|_{\infty},$$

where  $f, g : [a, b] \rightarrow \mathbb{R}$  are absolutely continuous function, whose first derivatives  $f'$  and  $g'$  are bounded,

$$T(f, g) = \frac{1}{b-a} \int_a^b f(x)g(x)dx - \left( \frac{1}{b-a} \int_a^b f(x)dx \right) \left( \frac{1}{b-a} \int_a^b g(x)dx \right)$$

and  $\|\cdot\|_{\infty}$  denotes the norm in  $L_{\infty}[a, b]$  defined as  $\|p\|_{\infty} = \operatorname{ess\,sup}_{t \in [a, b]} |p(t)|$ .

For a differentiable function  $f : [a, b] \rightarrow \mathbb{R}$ ,  $a \cdot b > 0$ , Pachpatte has in [6] proved, using Pompeiu's mean value theorem [9], the following Grüss type inequality:

$$\begin{aligned} & \left| \int_a^b f(t)g(t)dt - \frac{1}{b^2 - a^2} \left( \int_a^b f(t)dt \cdot \int_a^b tg(t)dt + \int_a^b g(t)dt \cdot \int_a^b tf(t)dt \right) \right| \\ & \leq \|f - \ell f'\|_{\infty} \int_a^b |g(t)| \left| \frac{1}{2} - \frac{t}{a+b} \right| dt + \|g - \ell g'\|_{\infty} \int_a^b |f(t)| \left| \frac{1}{2} - \frac{t}{a+b} \right| dt \end{aligned}$$

where  $\ell(t) = t$ ,  $t \in [a, b]$ .

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In [7], Pecaric and Ungar proved a general estimate with the  $p$ -norm,  $1 < p < \infty$ , which will for  $p = \infty$  give the Pachpatte [6] result.

The interested reader is also referred to ([1], [3], [5]-[11]) for integral inequalities by using Pompeiu's mean value theorem. In this paper, we establish a new integral inequality of Grüss like via Pompeiu's mean value theorem.

## 2. MAIN RESULTS

Before stating the main results, we will give the following lemma proved by Pecaric and Ungar in [7]:

**Lemma 1.** For  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $1 \leq p, q \leq \infty$ , and  $0 < a \leq x \leq b$ , denote

$$(2.1) \quad A(x, q) := \left( \int_a^x \left( \int_t^x \frac{t^q du}{u^{2q}} \right) dt \right)^{\frac{1}{q}} + \left( \int_x^b \left( \int_x^t \frac{t^q du}{u^{2q}} \right) dt \right)^{\frac{1}{q}}$$

where for  $p = 1$ , i.e.  $q = \infty$ , the integrals are to be interpreted as the  $\infty$ -norms, i.e. as maxima of the function  $(u, t) \mapsto \frac{1}{u^2}$  on the corresponding domains of integration. Then,

$$A(x, q) = \left( \frac{a^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - a^{1+q}x^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}} + \left( \frac{b^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - b^{1+q}x^{1-2q}}{(1-2q)(1+q)} \right)^{\frac{1}{q}},$$

for  $1 < p, q < \infty$ ,  $p, q \neq 2$ ;

$$A(x, 2) = \frac{1}{3} \left[ \left( \ln \left( \frac{x}{a} \right)^3 + \frac{a^3}{x^3} - 1 \right)^{\frac{1}{2}} + \left( \ln \left( \frac{x}{b} \right)^3 + \frac{b^3}{x^3} - 1 \right)^{\frac{1}{2}} \right] = \lim_{q \rightarrow 2} A(x, q);$$

$$A(x, \infty) = \frac{a^2 + b^2}{2x} + x - a - b = \lim_{q \rightarrow \infty} A(x, q);$$

$$A(x, 1) = \frac{1}{a} + \frac{b}{x^2} = \lim_{q \rightarrow 1} A(x, q).$$

To prove our theorems, we need the following lemma proved by Sarikaya in [12]:

**Lemma 2.**  $f : [a, b] \rightarrow \mathbb{R}$  be continuous function on  $[a, b]$  and twice order differentiable function on  $(a, b)$  with  $0 < a < b$ . Then for any  $t, x \in [a, b]$ , we have

$$(2.2) \quad tf(x) - xf(t) + xt \frac{f'(t) - f'(x)}{2} = \frac{xt}{2} \int_x^t [2f(u) - 2uf'(u) + u^2 f''(u)] \frac{1}{u^2} du.$$

**Theorem 1.**  $f, g : [a, b] \rightarrow \mathbb{R}$  be continuous function on  $[a, b]$  and twice order differentiable function on  $(a, b)$  with  $0 < a < b$ . Then for  $\frac{1}{p} + \frac{1}{q} = 1$ , with  $1 < p, q < \infty$

any  $t, x \in [a, b]$ , we have

$$\begin{aligned}
(2.3) \quad & \left| \frac{1}{b-a} \int_a^b f(x)g(x)dx \right. \\
& - \left[ \frac{3}{b-a} \int_a^b f(t)dt - \frac{bf(b) - af(a)}{b-a} \right] \left( \frac{2}{5(b^2 - a^2)} \int_a^b xg(x)dx \right) \\
& - \left[ \frac{3}{b-a} \int_a^b g(t)dt - \frac{bg(b) - ag(a)}{b-a} \right] \left( \frac{2}{5(b^2 - a^2)} \int_a^b xf(x)dx \right) \\
& \left. - \frac{bf(b)g(b) - af(a)g(a)}{5(b-a)} \right| \\
& \leq \frac{2(b-a)^{\frac{1}{p}-2}}{5(b+a)} \left\{ \|2f - 2lf' + l^2 f''\|_p \int_a^b xg(x)A(x, q)dx \right. \\
& \quad \left. + \|2g - 2lg' + l^2 g''\|_p \int_a^b xf(x)A(x, q)dx \right\}
\end{aligned}$$

where  $l(t) = t$  for  $t \in [a, b]$ .

*Proof.* Applying (2.2) to the function  $g$ , we have

$$(2.4) \quad tg(x) - xg(t) + xt \frac{g'(t) - g'(x)}{2} = \frac{xt}{2} \int_x^t [2g(u) - 2ug'(u) + u^2 g''(u)] \frac{1}{u^2} du.$$

Multiplying (2.2) by  $g(x)$ , (2.4) by  $f(x)$ , summing the resultant equalities, then integrating with respect to  $t$  on  $[a, b]$ , we have

$$\begin{aligned}
(2.5) \quad & (b^2 - a^2) f(x)g(x) - \frac{3xg(x)}{2} \int_a^b f(t)dt - \frac{3xf(x)}{2} \int_a^b g(t)dt + \frac{xg(x)}{2} [bf(b) - af(a)] \\
& - \frac{b^2 - a^2}{4} xg(x)f'(x) + \frac{xf(x)}{2} [bg(b) - ag(a)] - \frac{b^2 - a^2}{4} xf(x)g'(x) \\
= & \frac{xg(x)}{2} \int_a^b \int_x^t [2f(u) - 2uf'(u) + u^2 f''(u)] \frac{1}{u^2} du \Bigg] dt \\
& + \frac{xf(x)}{2} \int_a^b \int_x^t [2g(u) - 2ug'(u) + u^2 g''(u)] \frac{1}{u^2} du \Bigg] dt.
\end{aligned}$$

Integrating with respect to  $x$  on  $[a, b]$  and adding notations  $F(u) = 2f(u) - 2uf'(u) + u^2f''(u)$  and  $G(u) = 2g(u) - 2ug'(u) + u^2g''(u)$ , we obtain

$$\begin{aligned}
(2.6) \quad & (b^2 - a^2) \int_a^b f(x)g(x)dx \\
& - \frac{3}{2} \left( \int_a^b xg(x)dx \right) \left( \int_a^b f(t)dt \right) - \frac{3}{2} \left( \int_a^b xf(x)dx \right) \left( \int_a^b g(t)dt \right) \\
& + \frac{bf(b) - af(a)}{2} \left( \int_a^b xg(x)dx \right) + \frac{bg(b) - ag(a)}{2} \left( \int_a^b xf(x)dx \right) \\
& - \frac{b^2 - a^2}{4} \int_a^b xg(x)f'(x)dx - \frac{b^2 - a^2}{4} \int_a^b xf(x)g'(x)dx \\
& = \frac{1}{2} \int_a^b xg(x) \left( \int_a^b t \left[ \int_x^t F(u) \frac{du}{u^2} \right] dt \right) dx + \frac{1}{2} \int_a^b xf(x) \left( \int_a^b t \left[ \int_x^t G(u) \frac{du}{u^2} \right] dt \right) dx.
\end{aligned}$$

$$(2.7) \quad \int_a^b xg(x)f'(x)dx = bf(b)g(b) - af(a)g(a) - \int_a^b f(x)g(x)dx - \int_a^b xf(x)g'(x)dx.$$

Adding (2.7) in (2.6), we have

$$\begin{aligned}
& \frac{5(b^2 - a^2)}{4} \int_a^b f(x)g(x)dx \\
& - \left[ \frac{3}{2} \int_a^b f(t)dt - \frac{bf(b) - af(a)}{2} \right] \left( \int_a^b xg(x)dx \right) \\
& - \left[ \frac{3}{2} \int_a^b g(t)dt - \frac{bg(b) - ag(a)}{2} \right] \left( \int_a^b xf(x)dx \right) \\
& - \frac{(b^2 - a^2)}{4} [bf(b)g(b) - af(a)g(a)] \\
& = \frac{1}{2} \int_a^b xg(x) \left( \int_a^b t \left[ \int_x^t F(u) \frac{du}{u^2} \right] dt \right) dx + \frac{1}{2} \int_a^b xf(x) \left( \int_a^b t \left[ \int_x^t G(u) \frac{du}{u^2} \right] dt \right) dx.
\end{aligned}$$

Taking modulus, we have

$$\begin{aligned}
(2.8) \quad & \left| \frac{5(b^2 - a^2)}{4} \int_a^b f(x)g(x)dx \right. \\
& - \left[ \frac{3}{2} \int_a^b f(t)dt - \frac{bf(b) - af(a)}{2} \right] \left( \int_a^b xg(x)dx \right) \\
& - \left[ \frac{3}{2} \int_a^b g(t)dt - \frac{bg(b) - ag(a)}{2} \right] \left( \int_a^b xf(x)dx \right) \\
& \left. - \frac{(b^2 - a^2)}{4} [bf(b)g(b) - af(a)g(a)] \right| \\
& \leq \frac{1}{2} \left| \int_a^b xg(x) \left( \int_a^b t \left[ \int_x^t F(u) \frac{du}{u^2} \right] dt \right) dx \right| + \frac{1}{2} \left| \int_a^b xf(x) \left( \int_a^b t \left[ \int_x^t G(u) \frac{du}{u^2} \right] dt \right) dx \right| \\
& \leq \frac{1}{2} \int_a^b |xg(x)| \left| \int_a^b t \left[ \int_x^t F(u) \frac{du}{u^2} \right] dt \right| dx + \frac{1}{2} \int_a^b |xf(x)| \left| \int_a^b t \left[ \int_x^t G(u) \frac{du}{u^2} \right] dt \right| dx \\
& \leq \frac{1}{2} \int_a^b |xg(x)| \left( \int_a^b \left| \int_x^t |F(u)| \frac{t}{u^2} du \right| dt \right) dx + \frac{1}{2} \int_a^b |xf(x)| \left( \int_a^b \left| \int_x^t |G(u)| \frac{t}{u^2} du \right| dt \right) dx.
\end{aligned}$$

In the last line (2.8), we have

$$(2.9) \quad \int_a^b \left| \int_x^t |F(u)| \frac{t}{u^2} du \right| dt = \int_a^x \int_t^x |F(u)| \frac{t}{u^2} dudt + \int_x^b \int_x^t |F(u)| \frac{t}{u^2} dudt.$$

Using Hölder's inequality in (2.9), we obtain

$$\begin{aligned}
(2.10) \quad & \int_a^b \left| \int_x^t |F(u)| \frac{t}{u^2} du \right| dt \\
& \leq \left( \int_a^x \int_t^x |F(u)|^p dudt \right)^{\frac{1}{p}} \left( \int_a^x \int_t^x \frac{t^q}{u^{2q}} dudt \right)^{\frac{1}{q}} \\
& \quad + \left( \int_x^b \int_x^t |F(u)|^p dudt \right)^{\frac{1}{p}} \left( \int_x^b \int_x^t \frac{t^q}{u^{2q}} dudt \right)^{\frac{1}{q}} \\
& \leq \left( \int_a^b \int_a^b |F(u)|^p dudt \right)^{\frac{1}{p}} \left\{ \left( \int_a^x \int_t^x \frac{t^q}{u^{2q}} dudt \right)^{\frac{1}{q}} + \left( \int_x^b \int_x^t \frac{t^q}{u^{2q}} dudt \right)^{\frac{1}{q}} \right\} \\
& = (b-a)^{\frac{1}{p}} \|2f - 2lf' + l^2 f''\|_p A(x, q).
\end{aligned}$$

Similarly, we get

$$(2.11) \quad \int_a^b \left| \int_x^t |G(u)| \frac{t}{u^2} du \right| dt \leq (b-a)^{\frac{1}{p}} \|2g - 2lg' + l^2g''\|_p A(x, q).$$

Adding (2.10) and (2.11) in (2.8), we obtain

$$(2.12) \quad \left| \frac{5(b^2 - a^2)}{4} \int_a^b f(x)g(x)dx - \left[ \frac{3}{2} \int_a^b f(t)dt - \frac{bf(b) - af(a)}{2} \right] \left( \int_a^b xg(x)dx \right) - \left[ \frac{3}{2} \int_a^b g(t)dt - \frac{bg(b) - ag(a)}{2} \right] \left( \int_a^b xf(x)dx \right) - \frac{(b^2 - a^2)}{4} [bf(b)g(b) - af(a)g(a)] \right| \leq \frac{1}{2} (b-a)^{\frac{1}{p}} \left\{ \|2f - 2lf' + l^2f''\|_p \int_a^b |xg(x)| A(x, q)dx + \|2g - 2lg' + l^2g''\|_p \int_a^b |xf(x)| A(x, q)dx \right\}.$$

Dividing (2.12) by  $\frac{5(b^2 - a^2)(b-a)}{4}$ , we obtain the required inequality (2.3).  $\square$

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