Received 14/07/14

SOME INEQUALITIES FOR DIFFERENTIABLE CO-ORDINATED CONVEX MAPPINGS

MEHMET ZEKI SARIKAYA

Abstract. In this paper, we indicate some inequalities for double integrals of functions whose partial derivatives in absolute value are convex on the co-ordinates on rectangle from the plane. Our established results generalize some recent results for functions whose partial derivatives in absolute value are convex on the co-ordinates on the rectangle from the plane.

1. Introduction

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. the following double inequality is well known in the literature as the Hermite-Hadamard inequality(see, e.g.,[12, p.137]):

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

Let us now consider a bidemensional interval $\Delta =: [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A mapping $f : \Delta \rightarrow \mathbb{R}$ is said to be convex on Δ if the following inequality:

$$f(tx + (1-t)z, ty + (1-t)w) \leq tf(x, y) + (1-t)f(z, w)$$

holds, for all $(x, y), (z, w) \in \Delta$ and $t \in [0, 1]$ .A function $f : \Delta \rightarrow \mathbb{R}$ is said to be on the co-ordinates on Δ if the partial mappings $f_y : [a, b] \rightarrow \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \rightarrow \mathbb{R}$, $f_x(v) = f(x, v)$ are convex where defined for all $x \in [a, b]$ and $y \in [c, d]$ (see [4]).

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1. *A function $f : \Delta \rightarrow \mathbb{R}$ will be called co-ordinated canvex on Δ, for all $t, s \in [0, 1]$ and $(x, y), (u, v) \in \Delta$,if the following inequality holds:*

$$\begin{aligned} & f(tx + (1-t)y, su + (1-s)v) \\ \leq\ & tsf(x, u) + s(1-t)f(y, u) + t(1-s)f(x, v) + (1-t)(1-s)f(y, v). \end{aligned}$$

Clearly, every convex function is co-ordinated convex. Furthermore, there exist co-ordinated convex function which is not convex, (see, [4]).

Also, in [4], Dragomir establish the following similar inequality of Hadamard's type for co-ordinated convex mapping on a rectangle from the plane $\mathbb{R}^2$.

2000 *Mathematics Subject Classification.* 26A51, 26D15.

Key words and phrases. convex function, co-ordinated convex mapping, Hermite-Hadamard inequality.

Theorem 1. *Suppose that $f:\Delta\rightarrow\mathbb{R}$ is co-ordinated convex on Δ. Then one has the inequalities:*

$$
\begin{aligned}
& f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
\leq\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)dy\right] \\
\leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\
\leq\ & \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx+\frac{1}{b-a}\int_a^b f(x,d)\,dx\right. \\
& \left.+\frac{1}{d-c}\int_c^d f(a,y)\,dy+\frac{1}{d-c}\int_c^d f(b,y)\,dy\right] \\
\leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned} \tag{1.1}
$$

The above inequalities are sharp.

In [13], Sarikaya et al. proved some new inequalities that give estimate of the deference between the middle and the right most terms in (1.1) for differentiable co-ordinated convex functions on rectangele from the plane $\mathbb{R}^2$. For several recent results concerning Hermite-Hadamard's inequality for some convex function on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$, we refer the reader to ([1]-[11], [13]).

The main purpose of this paper is to establish some weighted Hermite-Hadamard-type inequalities of convex functions of 2-variables on the co-ordinates.

2. Inequalities for co-ordinated convex functions

Firstly, we give the following notations used to simplify the details of presentation,

$$
\begin{aligned}
A(x,y) =\ & \left\{\left[\frac{(b-x)(x-a)^2}{2}+\frac{(x-a)^3}{3}\right]\left[\frac{(d-y)(y-c)^2}{2}+\frac{(y-c)^3}{3}\right]\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|\right. \\
& +\left[\frac{(b-x)(x-a)^2}{2}+\frac{(x-a)^3}{3}\right]\frac{(y-c)^3}{6}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right| \\
& \left.+\frac{(x-a)^3}{6}\left[\frac{(d-y)(y-c)^2}{2}+\frac{(y-c)^3}{3}\right]\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\frac{(x-a)^3(y-c)^3}{36}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right\},
\end{aligned}
$$

$$
\begin{aligned}
B(x,y) =\ & \left\{\left[\frac{(b-x)(x-a)^2}{2}+\frac{(x-a)^3}{3}\right]\frac{(d-y)^3}{6}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|\right. \\
& +\left[\frac{(b-x)(x-a)^2}{2}+\frac{(x-a)^3}{3}\right]\left[\frac{(d-y)^2(y-c)}{2}+\frac{(d-y)^3}{3}\right]\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right| \\
& \left.+\frac{(x-a)^3(d-y)^3}{36}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\frac{(x-a)^3}{6}\left[\frac{(d-y)^2(y-c)}{2}+\frac{(d-y)^3}{3}\right]\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right\},
\end{aligned}
$$

$$\begin{aligned} C(x,y) &= \left\{ \frac{(b-x)^3}{6}\left[\frac{(d-y)(y-c)^2}{2}+\frac{(y-c)^3}{3}\right]\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right| \right. \\ &+\frac{(b-x)^3(y-c)^3}{36}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right| \\ &+\left[\frac{(b-x)^2(x-a)}{2}+\frac{(b-x)^3}{3}\right]\left[\frac{(d-y)(y-c)^2}{2}+\frac{(y-c)^3}{3}\right]\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right| \\ &\left. +\left[\frac{(b-x)^2(x-a)}{2}+\frac{(b-x)^3}{3}\right]\frac{(y-c)^3}{6}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right\}, \end{aligned}$$

$$\begin{aligned} D(x,y) &= \left\{ \frac{(b-x)^3(d-y)^3}{36}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right| \right. \\ &+\frac{(b-x)^3}{6}\left[\frac{(d-y)^2(y-c)}{2}+\frac{(d-y)^3}{3}\right]\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right| \\ &+\left[\frac{(b-x)^2(x-a)}{2}+\frac{(b-x)^3}{3}\right]\frac{(d-y)^3}{6}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right| \\ &\left. +\left[\frac{(b-x)^2(x-a)}{2}+\frac{(b-x)^3}{3}\right]\left[\frac{(d-y)^2(y-c)}{2}+\frac{(d-y)^3}{3}\right]\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right\}, \end{aligned}$$

$$\begin{aligned} A_1(x,y) &= \left\{ \frac{\left[(b-a)^2-(b-x)^2\right]\left[(d-c)^2-(d-y)^2\right]}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q \right. \\ &+\frac{\left[(b-a)^2-(b-x)^2\right](y-c)^2}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q \\ &+\frac{(x-a)^2\left[(d-c)^2-(d-y)^2\right]}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q \\ &\left. +\frac{(x-a)^2(y-c)^2}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}, \end{aligned}$$

$$\begin{aligned} B_1(x,y) &= \left\{ \frac{\left[(b-a)^2-(b-x)^2\right](d-y)^2}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q \right. \\ &+\frac{\left[(b-a)^2-(b-x)^2\right]\left[(d-c)^2-(y-c)^2\right]}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q \\ &+\frac{(x-a)^2(d-y)^2}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q \\ &\left. +\frac{(x-a)^2\left[(d-c)^2-(y-c)^2\right]}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}, \end{aligned}$$

$$
\begin{aligned}
C_1(x,y) &= \left\{ \frac{(b-x)^2\left[(d-c)^2-(d-y)^2\right]}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q \right. \\
&+\frac{(b-x)^2(y-c)^2}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q \\
&+\frac{\left[(b-a)^2-(x-a)^2\right]\left[(d-c)^2-(d-y)^2\right]}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q \\
&\left. +\frac{\left[(b-a)^2-(x-a)^2\right](y-c)^2}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q \right\},
\end{aligned}
$$

$$
\begin{aligned}
D_1(x,y) &= \left\{ \frac{(b-x)^2(d-y)^2}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q \right. \\
&+\frac{(b-x)^2\left[(d-c)^2-(y-c)^2\right]}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q \\
&+\frac{\left[(b-a)^2-(x-a)^2\right](d-y)^2}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q \\
&\left. +\frac{\left[(b-a)^2-(x-a)^2\right]\left[(d-c)^2-(y-c)^2\right]}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q \right\}.
\end{aligned}
$$

Lemma 1. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}^2$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. If $\frac{\partial^2 f}{\partial t\partial s}\in L(\Delta)$, then the following equality holds:*

$$
\begin{aligned}
(2.1)\qquad & \int_a^b\int_c^d\left[\int_x^t\int_y^s w(u,v)dvdu\right]\frac{\partial^2 f}{\partial t\partial s}(t,s)\,dsdt \\
= & \left(\int_x^b\int_y^d w(u,v)dvdu\right)f(b,d)-\left(\int_x^b\int_y^c w(u,v)dvdu\right)f(b,c) \\
& -\left(\int_x^a\int_y^d w(u,v)dvdu\right)f(a,d)+\left(\int_x^a\int_y^c w(u,v)dvdu\right)f(a,c) \\
& -\int_a^b\left[\left(\int_y^d w(t,v)dv\right)f(t,d)-\left(\int_y^c w(t,v)dv\right)f(t,c)\right]dt \\
& -\int_c^d\left[\left(\int_x^b w(u,s)du\right)f(b,s)-\left(\int_x^a w(u,s)du\right)f(a,s)\right]ds \\
& +\int_a^b\int_c^d w(t,s)f(t,s)\,dsdt
\end{aligned}
$$

for all $(x,y)\in\Delta$.

Proof. By integration by parts, we get

$$
\begin{aligned}
&\int_a^b\int_c^d\left[\int_x^t\int_y^s w(u,v)dvdu\right]\frac{\partial^2 f}{\partial t\partial s}(t,s)\,dsdt \qquad (2.2)\\
=\;&\int_a^b\left(\left[\int_x^t\int_y^s w(u,v)dvdu\right]\frac{\partial f}{\partial s}(t,s)\Big|_c^d-\int_c^d\left[\int_x^t w(u,s)du\right]\frac{\partial f}{\partial s}(t,s)\,ds\right)dt\\
=\;&\int_a^b\left\{\left[\int_x^t\int_y^d w(u,v)dvdu\right]\frac{\partial f}{\partial s}(t,d)-\left[\int_x^t\int_y^c w(u,v)dvdu\right]\frac{\partial f}{\partial s}(t,c)\right\}dt\\
&-\int_a^b\int_c^d\left[\int_x^t w(u,s)dv\right]\frac{\partial f}{\partial t}(t,s)\,dsdt.
\end{aligned}
$$

Thus, again by integration by parts in the right hand side of (2.2), it follows that

$$
\begin{aligned}
&\int_a^b\int_c^d\left[\int_x^t\int_y^s w(u,v)dvdu\right]\frac{\partial^2 f}{\partial t\partial s}(t,s)\,dsdt\\
=\;&\left[\left(\int_x^t\int_y^d w(u,v)dvdu\right)f(t,d)-\left(\int_x^t\int_y^c w(u,v)dvdu\right)f(t,c)\right]\Bigg|_a^b\\
&-\int_a^b\left[\left(\int_y^d w(t,v)dv\right)f(t,d)-\left(\int_y^c w(t,v)dvdu\right)f(t,c)\right]dt\\
&-\int_c^d\left\{\left(\int_x^t w(u,s)dv\right)f(t,s)\Big|_a^b-\int_a^b w(t,s)f(t,s)dt\right\}ds\\
=\;&\left(\int_x^b\int_y^d w(u,v)dvdu\right)f(b,d)-\left(\int_x^b\int_y^c w(u,v)dvdu\right)f(b,c)\\
&-\left(\int_x^a\int_y^d w(u,v)dvdu\right)f(a,d)+\left(\int_x^a\int_y^c w(u,v)dvdu\right)f(a,c)\\
&-\int_a^b\left[\left(\int_y^d w(t,v)dv\right)f(t,d)-\left(\int_y^c w(t,v)dv\right)f(t,c)\right]dt\\
&-\int_c^d\left[\left(\int_x^b w(u,s)du\right)f(b,s)-\left(\int_x^a w(u,s)du\right)f(a,s)\right]ds\\
&+\int_a^b\int_c^d w(t,s)\,f(t,s)\,dsdt,
\end{aligned}
$$

which completes the proof. □

Corollary 1. *Under the same assumptions of Lemma 1 with $w(u,v)=1$, then the following identity holds:*

$$\begin{aligned}
&\int_a^b\int_c^d (t-x)(s-y)\frac{\partial^2 f}{\partial t\partial s}(t,s)\,dsdt \quad (2.3)\\
&= (b-x)(d-y)f(b,d)+(b-x)(y-c)f(b,c)\\
&\quad +(x-a)(d-y)f(a,d)+(x-a)(y-c)f(a,c)\\
&\quad -\int_a^b \left[(d-y)f(t,d)+(y-c)f(t,c)\right]dt\\
&\quad -\int_c^d \left[(b-x)f(b,s)+(x-a)f(a,s)\right]ds\\
&\quad +\int_a^b\int_c^d w(t,s)f(t,s)\,dsdt
\end{aligned}$$

Remark 1. *If we take $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ in (2.3), the identity (2.3) reduces to*

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \left(t-\frac{a+b}{2}\right)\left(s-\frac{c+d}{2}\right)\frac{\partial^2 f}{\partial t\partial s}(t,s)\,dsdt\\
&= \frac{f(b,d)+f(b,c)+f(a,d)+f(a,c)}{4}-\frac{1}{2(b-a)}\int_a^b\left[f(t,d)+f(t,c)\right]dt\\
&\quad -\frac{1}{2(d-c)}\int_c^d\left[f(b,s)+f(a,s)\right]ds+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt
\end{aligned}$$

which is given by Sarikaya et al. in [13].

Theorem 2. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}^2$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ is a convex function on the co-ordinates on Δ, then one has the inequalities:*

$$\begin{aligned}
&\Bigg|\left(\int_x^b\int_y^d w(u,v)dvdu\right)f(b,d)-\left(\int_x^b\int_y^c w(u,v)dvdu\right)f(b,c) \quad (2.4)\\
&-\left(\int_x^a\int_y^d w(u,v)dvdu\right)f(a,d)+\left(\int_x^a\int_y^c w(u,v)dvdu\right)f(a,c)\\
&-\int_a^b\left[\left(\int_y^d w(t,v)dv\right)f(t,d)-\left(\int_y^c w(t,v)dv\right)f(t,c)\right]dt\\
&-\int_c^d\left[\left(\int_x^b w(u,s)du\right)f(b,s)-\left(\int_x^a w(u,s)du\right)f(a,s)\right]ds\\
&+\int_a^b\int_c^d w(t,s)f(t,s)\,dsdt\Bigg|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\|w\|_{[a,x]\times[c,y]}}{(b-a)(d-c)}A(x,y)+\frac{\|w\|_{[a,x]\times[y,d]}}{(b-a)(d-c)}B(x,y) \\
&\quad +\frac{\|w\|_{[x,b]\times[c,y]}}{(b-a)(d-c)}C(x,y)+\frac{\|w\|_{[x,b]\times[y,d]}}{(b-a)(d-c)}D(x,y) \\
&\leq \frac{\|w\|_{[a,b]\times[c,d]}}{(b-a)(d-c)}\left[A(x,y)+B(x,y)+C(x,y)+D(x,y)\right]
\end{aligned}$$

for all $(x,y)\in\Delta$.

Proof. From Lemma 1, we have

$$\begin{aligned}
(2.5)\qquad &\left|\left(\int_x^b\int_y^d w(u,v)dvdu\right)f(b,d)-\left(\int_x^b\int_y^c w(u,v)dvdu\right)f(b,c)\right. \\
&-\left(\int_x^a\int_y^d w(u,v)dvdu\right)f(a,d)+\left(\int_x^a\int_y^c w(u,v)dvdu\right)f(a,c) \\
&-\int_a^b\left[\left(\int_y^d w(t,v)dv\right)f(t,d)-\left(\int_y^c w(t,v)dv\right)f(t,c)\right]dt \\
&-\int_c^d\left[\left(\int_x^b w(u,s)du\right)f(b,s)-\left(\int_x^a w(u,s)du\right)f(a,s)\right]ds \\
&\left.+\int_a^b\int_c^d w(t,s)f(t,s)dsdt\right| \\
\leq\ &\int_a^x\int_c^y\left[\left|\int_x^t\int_y^s w(u,v)dvdu\right|\right]\left|\frac{\partial^2 f}{\partial t\partial s}(t,s)\right|dsdt \\
&+\int_a^x\int_y^d\left[\left|\int_x^t\int_y^s w(u,v)dvdu\right|\right]\left|\frac{\partial^2 f}{\partial t\partial s}(t,s)\right|dsdt \\
&+\int_x^b\int_c^y\left[\left|\int_x^t\int_y^s w(u,v)dvdu\right|\right]\left|\frac{\partial^2 f}{\partial t\partial s}(t,s)\right|dsdt \\
&+\int_x^b\int_y^d\left[\left|\int_x^t\int_y^s w(u,v)dvdu\right|\right]\left|\frac{\partial^2 f}{\partial t\partial s}(t,s)\right|dsdt \\
\leq\ &\|w\|_{[a,x]\times[c,y]}\int_a^x\int_c^y\left(\int_t^x\int_s^y dvdu\right)\left|\frac{\partial^2 f}{\partial t\partial s}(t,s)\right|dsdt \\
&+\|w\|_{[a,x]\times[y,d]}\int_a^x\int_y^d\left(\int_t^x\int_y^s dvdu\right)\left|\frac{\partial^2 f}{\partial t\partial s}(t,s)\right|dsdt \\
&+\|w\|_{[x,b]\times[c,y]}\int_x^b\int_c^y\left(\int_x^t\int_s^y dvdu\right)\left|\frac{\partial^2 f}{\partial t\partial s}(t,s)\right|dsdt \\
&+\|w\|_{[x,b]\times[y,d]}\int_x^b\int_y^d\left(\int_x^t\int_y^s dvdu\right)\left|\frac{\partial^2 f}{\partial t\partial s}(t,s)\right|dsdt
\end{aligned}$$

for all $(x,y)\in\Delta$. Firstly, by calculating the last integral in (2.5), since $f:\Delta\rightarrow\mathbb{R}$ is co-ordinated convex on Δ, then one has:

$$
\begin{aligned}
(2.6)\quad & \int_a^x\int_c^y\left(\int_t^x\int_s^y dvdu\right)\left|\frac{\partial^2 f}{\partial t\partial s}(t,s)\right|dsdt \\
= & \int_a^x\int_c^y (x-t)(y-s)\left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b,\frac{d-s}{d-c}c+\frac{s-c}{d-c}d\right)\right|dsdt \\
\leq & \frac{1}{(b-a)(d-c)}\int_a^x\int_c^y (x-t)(y-s)\left[(b-t)(d-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|\right. \\
& +(b-t)(s-c)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+(t-a)(d-s)\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right| \\
& \left.+(t-a)(s-c)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right]dsdt \\
= & \frac{1}{(b-a)(d-c)}\left\{\left[\frac{(b-x)(x-a)^2}{2}+\frac{(x-a)^3}{3}\right]\frac{(y-c)^3}{6}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|\right. \\
& +\left[\frac{(b-x)(x-a)^2}{2}+\frac{(x-a)^3}{3}\right]\left[\frac{(d-y)(y-c)^2}{2}+\frac{(y-c)^3}{3}\right]\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right| \\
& +\frac{(x-a)^3}{6}\left[\frac{(d-y)(y-c)^2}{2}+\frac{(y-c)^3}{3}\right]\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right| \\
& \left.+\frac{(x-a)^3(y-c)^3}{36}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right\}.
\end{aligned}
$$

A similar way for other integrals, since $f:\Delta\rightarrow\mathbb{R}$ is co-ordinated convex on Δ, we get

$$
\begin{aligned}
(2.7)\quad & \int_a^x\int_y^d\left(\int_t^x\int_y^s dvdu\right)\left|\frac{\partial^2 f}{\partial t\partial s}(t,s)\right|dsdt \\
\leq & \frac{1}{(b-a)(d-c)}\left\{\left[\frac{(b-x)(x-a)^2}{2}+\frac{(x-a)^3}{3}\right]\frac{(d-y)^3}{6}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|\right. \\
& +\left[\frac{(b-x)(x-a)^2}{2}+\frac{(x-a)^3}{3}\right]\left[\frac{(d-y)^2(y-c)}{2}+\frac{(d-y)^3}{3}\right]\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right| \\
& +\frac{(x-a)^3(d-y)^3}{36}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right| \\
& \left.+\frac{(x-a)^3}{6}\left[\frac{(d-y)^2(y-c)}{2}+\frac{(d-y)^3}{3}\right]\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right\},
\end{aligned}
$$

$$
\begin{aligned}
(2.8)\quad & \int_x^b\int_c^y\left(\int_x^t\int_s^y dvdu\right)\left|\frac{\partial^2 f}{\partial t\partial s}(t,s)\right|dsdt \\
\leq\quad & \frac{1}{(b-a)(d-c)}\left\{\frac{(b-x)^3}{6}\left[\frac{(d-y)(y-c)^2}{2}+\frac{(y-c)^3}{3}\right]\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|\right. \\
& +\frac{(b-x)^3(y-c)^3}{36}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right| \\
& +\left[\frac{(b-x)^2(x-a)}{2}+\frac{(b-x)^3}{3}\right]\left[\frac{(d-y)(y-c)^2}{2}+\frac{(y-c)^3}{3}\right]\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right| \\
& \left.+\left[\frac{(b-x)^2(x-a)}{2}+\frac{(b-x)^3}{3}\right]\frac{(y-c)^3}{6}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right\},
\end{aligned}
$$

$$
\begin{aligned}
(2.9)\quad & \int_x^b\int_y^d\left(\int_x^t\int_y^s dvdu\right)\left|\frac{\partial^2 f}{\partial t\partial s}(t,s)\right|dsdt \\
\leq\quad & \frac{1}{(b-a)(d-c)}\left\{\frac{(b-x)^3(d-y)^3}{36}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|\right. \\
& +\frac{(b-x)^3}{6}\left[\frac{(d-y)^2(y-c)}{2}+\frac{(d-y)^3}{3}\right]\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right| \\
& +\left[\frac{(b-x)^2(x-a)}{2}+\frac{(b-x)^3}{3}\right]\frac{(d-y)^3}{6}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right| \\
& \left.+\left[\frac{(b-x)^2(x-a)}{2}+\frac{(b-x)^3}{3}\right]\left[\frac{(d-y)^2(y-c)}{2}+\frac{(d-y)^3}{3}\right]\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right\},
\end{aligned}
$$

Thus, putting (2.6)-(2.9) into (2.5), we obtain

$$
\begin{aligned}
(2.10)\qquad & \left|\left(\int_x^b\int_y^d w(u,v)dvdu\right)f(b,d)-\left(\int_x^b\int_y^c w(u,v)dvdu\right)f(b,c)\right. \\
& -\left(\int_x^a\int_y^d w(u,v)dvdu\right)f(a,d)+\left(\int_x^a\int_y^c w(u,v)dvdu\right)f(a,c) \\
& -\int_a^b\left[\left(\int_y^d w(t,v)dv\right)f(t,d)-\left(\int_y^c w(t,v)dv\right)f(t,c)\right]dt \\
& -\int_c^d\left[\left(\int_x^b w(u,s)du\right)f(b,s)-\left(\int_x^a w(u,s)du\right)f(a,s)\right]ds \\
& \left.+\int_a^b\int_c^d w(t,s)f(t,s)dsdt\right|
\end{aligned}
$$

$$
\begin{aligned}
\leq\; & \frac{\|w\|_{[a,x]\times[c,y]}}{(b-a)(d-c)}\left\{\left[\frac{(b-x)(x-a)^2}{2}+\frac{(x-a)^3}{3}\right]\right. \\
& \times\left[\frac{(d-y)(y-c)^2}{2}+\frac{(y-c)^3}{3}\right]\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right| \\
& +\left[\frac{(b-x)(x-a)^2}{2}+\frac{(x-a)^3}{3}\right]\frac{(y-c)^3}{6}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right| \\
& +\frac{(x-a)^3}{6}\left[\frac{(d-y)(y-c)^2}{2}+\frac{(y-c)^3}{3}\right]\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right| \\
& \left.+\frac{(x-a)^3(y-c)^3}{36}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right\} \\
& +\frac{\|w\|_{[a,x]\times[y,d]}}{(b-a)(d-c)}\left\{\left[\frac{(b-x)(x-a)^2}{2}+\frac{(x-a)^3}{3}\right]\frac{(d-y)^3}{6}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|\right. \\
& +\left[\frac{(b-x)(x-a)^2}{2}+\frac{(x-a)^3}{3}\right]\left[\frac{(d-y)^2(y-c)}{2}+\frac{(d-y)^3}{3}\right]\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right| \\
& +\frac{(x-a)^3(d-y)^3}{36}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right| \\
& \left.+\frac{(x-a)^3}{6}\left[\frac{(d-y)^2(y-c)}{2}+\frac{(d-y)^3}{3}\right]\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right\} \\
& +\frac{\|w\|_{[x,b]\times[c,y]}}{(b-a)(d-c)}\left\{\frac{(b-x)^3}{6}\left[\frac{(d-y)(y-c)^2}{2}+\frac{(y-c)^3}{3}\right]\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|\right. \\
& +\frac{(b-x)^3(y-c)^3}{36}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right| \\
& +\left[\frac{(b-x)^2(x-a)}{2}+\frac{(b-x)^3}{3}\right]\left[\frac{(d-y)(y-c)^2}{2}+\frac{(y-c)^3}{3}\right]\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right| \\
& \left.+\left[\frac{(b-x)^2(x-a)}{2}+\frac{(b-x)^3}{3}\right]\frac{(y-c)^3}{6}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right\} \\
& +\frac{\|w\|_{[x,b]\times[y,d]}}{(b-a)(d-c)}\left\{\frac{(b-x)^3(d-y)^3}{36}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|\right. \\
& +\frac{(b-x)^3}{6}\left[\frac{(d-y)^2(y-c)}{2}+\frac{(d-y)^3}{3}\right]\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right| \\
& +\left[\frac{(b-x)^2(x-a)}{2}+\frac{(b-x)^3}{3}\right]\frac{(d-y)^3}{6}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right| \\
& \left.+\left[\frac{(b-x)^2(x-a)}{2}+\frac{(b-x)^3}{3}\right]\left[\frac{(d-y)^2(y-c)}{2}+\frac{(d-y)^3}{3}\right]\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right\}
\end{aligned}
$$

By using $\|w\|_{[a,x]\times[c,y]} \leq \|w\|_{[a,b]\times[c,d]}$ in the last expression in (2.10), we get the inequalities (2.4). □

Corollary 2. *Under the same assumptions of Theorem 2 with $w(u,v)=1$, then the following inequality holds:*

$$\begin{aligned}
(2.11)\qquad & \Big|(b-x)(d-y)f(b,d)+(b-x)(y-c)f(b,c) \\
& +(x-a)(d-y)f(a,d)+(x-a)(y-c)f(a,c) \\
& -\int_a^b [(d-y)f(t,d)+(y-c)f(t,c)]\,dt \\
& -\int_c^d [(b-x)f(b,s)+(x-a)f(a,s)]\,ds \\
& +\int_a^b\int_c^d f(t,s)\,dsdt\Big| \\
\leq\ & \frac{1}{(b-a)(d-c)}\left[A(x,y)+B(x,y)+C(x,y)+D(x,y)\right].
\end{aligned}$$

Corollary 3. *Let $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ in Corollary 2. Then the following inequality holds:*

$$\begin{aligned}
& \Bigg|\frac{f(b,d)+f(b,c)+f(a,d)+f(a,c)}{4}-\frac{1}{2(b-a)}\int_a^b [f(t,d)+f(t,c)]\,dt \\
& -\frac{1}{2(d-c)}\int_c^d [f(b,s)+f(a,s)]\,ds+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\Bigg| \\
\leq\ & \frac{(b-a)^2(d-c)^2}{16} \\
& \times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|}{4}\right)
\end{aligned}$$

which is proved by Sarikaya et al. in [13].

Theorem 3. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}^2$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $a<b$ and $c<d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q>1$, is a convex function* *on the co-ordinates on* Δ, *then for all* $(x,y)\in\Delta$ *one has the inequalities:*

$$
\begin{aligned}
(2.12)\quad & \left|\left(\int_x^b\int_y^d w(u,v)dvdu\right)f(b,d)-\left(\int_x^b\int_y^c w(u,v)dvdu\right)f(b,c)\right.\\
&-\left(\int_x^a\int_y^d w(u,v)dvdu\right)f(a,d)+\left(\int_x^a\int_y^c w(u,v)dvdu\right)f(a,c)\\
&-\int_a^b\left[\left(\int_y^d w(t,v)dv\right)f(t,d)-\left(\int_y^c w(t,v)dv\right)f(t,c)\right]dt\\
&-\int_c^d\left[\left(\int_x^b w(u,s)du\right)f(b,s)-\left(\int_x^a w(u,s)du\right)f(a,s)\right]ds\\
&\left.+\int_a^b\int_c^d w(t,s)f(t,s)dsdt\right|\\
\leq\ & \frac{1}{[(b-a)(d-c)]^{\frac{1}{p}}(p+1)^{\frac{2}{p}}}\left\{(x-a)^{1+\frac{1}{p}}(y-c)^{1+\frac{1}{p}}(A_1(x,y))^{\frac{1}{q}}\|w\|_{[a,x]\times[c,y]}\right.\\
&+(x-a)^{1+\frac{1}{p}}(d-y)^{1+\frac{1}{p}}(B_1(x,y))^{\frac{1}{q}}\|w\|_{[a,x]\times[y,d]}\\
&+(b-x)^{1+\frac{1}{p}}(y-c)^{1+\frac{1}{p}}(C_1(x,y))^{\frac{1}{q}}\|w\|_{[x,b]\times[c,y]}\\
&\left.+(b-x)^{1+\frac{1}{p}}(d-y)^{1+\frac{1}{p}}(D_1(x,y))^{\frac{1}{q}}\|w\|_{[x,b]\times[y,d]}\right\}\\
\leq\ & \frac{\|w\|_{[a,b]\times[c,d]}}{[(b-a)(d-c)]^{\frac{1}{p}}(p+1)^{\frac{2}{p}}}\left\{(x-a)^{1+\frac{1}{p}}(y-c)^{1+\frac{1}{p}}(A_1(x,y))^{\frac{1}{q}}\right.\\
&+(x-a)^{1+\frac{1}{p}}(d-y)^{1+\frac{1}{p}}(B_1(x,y))^{\frac{1}{q}}+(b-x)^{1+\frac{1}{p}}(y-c)^{1+\frac{1}{p}}(C_1(x,y))^{\frac{1}{q}}\\
&\left.+(b-x)^{1+\frac{1}{p}}(d-y)^{1+\frac{1}{p}}(D_1(x,y))^{\frac{1}{q}}\right\}
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1, we have

$$
\begin{aligned}
(2.13)\quad & \left|\left(\int_x^b\int_y^d w(u,v)dvdu\right)f(b,d)-\left(\int_x^b\int_y^c w(u,v)dvdu\right)f(b,c)\right.\\
&-\left(\int_x^a\int_y^d w(u,v)dvdu\right)f(a,d)+\left(\int_x^a\int_y^c w(u,v)dvdu\right)f(a,c)\\
&-\int_a^b\left[\left(\int_y^d w(t,v)dv\right)f(t,d)-\left(\int_y^c w(t,v)dv\right)f(t,c)\right]dt\\
&-\int_c^d\left[\left(\int_x^b w(u,s)du\right)f(b,s)-\left(\int_x^a w(u,s)du\right)f(a,s)\right]ds\\
&\left.+\int_a^b\int_c^d w(t,s)f(t,s)dsdt\right|
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \int_a^x \int_c^y \left[\left|\int_x^t \int_y^s w(u,v)dvdu\right|\right] \left|\frac{\partial^2 f}{\partial t \partial s}(t,s)\right| dsdt \\
&+\int_a^x \int_y^d \left[\left|\int_x^t \int_y^s w(u,v)dvdu\right|\right] \left|\frac{\partial^2 f}{\partial t \partial s}(t,s)\right| dsdt \\
&+\int_x^b \int_c^y \left[\left|\int_x^t \int_y^s w(u,v)dvdu\right|\right] \left|\frac{\partial^2 f}{\partial t \partial s}(t,s)\right| dsdt \\
&+\int_x^b \int_y^d \left[\left|\int_x^t \int_y^s w(u,v)dvdu\right|\right] \left|\frac{\partial^2 f}{\partial t \partial s}(t,s)\right| dsdt
\end{aligned}
$$

By using the well known Hölder inequality for double integrals, then one has:

$$
\begin{aligned}
(2.14)\quad & \left|\left(\int_x^b \int_y^d w(u,v)dvdu\right) f(b,d) - \left(\int_x^b \int_y^c w(u,v)dvdu\right) f(b,c)\right. \\
&-\left(\int_x^a \int_y^d w(u,v)dvdu\right) f(a,d) + \left(\int_x^a \int_y^c w(u,v)dvdu\right) f(a,c) \\
&-\int_a^b \left[\left(\int_y^d w(t,v)dv\right) f(t,d) - \left(\int_y^c w(t,v)dv\right) f(t,c)\right] dt \\
&-\int_c^d \left[\left(\int_x^b w(u,s)du\right) f(b,s) - \left(\int_x^a w(u,s)du\right) f(a,s)\right] ds \\
&\left.+\int_a^b \int_c^d w(t,s) f(t,s) dsdt\right| \\
\leq\ & \left(\int_a^x \int_c^y \left|\int_x^t \int_y^s w(u,v)dvdu\right|^p dsdt\right)^{\frac{1}{p}} \left(\int_a^x \int_c^y \left|\frac{\partial^2 f}{\partial t \partial s}(t,s)\right|^q dsdt\right)^{\frac{1}{q}} \\
&+\left(\int_a^x \int_y^d \left|\int_x^t \int_y^s w(u,v)dvdu\right|^p dsdt\right)^{\frac{1}{p}} \left(\int_a^x \int_y^d \left|\frac{\partial^2 f}{\partial t \partial s}(t,s)\right|^q dsdt\right)^{\frac{1}{q}} \\
&+\left(\int_x^b \int_c^y \left|\int_x^t \int_y^s w(u,v)dvdu\right|^p dsdt\right)^{\frac{1}{p}} \left(\int_x^b \int_c^y \left|\frac{\partial^2 f}{\partial t \partial s}(t,s)\right|^q dsdt\right)^{\frac{1}{q}} \\
&+\left(\int_x^b \int_y^d \left|\int_x^t \int_y^s w(u,v)dvdu\right|^p dsdt\right)^{\frac{1}{p}} \left(\int_x^b \int_y^d \left|\frac{\partial^2 f}{\partial t \partial s}(t,s)\right|^q dsdt\right)^{\frac{1}{q}}
\end{aligned}
$$

$$\begin{aligned}
&\leq \|w\|_{[a,x]\times[c,y]}\left(\int_a^x\int_c^y\left|\int_x^t\int_y^s dvdu\right|^p dsdt\right)^{\frac{1}{p}}\left(\int_a^x\int_c^y\left|\frac{\partial^2 f}{\partial t\partial s}(t,s)\right|^q dsdt\right)^{\frac{1}{q}}\\
&+\|w\|_{[a,x]\times[y,d]}\left(\int_a^x\int_y^d\left|\int_x^t\int_y^s dvdu\right|^p dsdt\right)^{\frac{1}{p}}\left(\int_a^x\int_y^d\left|\frac{\partial^2 f}{\partial t\partial s}(t,s)\right|^q dsdt\right)^{\frac{1}{q}}\\
&+\|w\|_{[x,b]\times[c,y]}\left(\int_x^b\int_c^y\left|\int_x^t\int_y^s dvdu\right|^p dsdt\right)^{\frac{1}{p}}\left(\int_x^b\int_c^y\left|\frac{\partial^2 f}{\partial t\partial s}(t,s)\right|^q dsdt\right)^{\frac{1}{q}}\\
&+\|w\|_{[x,b]\times[y,d]}\left(\int_x^b\int_y^d\left|\int_x^t\int_y^s dvdu\right|^p dsdt\right)^{\frac{1}{p}}\left(\int_x^b\int_y^d\left|\frac{\partial^2 f}{\partial t\partial s}(t,s)\right|^q dsdt\right)^{\frac{1}{q}}
\end{aligned}$$

The first factors in the last integral in (2.14) are

$$\begin{aligned}
\int_a^x\int_c^y\left|\int_x^t\int_y^s dvdu\right|^p dsdt &= \int_a^x\int_c^y (x-t)^p(y-s)^p dsdt \\
&= \frac{(x-a)^{p+1}(y-c)^{p+1}}{(p+1)^2}
\end{aligned} \tag{2.15}$$

and similarly

$$\begin{aligned}
\int_a^x\int_y^d\left|\int_x^t\int_y^s dvdu\right|^p dsdt &= \frac{(x-a)^{p+1}(d-y)^{p+1}}{(p+1)^2},\\
\int_x^b\int_c^y\left|\int_x^t\int_y^s dvdu\right|^p dsdt &= \frac{(b-x)^{p+1}(y-c)^{p+1}}{(p+1)^2},\\
\int_x^b\int_y^d\left|\int_x^t\int_y^s dvdu\right|^p dsdt &= \frac{(b-x)^{p+1}(d-y)^{p+1}}{(p+1)^2}.
\end{aligned} \tag{2.16}$$

On the other hand, since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is convex function on the co-ordinates on Δ, we know that

$$\begin{aligned}
\left|\frac{\partial^2 f}{\partial t\partial s}(t,s)\right|^q &= \left|\frac{\partial^2 f}{\partial t\partial s}\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b,\frac{d-s}{d-c}c+\frac{s-c}{d-c}d\right)\right|^q\\
&\leq \frac{(b-t)(d-s)}{(b-a)(d-c)}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\frac{(b-t)(s-c)}{(b-a)(d-c)}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q\\
&+\frac{(t-a)(d-s)}{(b-a)(d-c)}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\frac{(t-a)(s-c)}{(b-a)(d-c)}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q
\end{aligned}$$

hence, the second factors in the last integral in (2.14) are

$$\int_a^x\int_c^y\left|\frac{\partial^2 f}{\partial t\partial s}(t,s)\right|^q dsdt \tag{2.17}$$
$$\leq \frac{1}{(b-a)(d-c)}\left\{\frac{\left[(b-a)^2-(b-x)^2\right]\left[(d-c)^2-(d-y)^2\right]}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\right.$$
$$+\frac{\left[(b-a)^2-(b-x)^2\right](y-c)^2}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q$$
$$+\frac{(x-a)^2\left[(d-c)^2-(d-y)^2\right]}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q$$
$$\left.+\frac{(x-a)^2(y-c)^2}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\},$$

$$\int_a^x\int_y^d\left|\frac{\partial^2 f}{\partial t\partial s}(t,s)\right|^q dsdt \tag{2.18}$$
$$\leq \frac{1}{(b-a)(d-c)}\left\{\frac{\left[(b-a)^2-(b-x)^2\right](d-y)^2}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\right.$$
$$+\frac{\left[(b-a)^2-(b-x)^2\right]\left[(d-c)^2-(y-c)^2\right]}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q$$
$$+\frac{(x-a)^2(d-y)^2}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q$$
$$\left.+\frac{(x-a)^2\left[(d-c)^2-(y-c)^2\right]}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\},$$

$$\int_x^b\int_c^y\left|\frac{\partial^2 f}{\partial t\partial s}(t,s)\right|^q dsdt \tag{2.19}$$
$$\leq \frac{1}{(b-a)(d-c)}\left\{\frac{(b-x)^2\left[(d-c)^2-(d-y)^2\right]}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q\right.$$
$$+\frac{(b-x)^2(y-c)^2}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q$$
$$+\frac{\left[(b-a)^2-(x-a)^2\right]\left[(d-c)^2-(d-y)^2\right]}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q$$
$$\left.+\frac{\left[(b-a)^2-(x-a)^2\right](y-c)^2}{4}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\},$$

$$
\begin{aligned}
(2.20) \quad & \int_x^b \int_y^d \left| \frac{\partial^2 f}{\partial t \partial s}(t,s) \right|^q dsdt \\
\leq \quad & \frac{1}{(b-a)(d-c)} \left\{ \frac{(b-x)^2 (d-y)^2}{4} \left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right|^q \right. \\
& + \frac{(b-x)^2 \left[(d-c)^2 - (y-c)^2 \right]}{4} \left| \frac{\partial^2 f}{\partial t \partial s}(a,d) \right|^q \\
& + \frac{\left[(b-a)^2 - (x-a)^2 \right] (d-y)^2}{4} \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right|^q \\
& \left. + \frac{\left[(b-a)^2 - (x-a)^2 \right] \left[(d-c)^2 - (y-c)^2 \right]}{4} \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right|^q \right\}.
\end{aligned}
$$

Putting (2.15)-(2.20) into (2.14) and by using $\|w\|_{[a,x]\times[c,y]} \leq \|w\|_{[a,b]\times[c,d]}$, we get required the inequalities(2.12). □

Corollary 4. *Under the same assumptions of Theorem 3 with $w(u,v)=1$, then the following inequality holds:*

$$
\begin{aligned}
(2.21) \quad & |(b-x)(d-y) f(b,d) + (b-x)(y-c) f(b,c) \\
& + (x-a)(d-y) f(a,d) + (x-a)(y-c) f(a,c) \\
& - \int_a^b \left[(d-y) f(t,d) + (y-c) f(t,c) \right] dt \\
& - \int_c^d \left[(b-x) f(b,s) + (x-a) f(a,s) \right] ds \\
& + \int_a^b \int_c^d f(t,s) \, dsdt \Bigg| \\
\leq \quad & \frac{1}{\left[(b-a)(d-c) \right]^{\frac{1}{p}} (p+1)^{\frac{2}{p}}} \left\{ (x-a)^{1+\frac{1}{p}} (y-c)^{1+\frac{1}{p}} \left(A_1(x,y) \right)^{\frac{1}{q}} \right. \\
& + (x-a)^{1+\frac{1}{p}} (d-y)^{1+\frac{1}{p}} \left(B_1(x,y) \right)^{\frac{1}{q}} + (b-x)^{1+\frac{1}{p}} (y-c)^{1+\frac{1}{p}} \left(C_1(x,y) \right)^{\frac{1}{q}} \\
& \left. + (b-x)^{1+\frac{1}{p}} (d-y)^{1+\frac{1}{p}} \left(D_1(x,y) \right)^{\frac{1}{q}} \right\}.
\end{aligned}
$$

Corollary 5. *Let $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in Corollary 4. Then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{f(b,d)+f(b,c)+f(a,d)+f(a,c)}{4}-\frac{1}{2(b-a)}\int_a^b [f(t,d)+f(t,c)]\,dt\right.\\
&\left.-\frac{1}{2(d-c)}\int_c^d [f(b,s)+f(a,s)]\,ds+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\right|\\
\leq\;& \frac{(b-a)^2(d-c)^2}{4^{4+\frac{1}{p}}(p+1)^{\frac{2}{p}}}\left\{\left(9\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right)^{\frac{1}{q}}\right.\\
&+\left(3\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+9\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right)^{\frac{1}{q}}\\
&+\left(3\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+9\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right)^{\frac{1}{q}}\\
&\left.+\left(\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+9\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Theorem 4. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a partial differentiable mapping on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q > 1$, is a convex function on the co-ordinates on Δ, then for all $(x,y) \in \Delta$ one has the inequalities:*

$$\begin{aligned}
&\left|\left(\int_x^b\int_y^d w(u,v)dvdu\right)f(b,d)-\left(\int_x^b\int_y^c w(u,v)dvdu\right)f(b,c)\right.\\
&-\left(\int_x^a\int_y^d w(u,v)dvdu\right)f(a,d)+\left(\int_x^a\int_y^c w(u,v)dvdu\right)f(a,c)\\
&-\int_a^b\left[\left(\int_y^d w(t,v)dv\right)f(t,d)-\left(\int_y^c w(t,v)dv\right)f(t,c)\right]dt\\
&-\int_c^d\left[\left(\int_x^b w(u,s)du\right)f(b,s)-\left(\int_x^a w(u,s)du\right)f(a,s)\right]ds\\
&\left.+\int_a^b\int_c^d w(t,s)\,f(t,s)\,dsdt\right|\\
\leq\;& \frac{[(b-a)(d-c)]^{\frac{1}{q}}\,\|w\|_{[a,b]\times[c,d]}}{4^{\frac{1}{q}}(p+1)^{\frac{2}{p}}}\\
&\times\left[(b-x)^{p+1}+(x-a)^{p+1}\right]^{\frac{1}{p}}\left[(d-y)^{p+1}+(y-c)^{p+1}\right]^{\frac{1}{p}}\\
&\times\left(\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right)^{\frac{1}{q}}
\end{aligned} \tag{2.22}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 1 and using the well known Hölder inequality for double integrals, we have

$$
\begin{aligned}
(2.23)\quad & \left| \left(\int_x^b \int_y^d w(u,v) dvdu \right) f(b,d) - \left(\int_x^b \int_y^c w(u,v) dvdu \right) f(b,c) \right. \\
& - \left(\int_x^a \int_y^d w(u,v) dvdu \right) f(a,d) + \left(\int_x^a \int_y^c w(u,v) dvdu \right) f(a,c) \\
& - \int_a^b \left[\left(\int_y^d w(t,v) dv \right) f(t,d) - \left(\int_y^c w(t,v) dv \right) f(t,c) \right] dt \\
& - \int_c^d \left[\left(\int_x^b w(u,s) du \right) f(b,s) - \left(\int_x^a w(u,s) du \right) f(a,s) \right] ds \\
& \left. + \int_a^b \int_c^d w(t,s) f(t,s) dsdt \right| \\
\leq & \left(\int_a^b \int_c^d \left| \int_x^t \int_y^s w(u,v) dvdu \right|^p dsdt \right)^{\frac{1}{p}} \left(\int_a^b \int_c^d \left| \frac{\partial^2 f}{\partial t \partial s}(t,s) \right|^q dsdt \right)^{\frac{1}{q}} \\
\leq & \|w\|_{[a,b]\times[c,d]} \left(\int_a^b \int_c^d \left| \int_x^t \int_y^s dvdu \right|^p dsdt \right)^{\frac{1}{p}} \left(\int_a^b \int_c^d \left| \frac{\partial^2 f}{\partial t \partial s}(t,s) \right|^q dsdt \right)^{\frac{1}{q}}.
\end{aligned}
$$

The first factors in the last integral in (2.23) is

$$
\begin{aligned}
(2.24)\quad & \int_a^b \int_c^d \left| \int_x^t \int_y^s dvdu \right|^p dsdt \\
= & \int_a^b \int_c^d |t-x|^p |s-y|^p dsdt \\
= & \frac{\left[(b-x)^{p+1} + (x-a)^{p+1} \right] \left[(d-y)^{p+1} + (y-c)^{p+1} \right]}{(p+1)^2}.
\end{aligned}
$$

On the other hand, since $\left| \frac{\partial^2 f}{\partial t \partial s} \right|^q$ is convex function on the co-ordinates on Δ, the second factors in the last integral in (2.14) is

$$
\begin{aligned}
(2.25)\quad & \int_a^b \int_c^d \left| \frac{\partial^2 f}{\partial t \partial s}(t,s) \right|^q dsdt \\
= & \int_a^b \int_c^d \left| \frac{\partial^2 f}{\partial t \partial s} \left(\frac{b-t}{b-a} a + \frac{t-a}{b-a} b, \frac{d-s}{d-c} c + \frac{s-c}{d-c} d \right) \right|^q dsdt \\
\leq & \frac{(b-a)(d-c)}{4} \left(\left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(a,d) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right|^q \right).
\end{aligned}
$$

Putting (2.24) and (2.25) in (2.23), we get the inequality (2.22). □

Corollary 6. *Under the same assumptions of Theorem 4 with $w(u,v)=1$, then the following inequality holds:*

$$
\begin{aligned}
(2.26)\qquad & \Big|(b-x)(d-y)f(b,d)+(b-x)(y-c)f(b,c) \\
& +(x-a)(d-y)f(a,d)+(x-a)(y-c)f(a,c) \\
& -\int_a^b\left[(d-y)f(t,d)+(y-c)f(t,c)\right]dt \\
& -\int_c^d\left[(b-x)f(b,s)+(x-a)f(a,s)\right]ds \\
& +\int_a^b\int_c^d f(t,s)\,dsdt\Big| \\
\leq\ & \frac{\left[(b-a)(d-c)\right]^{\frac{1}{q}}}{4^{\frac{1}{q}}(p+1)^{\frac{2}{p}}} \\
& \times\left[(b-x)^{p+1}+(x-a)^{p+1}\right]^{\frac{1}{p}}\left[(d-y)^{p+1}+(y-c)^{p+1}\right]^{\frac{1}{p}} \\
& \times\left(\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right)^{\frac{1}{q}}.
\end{aligned}
$$

Remark 2. *If we take $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ in (2.26), the inequality (2.26) reduces to*

$$
\begin{aligned}
& \Big|\frac{f(b,d)+f(b,c)+f(a,d)+f(a,c)}{4}-\frac{1}{2(b-a)}\int_a^b\left[f(t,d)+f(t,c)\right]dt \\
& -\frac{1}{2(d-c)}\int_c^d\left[f(b,s)+f(a,s)\right]ds+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt\Big| \\
\leq\ & \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}} \\
& \times\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{4}\right)^{\frac{1}{q}}
\end{aligned}
$$

which is proved by Sarikaya et al. in [13].

References

[1] M. Alomari and M. Darus, Co-ordinated s-convex function in the first sense with some Hadamard-type inequalities, *Int. J. Contemp. Math. Sciences,* 3 (32) (2008), 1557-1567.

[2] M. Alomari and M. Darus, On the Hadamard's inequality for *log*-convex functions on the coordinates, *J. of Inequal. and Appl,* Article ID 283147, (2009), 13 pages.

[3] F. Chen, On Hermite-Hadamard type inequalities for s-convex functions on the coordinates via Riemann-Liouville fractional integrals, *Journal of Applied Mathematics*, Volume 2014, Article ID 248710, 8 pages.

[4] S.S. Dragomir, On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane, *Taiwanese Journal of Mathematics,* 4 (2001), 775-788.
[5] M. A. Latif and M. Alomari, Hadamard-type inequalities for product two convex functions on the co-ordinetes, *Int. Math. Forum*, 4(47), 2009, 2327-2338.
[6] M. A. Latif and M. Alomari, On the Hadamard-type inequalities for h-convex functions on the co-ordinetes, *Int. J. of Math. Analysis*, 3(33), 2009, 1645-1656.
[7] M. A. Latif and S.S. Dragomir, On some new inequalities for differentiable co-ordinated convex functions, *Journal of Inequalities and Applications* 2012, 2012:28
[8] M.E. Özdemir, E. Set and M.Z. Sarikaya, New some Hadamard's type inequalities for co-ordinated m-convex and (α, m)-convex functions, *Hacettepe Journal of Mathematics and Statistics*, 40(2), 2011, 219-229.
[9] J. Park, Generalizations of the Simpson-like type inequalities for co-ordinated s-convex mappings in the second sense, *International Journal of Mathematics and Mathematical Sciences* Volume 2012, Article ID 715751, 16 pages.
[10] J. Park, Some Hadamard's type inequalities for co-ordinated (s, m)-convex mappings in the second sense, *Far East Journal of Mathematical Sciences*, vol. 51, no. 2, pp. 205–216, 2011.
[11] J. Park, New generalizations of Simpson's type inequalities for twice differentiable convex mappings, *Far East Journal of Mathematical Sciences*, vol. 52, no. 1, pp. 43–55, 2011.
[12] J.E. Pečarić, F. Proschan and Y.L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Boston, 1992.
[13] M. Z. Sarikaya, E. Set, M. E. Ozdemir, and S. S. Dragomir, New some Hadamard's type inequalities for co-ordinated convex functions, *Tamsui Oxford Journal of Information and Mathematical Sciences,* 28(2) (2012) 137-152.

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey

E-mail address: sarikayamz@gmail.com