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ON THE WEIGHTED OSTROWSKI TYPE INEQUALITIES FOR DOUBLE INTEGRALS

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Abstract. In this paper, we obtain weighted Ostrowski type inequalities for function whose second order partial derivatives are bounded.

1. Introduction

In 1938, the classical integral inequality established by Ostrowski [8] as follows:

Theorem 1. *Let $f:[a,b]\rightarrow\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f^{'}:(a,b)\rightarrow\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty=\sup\limits_{t\in(a,b)}|f'(t)|<\infty$. Then, the inequality holds:*

$$\left|f(x)-\frac{1}{b-a}\int\limits_a^b f(t)dt\right|\leq\left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.1}$$

for all $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

Inequality (1.1) has wide applications in numerical analysis and in the theory of some special means; estimating error bounds for some special means, some mid-point, trapezoid and Simpson rules and quadrature rules, etc. Hence inequality (1.1) has attracted considerable attention and interest from mathematicans and researchers. Due to this, over the years, the interested reader is also refered to ([1]-[7],[9]-[20]) for integral inequalities in several independent variables. In addition, the current approach of obtaining the bounds, for a particular quadrature rule, have depended on the use of Peano kernel. The general approach in the past has involved the assumption of bounded derivatives of degree greater than one.

If $f:[a,b]\rightarrow\mathbb{R}$ is differentiable on $[a,b]$ with the first derivative f' integrable on $[a,b]$, then Montgomery identity holds:

$$f(x)=\frac{1}{b-a}\int\limits_a^b f(t)dt+\int\limits_a^b P(x,t)f'(t)dt, \tag{1.2}$$

where $P(x,t)$ is the Peano kernel defined by

$$P(x,t):=\begin{cases}\dfrac{t-a}{b-a}, & a\leq t<x\\[2ex] \dfrac{t-b}{b-a}, & x\leq t\leq b.\end{cases}$$

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In [3] and [5], the authors obtain two identities which generalize (1.2) for functions of two variables. In fact, for $f:[a,b]\times[c,d]\to\mathbb{R}$ such that the partial derivative $\frac{\partial f(t,s)}{\partial t}$, $\frac{\partial f(t,s)}{\partial s}$, and $\frac{\partial f^2(t,s)}{\partial t\partial s}$ all exist and are continuous on $[a,b]\times[c,d]$ and for all $(x,y)\in[a,b]\times[c,d]$, they obtain:

$$
\begin{aligned}
&(d-c)(b-a)f(x,y) \\
&= -\int_a^b\int_c^d f(t,s)dsdt+(d-c)\int_a^b f(t,y)dt \\
&\quad +(b-a)\int_c^d f(x,s)ds+\int_a^b\int_c^d p(x,t)p(y,s)\frac{\partial f^2(t,s)}{\partial t\partial s}dsdt
\end{aligned}
\tag{1.3}
$$

and

$$
\begin{aligned}
&(d-c)(b-a)f(x,y) \\
&= \int_a^b\int_c^d f(t,s)dsdt+\int_a^b\int_c^d p(x,t)\frac{\partial f(t,s)}{\partial t}dsdt \\
&\quad +\int_a^b\int_c^d q(y,s)\frac{\partial f(t,s)}{\partial s}dsdt+\int_a^b\int_c^d p(x,t)p(y,s)\frac{\partial f^2(t,s)}{\partial t\partial s}dsdt
\end{aligned}
\tag{1.4}
$$

where

$$p(x,t)=\begin{cases} t-a, & a\le t<x \\ t-b, & x\le t\le b\end{cases}$$

and

$$q(y,s)=\begin{cases} s-c, & c\le s<y \\ s-d, & y\le s\le d.\end{cases}$$

Definition 1. *Let $w:(a,b)\to[0,\infty)$ be an integrable function, i.e. $\int_a^b w(t)dt<\infty$, then define*

$$m_i(a,b)=\int_a^b t^i w(t)dt,\ \ i=0,1,...$$

as the i^{th} moment of w.

Definition 2. *Define the mean of the interval $[a,b]$ with respect to the density w as*

$$\mu(a,b)=\frac{m_1(a,b)}{m_0(a,b)} \tag{1.5}$$

and the variance by

$$\sigma^2(a,b)=\frac{m_2(a,b)}{m_0(a,b)}-\mu^2(a,b). \tag{1.6}$$

The main aim of this paper is to establish weighted Ostrowski type inequalities for function whose second order partial derivatives are bounded.

2. Main Results

In order to prove main results we need the following lemma:

Lemma 1. *Let $f:[a,b]\times[c,d]\rightarrow\mathbb{R}$ be an absolutely continuous function such that the partial derivative of order 2 exists for all $(t,s)\in[a,b]\times[c,d]$. Then, we have*

$$
\begin{aligned}
& m_0(a,b)\,m_0(c,d)\,(x-\mu(a,b))\,(y-\mu(c,d))\,f(x,y) \\
& -m_0(c,d)\,(y-\mu(c,d))\left[\int_a^x\left(\int_a^t w(u)\,du\right)f(t,y)\,dt+\int_x^b\left(\int_b^t w(u)\,du\right)f(t,y)\,dt\right] \\
& -m_0(a,b)\,(x-\mu(a,b))\left[\int_c^y\left(\int_c^s w(v)\,dv\right)f(x,s)\,ds+\int_y^d\left(\int_d^s w(v)\,dv\right)f(x,s)\,ds\right] \\
& +m_0(a,b)\,m_0(c,d)\int_a^b\int_c^d f(t,s)\,dsdt \\
= & \int_a^b\int_c^d P(x,t)\,Q(y,s)\,f_{ts}(t,s)\,dsdt.
\end{aligned}
$$

Proof. We define the following functions:

$$
P(x,t)=\begin{cases}\int_a^t (t-u)\,w(u)du, & a\le t<x\\ \int_b^t (t-u)\,w(u)du, & x\le t\le b\end{cases}
$$

and

$$
Q(y,s)=\begin{cases}\int_c^s (s-v)\,w(v)dv, & c\le s<y\\ \int_d^s (s-v)\,w(v)dv, & y\le s\le d\end{cases}
$$

for all $(x,y)\in[a,b]\times[c,d]$. Thus, by definitions of $P(x,t)$ and $Q(y,s)$, we have

$$\begin{aligned}
&\int_a^b\int_c^d P(x,t)\,Q(y,s)\,f_{ts}(t,s)\,dsdt\\
&= \int_a^x\int_c^y\left(\int_a^t (t-u)\,w(u)\,du\right)\left(\int_c^s (s-v)\,w(v)\,dv\right)f_{ts}(t,s)\,dsdt\\
&\quad+\int_a^x\int_y^d\left(\int_a^t (t-u)\,w(u)\,du\right)\left(\int_d^s (s-v)\,w(v)\,dv\right)f_{ts}(t,s)\,dsdt\\
&\quad+\int_x^b\int_c^y\left(\int_b^t (t-u)\,w(u)\,du\right)\left(\int_c^s (s-v)\,w(v)\,dv\right)f_{ts}(t,s)\,dsdt\\
&\quad+\int_x^b\int_y^d\left(\int_b^t (t-u)\,w(u)\,du\right)\left(\int_d^s (s-v)\,w(v)\,dv\right)f_{ts}(t,s)\,dsdt\\
&= I_1+I_2+I_3+I_4.
\end{aligned}$$

Integrating by parts, we can state:

$$\begin{aligned}
I_1 &= \int_a^x\left(\int_a^t (t-u)\,w(u)\,du\right)\left\{\left(\int_c^s (s-v)\,w(v)\,dv\right)f_t(t,s)\Big|_{s=c}^{y}-\int_c^y\left(\int_c^s w(v)\,dv\right)f_t(t,s)\,ds\right\}dt\\
&= \int_a^x\left(\int_a^t (t-u)\,w(u)\,du\right)\left\{\left(\int_c^y (y-v)\,w(v)\,dv\right)f_t(t,y)-\int_c^y\left(\int_c^s w(v)\,dv\right)f_t(t,s)\,ds\right\}dt\\
&= \left(\int_c^y (y-v)\,w(v)\,dv\right)\int_a^x\left(\int_a^t (t-u)\,w(u)\,du\right)f_t(t,y)\,dt\\
&\quad-\int_c^y\left(\int_c^s w(v)\,dv\right)\left[\int_a^x\left(\int_a^t (t-u)\,w(u)\,du\right)f_t(t,s)\,dt\right]ds\\
&= \left(\int_c^y (y-v)\,w(v)\,dv\right)\left[\left(\int_a^t (t-u)\,w(u)\,du\right)f(t,y)\Big|_{t=a}^{x}-\int_a^x\left(\int_a^t w(u)\,du\right)f(t,y)\,dt\right]\\
&\quad-\int_c^y\left(\int_c^s w(v)\,dv\right)\left[\left(\int_a^t (t-u)\,w(u)\,du\right)f(t,s)\Big|_{t=a}^{x}-\int_a^x\left(\int_a^t w(u)\,du\right)f(t,s)\,dt\right]ds
\end{aligned}$$

$$\begin{aligned}
&= \left(\int_a^x (x-u)\,w(u)\,du\right)\left(\int_c^y (y-v)\,w(v)\,dv\right) f(x,y) \\
&\quad -\left(\int_c^y (y-v)\,w(v)\,dv\right)\left(\int_a^x \left(\int_a^t w(u)\,du\right) f(t,y)\,dt\right) \\
&\quad -\left[\int_c^y \left(\int_c^s w(v)\,dv\right)\left(\int_a^x (x-u)\,w(u)\,du\right) f(x,s)\,ds\right] \\
&\quad +\int_a^x\int_c^y \left(\int_a^t w(u)\,du\right)\left(\int_c^s w(v)\,dv\right) f(t,s)\,dsdt
\end{aligned}$$

with similar methods

$$\begin{aligned}
I_2 &= \left(\int_a^x (x-u)\,w(u)\,du\right)\left(\int_y^d (y-v)\,w(v)\,dv\right) f(x,y) \\
&\quad -\left(\int_y^d (y-v)\,w(v)\,dv\right)\left(\int_a^x \left(\int_a^t w(u)\,du\right) f(t,y)\,dt\right) \\
&\quad -\left[\int_y^d \left(\int_c^s w(v)\,dv\right)\left(\int_a^x (x-u)\,w(u)\,du\right) f(x,s)\,ds\right] \\
&\quad +\int_a^x\int_y^d \left(\int_a^t w(u)\,du\right)\left(\int_d^s w(v)\,dv\right) f(t,s)\,dsdt,
\end{aligned}$$

$$\begin{aligned}
I_3 &= \left(\int_x^b (x-u)\,w(u)\,du\right)\left(\int_c^y (y-v)\,w(v)\,dv\right) f(x,y) \\
&\quad -\left(\int_c^y (y-v)\,w(v)\,dv\right)\left(\int_x^b \left(\int_b^t w(u)\,du\right) f(t,y)\,dt\right) \\
&\quad -\left[\int_c^y \left(\int_c^s w(v)\,dv\right)\left(\int_x^b (x-u)\,w(u)\,du\right) f(x,s)\,ds\right] \\
&\quad +\int_x^b\int_c^y \left(\int_b^t w(u)\,du\right)\left(\int_c^s w(v)\,dv\right) f(t,s)\,dsdt,
\end{aligned}$$

$$\begin{aligned} I_4 &= \left(\int_x^b (x-u)\, w(u)\, du\right)\left(\int_y^d (y-v)\, w(v)\, dv\right) f(x,y) \\ &\quad -\left(\int_y^d (y-v)\, w(v)\, dv\right)\left(\int_x^b \left(\int_b^t w(u)\, du\right) f(t,y)\, dt\right) \\ &\quad -\left[\int_y^d \left(\int_d^s w(v)\, dv\right)\left(\int_x^b (x-u)\, w(u)\, du\right) f(x,s)\, ds\right] \\ &\quad +\int_x^b\int_y^d \left(\int_b^t w(u)\, du\right)\left(\int_d^s w(v)\, dv\right) f(t,s)\, dsdt. \end{aligned}$$

Adding $I_1,\ I_2,\ I_3$ and I_4 and rewriting, we easily deduce:

$$\begin{aligned} &\left(\int_a^b (x-u)\, w(u)\, du\right)\left(\int_c^d (y-v)\, w(v)\, dv\right) f(x,y) \\ &-\left(\int_c^d (y-v)\, w(v)\, dv\right)\left[\int_a^x \left(\int_a^t w(u)\, du\right) f(t,y)\, dt + \int_x^b \left(\int_b^t w(u)\, du\right) f(t,y)\, dt\right] \\ &-\left(\int_a^b (x-u)\, w(u)\, du\right)\left[\int_c^y \left(\int_c^s w(v)\, dv\right) f(x,s)\, ds + \int_y^d \left(\int_d^s w(v)\, dv\right) f(x,s)\, ds\right] \\ &+\int_a^b\int_c^d \left(\int_a^b w(u)\, du\right)\left(\int_c^d w(v)\, dv\right) f(t,s)\, dsdt \\ = &\int_a^b\int_c^d P(x,t)\, Q(y,s)\, f_{ts}(t,s)\, dsdt \end{aligned}$$

which completes the proof. □

Theorem 2. *Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be an absolutely continuous function such that the partial derivative of order 2 exists and is bounded, i.e.,*

$$\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty = \sup_{(x,y)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right| < \infty$$

for all $(t,s)\in[a,b]\times[c,d]$. Then, we have

$$\begin{aligned} &(2.1) \\ &|F(x,y)| \\ \leq\ &\frac{m_0(a,b)m_0(c,d)}{4}\left[(x-\mu(a,b))^2+\sigma^2(a,b)\right]\left[(y-\mu(c,d))^2+\sigma^2(c,d)\right]\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty \\ \leq\ &\frac{m_0(a,b)m_0(c,d)}{4}\left(\left|x-\frac{a+b}{2}\right|+\frac{b-a}{2}\right)^2\left(\left|y-\frac{c+d}{2}\right|+\frac{d-c}{2}\right)^2\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty \end{aligned}$$

where

$$\begin{aligned} F(x,y) = \; & m_0(a,b)\,m_0(c,d)\,(x-\mu(a,b))\,(y-\mu(c,d))\,f(x,y) \\ & -m_0(c,d)\,(y-\mu(c,d))\left[\int_a^x\left(\int_a^t w(u)\,du\right)f(t,y)\,dt+\int_x^b\left(\int_b^t w(u)\,du\right)f(t,y)\,dt\right] \\ & -m_0(a,b)\,(x-\mu(a,b))\left[\int_c^y\left(\int_c^s w(v)\,dv\right)f(x,s)\,ds+\int_y^d\left(\int_d^s w(v)\,dv\right)f(x,s)\,ds\right] \\ & +m_0(a,b)\,m_0(c,d)\int_a^b\int_c^d f(t,s)\,dsdt. \end{aligned}$$

Proof. From Lemma 1 and using the properties of modulus, we observe that

$$\begin{aligned} & \text{(2.2)} \\ & \Big| m_0(a,b)\,m_0(c,d)\,(x-\mu(a,b))\,(y-\mu(c,d))\,f(x,y) \\ & -m_0(c,d)\,(y-\mu(c,d))\left[\int_a^x\left(\int_a^t w(u)\,du\right)f(t,y)\,dt+\int_x^b\left(\int_b^t w(u)\,du\right)f(t,y)\,dt\right] \\ & -m_0(a,b)\,(x-\mu(a,b))\left[\int_c^y\left(\int_c^s w(v)\,dv\right)f(x,s)\,ds+\int_y^d\left(\int_d^s w(v)\,dv\right)f(x,s)\,ds\right] \\ & +m_0(a,b)\,m_0(c,d)\int_a^b\int_c^d f(t,s)\,dsdt\Big| \\ \leq \; & \int_a^b\int_c^d |P(x,t)|\,|Q(y,s)|\left|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right| dsdt \\ \leq \; & \left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty \int_a^b\int_c^d |P(x,t)|\,|Q(y,s)|\,dsdt. \end{aligned}$$

Now, using the change of order of integration we get

$$\int_a^b |P(x,t)|\,dt = \int_a^x\int_a^t (t-u)\,w(u)\,dudt+\int_x^b\int_b^t (t-u)\,w(u)\,dudt$$

$$\begin{aligned} \text{(2.3)} \qquad & = \frac{1}{2}\int_a^b (x-t)^2\,w(t)dt \\ & = \frac{1}{2}\left[x^2 m_0(a,b)-2x m_1(a,b)+m_2(a,b)\right] \\ & = \frac{m_0(a,b)}{2}\left[(x-\mu(a,b))^2+\sigma^2(a,b)\right] \end{aligned}$$

and similarly,

$$\int_c^d |Q(y,s)|\,ds = \int_c^y \left(\int_c^s (s-v)\,w(v)\,dv \right) ds + \int_y^d \left(\int_d^s (s-v)\,w(v)\,dv \right) ds$$

$$(2.4) = \frac{1}{2}\int_c^d (y-s)^2\, w(s)ds$$

$$= \frac{m_0(c,d)}{2}\left[(y-\mu(c,d))^2 + \sigma^2(c,d) \right].$$

Thus, using (2.3) and (2.4) in (2.2), we obtain the first inequality of (2.1). To obtain the second inequality of (2.1) note that

$$\begin{aligned}
\int_a^b (x-t)^2\, w(t)dt &\le \sup_{t\in[a,b]} (x-t)^2 .m_0(a,b) \\
&= m_0(a,b)\max\left\{ (x-a)^2, (x-b)^2 \right\} \\
&= m_0(a,b)\frac{1}{2}\left((x-a)^2 + (x-b)^2 + \left| (x-a)^2 - (x-b)^2 \right| \right) \\
&= m_0(a,b)\left(\left| x - \frac{a+b}{2} \right| + \frac{b-a}{2} \right)^2
\end{aligned}$$

and similarly,

$$\int_c^d (y-s)^2\, w(s)ds \le m_0(c,d)\left(\left| y - \frac{c+d}{2} \right| + \frac{d-c}{2} \right)^2$$

which upon substitution into (2.2) the proof is completed. □

Remark 1. *If we choose* $(x,y)=(\frac{\mu(a,b)}{2},\frac{\mu(c,d)}{2})$ *in Theorem 2, then the inequalities (2.1) reduce the following inequalities*

$$\begin{aligned}
(2.5)\quad & \left|\frac{m_1(a,b)\,m_1(c,d)}{4}f\left(\frac{\mu(a,b)}{2},\frac{\mu(c,d)}{2}\right)\right. \\
& -\frac{m_1(c,d)}{2}\left[\int_a^{\frac{\mu(a,b)}{2}}\left(\int_a^t w(u)\,du\right)f\left(t,\frac{\mu(c,d)}{2}\right)dt+\int_{\frac{\mu(a,b)}{2}}^b\left(\int_b^t w(u)\,du\right)f\left(t,\frac{\mu(c,d)}{2}\right)dt\right] \\
& -\frac{m_1(a,b)}{2}\left[\int_c^{\frac{\mu(c,d)}{2}}\left(\int_c^s w(v)\,dv\right)f\left(\frac{\mu(a,b)}{2},s\right)ds+\int_{\frac{\mu(c,d)}{2}}^d\left(\int_d^s w(v)\,dv\right)f\left(\frac{\mu(a,b)}{2},s\right)ds\right] \\
& \left.+\,m_0(a,b)\,m_0(c,d)\int_a^b\int_c^d f(t,s)\,dsdt\right| \\
\leq\; & \frac{m_0(a,b)m_0(c,d)}{4}\left[\frac{\mu^2(a,b)}{4}+\sigma^2(a,b)\right]\left[\frac{\mu^2(c,d)}{4}+\sigma^2(c,d)\right]\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty \\
\leq\; & \frac{m_0(a,b)m_0(c,d)}{4}\left(\left|\frac{\mu(a,b)}{2}-\frac{a+b}{2}\right|+\frac{b-a}{2}\right)^2\left(\left|\frac{\mu(c,d)}{2}-\frac{c+d}{2}\right|+\frac{d-c}{2}\right)^2\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty.
\end{aligned}$$

Substituting $w(u)=1$ in (1.5) and (1.6) it follows that

$$m_0(a,b)=b-a$$

$$m_1(a,b)=\int_a^b udu=\frac{b^2-a^2}{2}$$

$$\mu(a,b)=\frac{\int_a^b udu}{\int_a^b du}=\frac{a+b}{2}$$

and

$$\sigma^2(a,b)=\frac{\int_a^b u^2du}{\int_a^b du}=\frac{(b-a)^2}{12}.$$

Substituting into (2.5) gives

$$\begin{aligned}
&\left|\frac{\left(b^2-a^2\right)\left(d^2-c^2\right)}{16}f\left(\frac{a+b}{4},\frac{c+d}{4}\right)\right.\\
&-\frac{d^2-c^2}{4}\left[\int_a^{\frac{a+b}{4}}(t-a)\,f\left(t,\frac{c+d}{4}\right)dt+\int_{\frac{a+b}{4}}^{b}(t-b)\,f\left(t,\frac{c+d}{4}\right)dt\right]\\
&-\frac{b^2-a^2}{4}\left[\int_c^{\frac{c+d}{4}}(s-c)\,f\left(\frac{a+b}{4},s\right)ds+\int_{\frac{c+d}{4}}^{d}(s-d)\,f\left(\frac{a+b}{4},s\right)ds\right]\\
&\left.+(b-a)(d-c)\int_a^b\int_c^d f(t,s)\,dsdt\right|\\
\leq\ &\frac{(b-a)(d-c)}{4}\left[\frac{(a+b)^2}{16}+\frac{(b-a)^2}{12}\right]\left[\frac{(c+d)^2}{16}+\frac{(d-c)^2}{12}\right]\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty\\
\leq\ &\frac{(b-a)^2(d-c)^2}{32}\left\|\frac{\partial^2 f(t,s)}{\partial t\partial s}\right\|_\infty.
\end{aligned}$$

Substituting $w(u)=\ln(\frac{1}{u})$, $a=c=0$, $b=d=1$ in (1.5) and (1.6) it follows that

$$m_0(0,1)=1$$

$$m_1(0,1)=\int_0^1 u\ln(\frac{1}{u})du=\frac{1}{4}$$

$$\mu(0,1)=\frac{\int_0^1 u\ln(\frac{1}{u})du}{\int_0^1\ln(\frac{1}{u})du}=\frac{1}{4}$$

and

$$\sigma^2(0,1)=\frac{\int_0^1 u^2\ln(\frac{1}{u})du}{\int_0^1\ln(\frac{1}{u})du}-\left(\frac{1}{4}\right)^2=\frac{7}{144}.$$

Substituting into (2.5) gives

$$\frac{1}{4^3} f\left(\frac{1}{8}, \frac{1}{8}\right) + \int_0^1 \int_0^1 f(t,s)\, dsdt$$
$$-\frac{1}{8}\left[\int_0^{\frac{1}{8}} \left(t\ln(\frac{1}{t}) + t\right) f\left(t, \frac{1}{8}\right) dt + \int_{\frac{1}{8}}^1 \left(t\ln(\frac{1}{t}) + t - 1\right) f\left(t, \frac{1}{8}\right) dt\right]$$
$$-\frac{1}{8}\left[\int_0^{\frac{1}{8}} \left(s\ln(\frac{1}{s}) + s\right) f\left(\frac{1}{8}, s\right) ds + \int_{\frac{1}{8}}^1 \left(s\ln(\frac{1}{s}) + s - 1\right) f\left(\frac{1}{8}, s\right) ds\right]$$
$$\leq \frac{1}{4^5}\left[1 + \frac{1}{36}\right]^2 \left\|\frac{\partial^2 f(t,s)}{\partial t \partial s}\right\|_\infty \leq \frac{81}{4^5}\left\|\frac{\partial^2 f(t,s)}{\partial t \partial s}\right\|_\infty .$$

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