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SOME INEQUALITIES ASSOCIATED WITH THE HERMITE-HADAMARD-FEJÉR TYPE FOR CONVEX FUNCTION

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Abstract. In this paper, we extend some estimates of the right hand side of a Hermite- Hadamard-Fejér type inequality for functions whose first derivatives absolute values are convex.The results presented here would provide extensions of those given in earlier works.

1. Introduction

Definition 1. *The function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$, is said to be convex if the following inequality holds*

$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. We say that f is concave if $(-f)$ is convex.

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality (see, [2], [4]):

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2} \tag{1.1}$$

where $f:I\subset\mathbb{R}\to\mathbb{R}$ is a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$.

In [1], Dragomir and Agarwal proved the following results connected with the right part of (1.1).

Lemma 1. *Let $f:I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$. If $f'\in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx=\frac{b-a}{2}\int_0^1(1-2t)f'(ta+(1-t)b)dt. \tag{1.2}$$

Theorem 1. *Let $f:I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)}{8}\left(|f'(a)|+|f'(b)|\right). \tag{1.3}$$

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Theorem 2. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, $f' \in L(a, b)$ and $p > 1$. If the mapping $|f'|^{p/(p-1)}$ is convex on $[a, b]$, then the following inequality holds:*

(1.4)
$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{2(p+1)^{1/p}} \left(\frac{|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}}{2}\right)^{(p-1)/p}.$$

The most well-known inequalities related to the integral mean of a convex function are the Hermite Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities (see, [5]-[12], [13], [14]). In [3], Fejer gave a weighted generalizatinon of the inequalities (1.1) as the following:

Theorem 3. *$f : [a, b] \to \mathbb{R}$, be a convex function, then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \le \frac{1}{b-a}\int_a^b f(x)w(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b w(x)dx \tag{1.5}$$

holds, where $w : [a, b] \to \mathbb{R}$ is nonnegative, integrable, and symmetric about $x = \frac{a+b}{2}$.

In [5], some inequalities of Hermite-Hadamard-Fejer type for differentiable convex mappings were proved using the following lemma.

Lemma 2. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and $w : [a, b] \to [0, \infty)$ be a differentiable mapping. If $f' \in L[a, b]$, then the following equality holds:*

(1.6)
$$\frac{f(a)+f(b)}{2}\int_a^b w(x)dx - \int_a^b f(x)w(x)dx = \frac{(b-a)^2}{2}\int_0^1 p(t)f'(ta+(1-t)b)dt$$

for each $t \in [0, 1]$, where

$$p(t) = \int_t^1 w(as+(1-s)b)ds - \int_0^t w(as+(1-s)b)ds.$$

In this article, using functions whose derivatives absolute values are convex, we obtained new inequalities of Hermite-Hadamard-Fejer type. The results presented here would provide extensions of those given in earlier works.

2. Main Results

We will establish some new results connected with the right-hand side of (1.5) and (1.1). Now, we prove our main theorems:

Theorem 4. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and let $w : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. If $|f'|$ is convex on $[a, b]$, then* *for all $x \in [a,b]$, the following inequalities hold:*

$$
\begin{aligned}
&\left| \left(\int_x^b w(s)ds \right)^{\alpha} f(b) - \left(\int_x^a w(s)ds \right)^{\alpha} f(a) - \alpha \int_a^b \left(\int_x^t w(s)ds \right)^{\alpha-1} w(t)f(t)dt \right| \\
\leq\ & \|w\|_{[a,x],\infty}^{\alpha} \left\{ \frac{\left|f^{'}(a)\right|}{b-a} \left[\frac{(x-a)^{\alpha+1}(b-x)}{\alpha+1} + \frac{(x-a)^{\alpha+2}}{\alpha+2} \right] + \frac{\left|f^{'}(b)\right|}{b-a} \frac{(x-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)} \right\} \\
& + \|w\|_{[x,b],\infty}^{\alpha} \left\{ \frac{\left|f^{'}(a)\right|}{b-a} \frac{(b-x)^{\alpha+2}}{(\alpha+1)(\alpha+2)} + \frac{\left|f^{'}(b)\right|}{b-a} \left[\frac{(b-x)^{\alpha+1}(x-a)}{\alpha+1} + \frac{(b-x)^{\alpha+2}}{\alpha+2} \right] \right\} \\
\leq\ & \frac{\|w\|_{[a,b],\infty}^{\alpha}}{(b-a)} \left\{ \left|f^{'}(a)\right| \left[\frac{(x-a)^{\alpha+1}(b-x)}{\alpha+1} + \frac{(b-x)^{\alpha+2}}{(\alpha+1)(\alpha+2)} + \frac{(x-a)^{\alpha+2}}{\alpha+2} \right] \right. \\
& \left. + \left|f^{'}(b)\right| \left[\frac{(b-x)^{\alpha+1}(x-a)}{\alpha+1} + \frac{(x-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)} + \frac{(b-x)^{\alpha+2}}{\alpha+2} \right] \right\}
\end{aligned}
$$

where $\alpha > 0$ and $\|w\|_{\infty} = \sup_{t \in [a,b]} |w(t)|$.

Proof. By integration by parts, we have the following equalities:

$$
\begin{aligned}
(2.1) & \int_a^b \left(\int_x^t w(s)ds \right)^{\alpha} f^{'}(t)dt \\
=\ & \left(\int_x^t w(s)ds \right)^{\alpha} f(t) \Bigg|_a^b - \alpha \int_a^b \left(\int_x^t w(s)ds \right)^{\alpha-1} w(t)f(t)dt \\
=\ & \left(\int_x^b w(s)ds \right)^{\alpha} f(b) - \left(\int_x^a w(s)ds \right)^{\alpha} f(a) - \alpha \int_a^b \left(\int_x^t w(s)ds \right)^{\alpha-1} w(t)f(t)dt.
\end{aligned}
$$

We take absolute value of (2.1) and using convexity of $\left|f'\right|$, we find that

$$\begin{aligned}
&\left|\left(\int_x^b w(s)ds\right)^{\alpha} f(b)-\left(\int_x^a w(s)ds\right)^{\alpha} f(a)-\alpha\int_a^b\left(\int_x^t w(s)ds\right)^{\alpha-1} w(t)f(t)dt\right| \\
\leq\ & \int_a^x\left(\left|\int_x^t w(s)ds\right|\right)^{\alpha}\left|f'(t)\right|dt+\int_x^b\left(\left|\int_x^t w(s)ds\right|\right)^{\alpha}\left|f'(t)\right|dt \\
\leq\ & \|w\|_{[a,x],\infty}^{\alpha}\int_a^x (x-t)^{\alpha}\left|f'(t)\right|dt+\|w\|_{[x,b],\infty}\int_x^b (t-x)^{\alpha}\left|f'(t)\right|dt \\
=\ & \|w\|_{[a,x],\infty}^{\alpha}\left[\int_a^x (x-t)^{\alpha}\left|f'(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b)\right|dt\right] \\
&+\|w\|_{[x,b],\infty}^{\alpha}\left[\int_x^b (t-x)^{\alpha}\left|f'(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b)\right|dt\right] \\
\leq\ & \|w\|_{[a,x],\infty}^{\alpha}\left\{\frac{\left|f'(a)\right|}{b-a}\left[\frac{(x-a)^{\alpha+1}(b-x)}{\alpha+1}+\frac{(x-a)^{\alpha+2}}{\alpha+2}\right]+\frac{\left|f'(b)\right|}{b-a}\frac{(x-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)}\right\} \\
&+\|w\|_{[x,b],\infty}^{\alpha}\left\{\frac{\left|f'(a)\right|}{b-a}\frac{(b-x)^{\alpha+2}}{(\alpha+1)(\alpha+2)}+\frac{\left|f'(b)\right|}{b-a}\left[\frac{(b-x)^{\alpha+1}(x-a)}{\alpha+1}+\frac{(b-x)^{\alpha+2}}{\alpha+2}\right]\right\} \\
\leq\ & \frac{\|w\|_{[a,b],\infty}^{\alpha}}{(b-a)}\left\{\left|f'(a)\right|\left[\frac{(x-a)^{\alpha+1}(b-x)}{\alpha+1}+\frac{(b-x)^{\alpha+2}}{(\alpha+1)(\alpha+2)}+\frac{(x-a)^{\alpha+2}}{\alpha+2}\right]\right. \\
&\left.+\left|f'(b)\right|\left[\frac{(b-x)^{\alpha+1}(x-a)}{\alpha+1}+\frac{(x-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)}+\frac{(b-x)^{\alpha+2}}{\alpha+2}\right]\right\}
\end{aligned}$$

for all $x\in[a,b]$. Hence, the proof of theorem is completed. □

Corollary 1. *Under the same assumptions of Theorem 4 with $w(s)=1$, then the following inequality holds:*

$$\begin{aligned}
&\left|(b-x)^{\alpha} f(b)-(a-x)^{\alpha} f(a)-\alpha\int_a^b (t-x)^{\alpha-1} f(t)dt\right| \\
\leq\ & \frac{1}{(b-a)}\left\{\left|f'(a)\right|\left[\frac{(x-a)^{\alpha+1}(b-x)}{\alpha+1}+\frac{(b-x)^{\alpha+2}}{(\alpha+1)(\alpha+2)}+\frac{(x-a)^{\alpha+2}}{\alpha+2}\right]\right. \\
&\left.+\left|f'(b)\right|\left[\frac{(b-x)^{\alpha+1}(x-a)}{\alpha+1}+\frac{(x-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)}+\frac{(b-x)^{\alpha+2}}{\alpha+2}\right]\right\}
\end{aligned}\tag{2.2}$$

for all $x\in[a,b]$.

Remark 1. *If we take $\alpha = 1$ and $x = \frac{a+b}{2}$ in (2.2), the inequality (2.2) reduces to (1.3).*

Corollary 2 (Fejer Type Inequality)**.** *Under the same assumptions of Theorem 4 with $\alpha = 1$, then the following inequalities hold:*

$$\begin{aligned}
&\left| f(b)\int_x^b w(s)ds + f(a)\int_a^x w(s)ds - \int_a^b w(t)f(t)dt \right| \\
\leq\ & \left| f^{'}(a) \right| \frac{(x-a)^2(3b-2a-x)\|w\|_{[a,x],\infty} + \|w\|_{[x,b],\infty}(b-x)^3}{6(b-a)} \\
&+ \left| f^{'}(b) \right| \frac{(b-x)^2(x-3a-2b)\|w\|_{[x,b],\infty} + (x-a)^3\|w\|_{[a,x],\infty}}{6(b-a)} \\
\leq\ & \left| f^{'}(a) \right| \left[\frac{(x-a)^2(3b-2a-x) + (b-x)^3}{6(b-a)} \right] \|w\|_{[a,b],\infty} \\
&+ \left| f^{'}(b) \right| \left[\frac{(b-x)^2(x-3a-2b) + (x-a)^3}{6(b-a)} \right] \|w\|_{[a,b],\infty}
\end{aligned}$$

which is proved by Tseng et. al. in [8].

Corollary 3 (Weighted Trapezoid Inequality)**.** *Let $w : [a,b] \to \mathbb{R}$ be symmetric to $\frac{a+b}{2}$ and $x = \frac{a+b}{2}$ in Corollary 2. Then the following inequalities hold:*

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2}\int_a^b w(s)ds - \int_a^b w(t)f(t)dt \right| \\
\leq\ & \frac{(b-a)^2}{48}\left[5\|w\|^{\alpha}_{\left[a,\frac{a+b}{2}\right],\infty} + \|w\|^{\alpha}_{\left[\frac{a+b}{2},b\right],\infty} \right] \left| f^{'}(a) \right| \\
&+ \left[\|w\|^{\alpha}_{\left[a,\frac{a+b}{2}\right],\infty} + 5\|w\|^{\alpha}_{\left[\frac{a+b}{2},b\right],\infty} \right] \left| f^{'}(b) \right| \\
\leq\ & (b-a)^2\|w\|^{\alpha}_{[a,b],\infty}\left(\frac{\left| f^{'}(a) \right| + \left| f^{'}(b) \right|}{8} \right)
\end{aligned}$$

which is proved by Tseng et. al. in [8].

Theorem 5. *Let $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$ and let $w : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. If $|f'|^q$ is convex on $[a,b]$,*

$q>1$, then for all $x\in[a,b]$, the following inequalities hold:

$$\left|\left(\int_x^b w(s)ds\right)^{\alpha} f(b)-\left(\int_x^a w(s)ds\right)^{\alpha} f(a)-\alpha\int_a^b\left(\int_x^t w(s)ds\right)^{\alpha-1} w(t)f(t)dt\right|$$

$$\leq \frac{(x-a)^{\alpha+\frac{1}{p}}\left\|w\right\|_{[a,x],\infty}^{\alpha}}{(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\left(\frac{(b-a)^2-(b-x)^2}{2}\left|f^{'}(a)\right|^q+\frac{(x-a)^2}{2}\left|f^{'}(b)\right|^q\right)^{\frac{1}{q}}$$

$$+\frac{(b-x)^{\alpha+\frac{1}{p}}\left\|w\right\|_{[x,b],\infty}^{\alpha}}{(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\left(\frac{(b-x)^2}{2}\left|f^{'}(a)\right|^q+\frac{(b-a)^2-(x-a)^2}{2}\left|f^{'}(b)\right|^q\right)^{\frac{1}{q}}$$

$$\leq \frac{\left\|w\right\|_{[a,b],\infty}^{\alpha}}{(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\left\{(x-a)^{\alpha+\frac{1}{p}}\left(\frac{(b-a)^2-(b-x)^2}{2}\left|f^{'}(a)\right|^q+\frac{(x-a)^2}{2}\left|f^{'}(b)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+(b-x)^{\alpha+\frac{1}{p}}\left(\left[\frac{(b-x)^2}{2}\left|f^{'}(a)\right|^q+\frac{(b-a)^2-(x-a)^2}{2}\left|f^{'}(b)\right|^q\right]\right)^{\frac{1}{q}}\right\}$$

where $\alpha>0$, $\frac{1}{p}+\frac{1}{q}=1$, and $\left\|w\right\|_{\infty}=\sup_{t\in[a,b]}\left|w(t)\right|$.

Proof. We take absolute value of (2.1). Using Holder's inequality, we find that

$$\left|\left(\int_x^b w(s)ds\right)^{\alpha} f(b)-\left(\int_x^a w(s)ds\right)^{\alpha} f(a)-\alpha\int_a^b\left(\int_x^t w(s)ds\right)^{\alpha-1} w(t)f(t)dt\right|$$

$$\leq \int_a^x\left(\left|\int_x^t w(s)ds\right|\right)^{\alpha} f^{'}(t)dt+\int_x^b\left(\left|\int_x^t w(s)ds\right|\right)^{\alpha} f^{'}(t)dt$$

$$\leq \left(\int_a^x\left|\int_x^t w(s)ds\right|^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_a^x\left|f^{'}(t)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_x^b\left|\int_x^t w(s)ds\right|^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_x^b\left|f^{'}(t)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $\left|f^{'}(t)\right|^q$ is convex on $[a,b]$

$$\left|f^{'}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|^q\leq\frac{b-t}{b-a}\left|f^{'}(a)\right|^q+\frac{t-a}{b-a}\left|f^{'}(b)\right|^q. \tag{2.3}$$

From (2.3), it follows that

$$\begin{aligned}
&\left|\left(\int_x^b w(s)ds\right)^{\alpha} f(b)-\left(\int_x^a w(s)ds\right)^{\alpha} f(a)-\alpha\int_a^b\left(\int_x^t w(s)ds\right)^{\alpha-1} w(t)f(t)dt\right| \\
\leq\ & \|w\|_{[a,x],\infty}^{\alpha}\left(\int_a^x (x-t)^{\alpha p}\,dt\right)^{\frac{1}{p}}\left(\int_a^x\left[\frac{b-t}{b-a}\left|f^{'}(a)\right|^{q}+\frac{t-a}{b-a}\left|f^{'}(b)\right|^{q}\right]dt\right)^{\frac{1}{q}} \\
&+\|w\|_{[x,b],\infty}^{\alpha}\left(\int_x^b (t-x)^{\alpha p}\,dt\right)^{\frac{1}{p}}\left(\int_x^b\left[\frac{b-t}{b-a}\left|f^{'}(a)\right|^{q}+\frac{t-a}{b-a}\left|f^{'}(b)\right|^{q}\right]dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{(x-a)^{\alpha+\frac{1}{p}}\|w\|_{[a,x],\infty}^{\alpha}}{(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\left(\frac{(b-a)^2-(b-x)^2}{2}\left|f^{'}(a)\right|^{q}+\frac{(x-a)^2}{2}\left|f^{'}(b)\right|^{q}\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^{\alpha+\frac{1}{p}}\|w\|_{[x,b],\infty}^{\alpha}}{(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\left(\left[\frac{(b-x)^2}{2}\left|f^{'}(a)\right|^{q}+\frac{(b-a)^2-(x-a)^2}{2}\left|f^{'}(b)\right|^{q}\right]\right)^{\frac{1}{q}} \\
\leq\ & \frac{\|w\|_{[a,b],\infty}^{\alpha}}{(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\left\{(x-a)^{\alpha+\frac{1}{p}}\left(\frac{(b-a)^2-(b-x)^2}{2}\left|f^{'}(a)\right|^{q}+\frac{(x-a)^2}{2}\left|f^{'}(b)\right|^{q}\right)^{\frac{1}{q}}\right. \\
&\left.+(b-x)^{\alpha+\frac{1}{p}}\left(\left[\frac{(b-x)^2}{2}\left|f^{'}(a)\right|^{q}+\frac{(b-a)^2-(x-a)^2}{2}\left|f^{'}(b)\right|^{q}\right]\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

which this completes the proof. $\square$

Corollary 4. *Under the same assumptions of Theorem 5 with $w(s)=1$, then the following inequalities hold:*

$$\begin{aligned}
&\left|(b-x)^{\alpha}f(b)-(a-x)^{\alpha}f(a)-\alpha\int_a^b (t-x)^{\alpha-1}f(t)dt\right| \\
\leq\ & \frac{(x-a)^{\alpha+\frac{1}{p}}}{(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\left(\frac{(b-a)^2-(b-x)^2}{2}\left|f^{'}(a)\right|^q+\frac{(x-a)^2}{2}\left|f^{'}(b)\right|^q\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^{\alpha+\frac{1}{p}}}{(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\left(\frac{(b-x)^2}{2}\left|f^{'}(a)\right|^q+\frac{(b-a)^2-(x-a)^2}{2}\left|f^{'}(b)\right|^q\right)^{\frac{1}{q}} \qquad (2.4)\\
\leq\ & \frac{1}{(b-a)^{\frac{1}{q}}(\alpha p+1)^{\frac{1}{p}}}\left\{(x-a)^{\alpha+\frac{1}{p}}\left(\frac{(b-a)^2-(b-x)^2}{2}\left|f^{'}(a)\right|^q+\frac{(x-a)^2}{2}\left|f^{'}(b)\right|^q\right)^{\frac{1}{q}}\right. \\
&\left.+(b-x)^{\alpha+\frac{1}{p}}\left(\left[\frac{(b-x)^2}{2}\left|f^{'}(a)\right|^q+\frac{(b-a)^2-(x-a)^2}{2}\left|f^{'}(b)\right|^q\right]\right)^{\frac{1}{q}}\right\}
\end{aligned}$$

Corollary 5. *Let the conditions of Corollary 4 hold. If we take $\alpha=1$ and $x=\frac{a+b}{2}$ in (2.4), then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \\
\leq\ & \frac{(b-a)}{4(p+1)^{\frac{1}{p}}}\left[\left(\frac{3\left|f^{'}(a)\right|^q+\left|f^{'}(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f^{'}(a)\right|^q+3\left|f^{'}(b)\right|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 6 (Fejer Type Inequality). *Under the same assumptions of Theorem 5 with $\alpha = 1$, then the following inequalities hold:*

$$\left| f(b) \int_x^b w(s)ds + f(a) \int_a^x w(s)ds - \int_a^b w(t)f(t)dt \right|$$

$$\leq \frac{(x-a)^{1+\frac{1}{p}} \|w\|_{[a,x],\infty}}{(b-a)^{\frac{1}{q}} (p+1)^{\frac{1}{p}}} \left(\frac{(b-a)^2 - (b-x)^2}{2} \left| f^{'}(a) \right|^q + \frac{(x-a)^2}{2} \left| f^{'}(b) \right|^q \right)^{\frac{1}{q}}$$

$$+ \frac{(b-x)^{1+\frac{1}{p}} \|w\|_{[x,b],\infty}}{(b-a)^{\frac{1}{q}} (p+1)^{\frac{1}{p}}} \left(\frac{(b-x)^2}{2} \left| f^{'}(a) \right|^q + \frac{(b-a)^2 - (x-a)^2}{2} \left| f^{'}(b) \right|^q \right)^{\frac{1}{q}}$$

$$\leq \frac{\|w\|_{[a,b],\infty}}{(b-a)^{\frac{1}{q}} (p+1)^{\frac{1}{p}}} \left\{ (x-a)^{1+\frac{1}{p}} \left(\frac{(b-a)^2 - (b-x)^2}{2} \left| f^{'}(a) \right|^q + \frac{(x-a)^2}{2} \left| f^{'}(b) \right|^q \right)^{\frac{1}{q}} \right.$$

$$\left. + (b-x)^{1+\frac{1}{p}} \left(\left[\frac{(b-x)^2}{2} \left| f^{'}(a) \right|^q + \frac{(b-a)^2 - (x-a)^2}{2} \left| f^{'}(b) \right|^q \right] \right)^{\frac{1}{q}} \right\}.$$

Corollary 7 (Weighted Trapezoid Inequality). *Let $w : [a,b] \to \mathbb{R}$ be symmetric to $\frac{a+b}{2}$ and $x = \frac{a+b}{2}$ in Corollary 6. Then the following inequalities hold:*

$$\left| \frac{f(a)+f(b)}{2} \int_a^b w(s)ds - \int_a^b w(t)f(t)dt \right|$$

$$\leq \frac{(b-a)^2}{4(p+1)^{\frac{1}{p}}} \left[\|w\|_{\left[a,\frac{a+b}{2}\right],\infty} \left(\frac{3\left|f^{'}(a)\right|^q + \left|f^{'}(b)\right|^q}{4} \right)^{\frac{1}{q}} + \|w\|_{\left[\frac{a+b}{2},b\right],\infty} \left(\frac{\left|f^{'}(a)\right|^q + 3\left|f^{'}(b)\right|^q}{4} \right)^{\frac{1}{q}} \right]$$

$$\leq \frac{(b-a)^2 \|w\|_{[a,b],\infty}}{4(p+1)^{\frac{1}{p}}} \left[\left(\frac{3\left|f^{'}(a)\right|^q + \left|f^{'}(b)\right|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{\left|f^{'}(a)\right|^q + 3\left|f^{'}(b)\right|^q}{4} \right)^{\frac{1}{q}} \right].$$

Theorem 6. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and let $w : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. If $|f'|^q$ is convex on $[a,b]$, $q > 1$, then for all $x \in [a,b]$, the following inequality holds:*

$$\left| \left(\int_x^b w(s)ds \right)^\alpha f(b) - \left(\int_x^a w(s)ds \right)^\alpha f(a) - \alpha \int_a^b \left(\int_x^t w(s)ds \right)^{\alpha-1} w(t)f(t)dt \right|$$

$$\leq \frac{(b-a)^{\frac{1}{q}} \|w\|_{[a,b],\infty}^\alpha}{(\alpha p+1)^{\frac{1}{p}}} \left[(x-a)^{\alpha p+1} + (b-x)^{\alpha p+1} \right]^{\frac{1}{p}} \left(\frac{\left|f^{'}(a)\right|^q + \left|f^{'}(b)\right|^q}{2} \right)^{\frac{1}{q}}$$

where $\alpha > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, and $\|w\|_\infty = \sup_{t \in [a,b]} |w(t)|$.

Proof. We take absolute value of (2.1). Using Holder's inequality and the convexity of $\left|f'\right|^q$, we find that

$$\begin{aligned}
&\left|\left(\int_x^b w(s)ds\right)^{\alpha} f(b)-\left(\int_x^a w(s)ds\right)^{\alpha} f(a)-\alpha\int_a^b\left(\int_x^t w(s)ds\right)^{\alpha-1} w(t)f(t)dt\right| \\
&\leq \left(\int_a^b\left|\int_x^t w(s)ds\right|^{\alpha p} dt\right)^{\frac{1}{p}}\left(\int_a^b\left|f'(t)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \|w\|^{\alpha}_{[a,b],\infty}\left(\int_a^b|t-x|^{\alpha p}\,dt\right)^{\frac{1}{p}}\left(\int_a^b\left[\frac{b-t}{b-a}\left|f'(a)\right|^q+\frac{t-a}{b-a}\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&= \frac{(b-a)^{\frac{1}{q}}\|w\|^{\alpha}_{[a,b],\infty}}{(\alpha p+1)^{\frac{1}{p}}}\left[(x-a)^{\alpha p+1}+(b-x)^{\alpha p+1}\right]^{\frac{1}{p}}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}
\end{aligned}$$

which this completes the proof. □

Corollary 8. *Under the same assumptions of Theorem 6 with $w(s)=1$, then the following inequality holds:*

$$\begin{aligned}
(2.5)\qquad &\left|(b-x)^{\alpha} f(b)-(a-x)^{\alpha} f(a)-\alpha\int_a^b (t-x)^{\alpha-1} f(t)dt\right| \\
&\leq \frac{(b-a)^{\frac{1}{q}}}{(\alpha p+1)^{\frac{1}{p}}}\left[(x-a)^{\alpha p+1}+(b-x)^{\alpha p+1}\right]^{\frac{1}{p}}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

Remark 2. *Let the conditions of Corollary 8 hold. If we take $\alpha=1$ and $x=\frac{a+b}{2}$ in (2.5), then the inequality becomes the inequality (1.4).*

Corollary 9 (Fejer Type Inequality)**.** *Under the same assumptions of Theorem 6 with $\alpha=1$, then the following inequality holds:*

$$\begin{aligned}
&\left|f(b)\int_x^b w(s)ds+f(a)\int_a^x w(s)ds-\int_a^b w(t)f(t)dt\right| \\
&\leq \frac{(b-a)^{\frac{1}{q}}\|w\|_{[a,b],\infty}}{(p+1)^{\frac{1}{p}}}\left[(x-a)^{p+1}+(b-x)^{p+1}\right]^{\frac{1}{p}}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

Corollary 10 (Weighted Trapezoid Inequality)**.** *Let $w:[a,b]\to\mathbb{R}$ be symmetric to $\frac{a+b}{2}$ and $x=\frac{a+b}{2}$ in Corollary 9. Then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b w(s)ds-\int_a^b w(t)f(t)dt\right|\leq\frac{(b-a)^2\|w\|_{[a,b],\infty}}{2(p+1)^{\frac{1}{p}}}\left(\frac{\left|f'(a)\right|^q+\left|f'(b)\right|^q}{2}\right)^{\frac{1}{q}}.$$

Theorem 7. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$ and let $w : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$. If $|f'|^q$ is convex on $[a, b]$, $q > 1$, then for all $x \in [a, b]$, the following inequality holds:*

$$\begin{aligned}
&\left| \left(\int_x^b w(s)ds \right)^\alpha f(b) - \left(\int_x^a w(s)ds \right)^\alpha f(a) - \alpha \int_a^b \left(\int_x^t w(s)ds \right)^{\alpha-1} w(t)f(t)dt \right| \\
\leq\ & \frac{\|w\|_{[a,b],\infty}^\alpha}{(\alpha+1)(\alpha+2)^{\frac{1}{q}}(b-a)^{\frac{1}{q}}} \left((x-a)^{\alpha+1} + (b-x)^{\alpha+1} \right)^{\frac{1}{p}} \\
& \times \Big(\left((\alpha+1)(b-a)(x-a)^{\alpha+1} + (b-x)\left[(x-a)^{\alpha+1} + (b-x)^{\alpha+1} \right] \right) \left| f'(a) \right|^q \\
& + \left((\alpha+1)(b-a)(b-x)^{\alpha+1} + (x-a)\left[(x-a)^{\alpha+1} + (b-x)^{\alpha+1} \right] \right) \left| f'(b) \right|^q \Big)^{\frac{1}{q}}
\end{aligned}$$

where $\alpha > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, and $\|w\|_\infty = \sup_{t\in[a,b]} |w(t)|$.

Proof. We take absolute value of (2.1). Using Holder's inequality and the convexity of $\left|f'\right|^q$, we find that

$$\begin{aligned}
&\left| \left(\int_x^b w(s)ds \right)^\alpha f(b) - \left(\int_x^a w(s)ds \right)^\alpha f(a) - \alpha \int_a^b \left(\int_x^t w(s)ds \right)^{\alpha-1} w(t)f(t)dt \right| \\
\leq\ & \left(\int_a^b \left| \int_x^t w(s)ds \right|^\alpha dt \right)^{\frac{1}{p}} \left(\int_a^b \left| \int_x^t w(s)ds \right|^\alpha \left| f'(t) \right|^q dt \right)^{\frac{1}{q}} \\
\leq\ & \|w\|_{[a,b],\infty}^\alpha \left(\int_a^b |t-x|^\alpha dt \right)^{\frac{1}{p}} \left(\int_a^b |t-x|^\alpha \left[\frac{b-t}{b-a} \left| f'(a) \right|^q + \frac{t-a}{b-a} \left| f'(b) \right|^q \right] dt \right)^{\frac{1}{q}} \\
=\ & \|w\|_{[a,b],\infty}^\alpha \left(\frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{\alpha+1} \right)^{\frac{1}{p}} \\
& \times \Bigg(\left(\frac{(b-x)(x-a)^{\alpha+1}}{\alpha+1} + \frac{(x-a)^{\alpha+2}}{\alpha+2} \right) \left| f'(a) \right|^q + \left(\frac{(x-a)^{\alpha+2}}{(\alpha+1)(\alpha+2)} \right) \left| f'(b) \right|^q \\
& + \left(\frac{(b-x)^{\alpha+2}}{(\alpha+1)(\alpha+2)} \right) \left| f'(a) \right|^q + \left(\frac{(x-a)(b-x)^{\alpha+1}}{\alpha+1} + \frac{(b-x)^{\alpha+2}}{\alpha+2} \right) \left| f'(b) \right|^q \Bigg)^{\frac{1}{q}} \\
=\ & \frac{\|w\|_{[a,b],\infty}^\alpha}{(\alpha+1)(\alpha+2)^{\frac{1}{q}}(b-a)^{\frac{1}{q}}} \left((x-a)^{\alpha+1} + (b-x)^{\alpha+1} \right)^{\frac{1}{p}} \\
& \times \Big(\left((\alpha+1)(b-a)(x-a)^{\alpha+1} + (b-x)\left[(x-a)^{\alpha+1} + (b-x)^{\alpha+1} \right] \right) \left| f'(a) \right|^q \\
& + \left((\alpha+1)(b-a)(b-x)^{\alpha+1} + (x-a)\left[(x-a)^{\alpha+1} + (b-x)^{\alpha+1} \right] \right) \left| f'(b) \right|^q \Big)^{\frac{1}{q}}
\end{aligned}$$

which this completes the proof. □

Corollary 11. *Under the same assumptions of Theorem 7 with $w(s)=1$, then the following inequality holds:*

$$\left|(b-x)^{\alpha} f(b)-(a-x)^{\alpha} f(a)-\alpha \int_{a}^{b}(t-x)^{\alpha-1} f(t) d t\right| \tag{2.6}$$

$$\begin{aligned}
\leq\ & \frac{1}{(\alpha+1)(\alpha+2)^{\frac{1}{q}}(b-a)^{\frac{1}{q}}}\left((x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right)^{\frac{1}{p}} \\
& \times\left(\left((\alpha+1)(b-a)(x-a)^{\alpha+1}+(b-x)\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right]\right)\left|f'(a)\right|^{q}\right. \\
& \left.+\left((\alpha+1)(b-a)(b-x)^{\alpha+1}+(x-a)\left[(x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right]\right)\left|f'(b)\right|^{q}\right)^{\frac{1}{q}} .
\end{aligned}$$

Corollary 12. *Let the conditions of Corollary 11 hold. If we take $\alpha=1$ and $x=\frac{a+b}{2}$ in (2.6), then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a} \int_{a}^{b} f(t) d t\right| \leq \frac{(b-a)}{4}\left(\frac{\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{2}\right)^{\frac{1}{q}} .$$

Corollary 13 (Fejer Type Inequality)**.** *Under the same assumptions of Theorem 7 with $\alpha=1$, then the following inequality holds:*

$$\begin{aligned}
& \left|f(b) \int_{x}^{b} w(s) d s+f(a) \int_{a}^{x} w(s) d s-\int_{a}^{b} w(t) f(t) d t\right| \\
\leq\ & \frac{\|w\|_{[a, b], \infty}^{\alpha}}{2 \cdot 3^{\frac{1}{q}}(b-a)^{\frac{1}{q}}}\left((x-a)^{2}+(b-x)^{2}\right)^{\frac{1}{p}} \\
& \times\left(\left(2(b-a)(x-a)^{2}+(b-x)\left[(x-a)^{2}+(b-x)^{2}\right]\right)\left|f'(a)\right|^{q}\right. \\
& \left.+\left(2(b-a)(b-x)^{2}+(x-a)\left[(x-a)^{2}+(b-x)^{2}\right]\right)\left|f'(b)\right|^{q}\right)^{\frac{1}{q}} .
\end{aligned}$$

Corollary 14 (Weighted Trapezoid Inequality)**.** *Let $w:[a,b]\to\mathbb{R}$ be symmetric to $\frac{a+b}{2}$ and $x=\frac{a+b}{2}$ in Corollary 13. Then the following inequality holds:*

$$\begin{aligned}
& \left|\frac{f(a)+f(b)}{2} \int_{a}^{b} w(s) d s-\int_{a}^{b} w(t) f(t) d t\right| \\
\leq\ & \frac{(b-a)^{2}\|w\|_{[a, b], \infty}^{\alpha}}{4}\left(\frac{\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{2}\right)^{\frac{1}{q}} .
\end{aligned}$$

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