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ON FEJÉR TYPE INEQUALITIES VIA FRACTIONAL INTEGRALS

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Abstract. In this paper, we extend some weighted version of the Hermite-Hadamard type and Fejér type inequalities for fractional integrals. The results presented here would provide extensions of those given in earlier works.

1. Introduction

Definition 1. *The function $f:[a,b]\subset\mathbb{R}\rightarrow\mathbb{R}$, is said to be convex if the following inequality holds*

$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. We say that f is concave if $(-f)$ is convex.

The following inequality is well known in the literature as the Hermite-Hadamard integral inequality (see, [8]):

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2} \tag{1.1}$$

where $f:I\subset\mathbb{R}\rightarrow\mathbb{R}$ is a convex function on the interval I of real numbers and $a,b\in I$ with $a<b$.

In [3] and [4], Dragomir et al. proved the following results connected with the Hermite-Hadamard inequality:

Theorem 1. *Let $f:[a,b]\rightarrow\mathbb{R}$ be a twice differentiable mapping such that there exists real constants m and M so that $m\leq f''\leq M$. Then, the following inequalities hold:*

$$m\frac{(b-a)^2}{24}\leq\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\leq M\frac{(b-a)^2}{24} \tag{1.2}$$

and

$$m\frac{(b-a)^2}{24}\leq\frac{1}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\leq M\frac{(b-a)^2}{24}. \tag{1.3}$$

The most well-known inequalities related to the integral mean of a convex function are the Hermite Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities (see, [6], [10]-[13], [16], [17]). In [5], Fejer gave a weighted generalizatinon of the inequalities (1.1) as the following:

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Theorem 2. *$f:[a,b]\to\mathbb{R}$, be a convex function, then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)dx \le \frac{1}{b-a}\int_a^b f(x)w(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b w(x)dx \tag{1.4}$$

holds, where $w:[a,b]\to\mathbb{R}$ is nonnegative, integrable, and symmetric about $x=\frac{a+b}{2}$ (i.e. $w(x)=w(a+b-x)$).

In [7], Minculete and Mitroi presented the following important inequalities;

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable mapping such that there exists real constants m and M so that $m\le f''\le M$. Then, the following inequalities hold:*

$$\begin{aligned} m\frac{\lambda(1-\lambda)}{2}(b-a)^2 &\le \lambda f(a)+(1-\lambda)f(b)-f(\lambda a+(1-\lambda)b) \\ &\le M\frac{\lambda(1-\lambda)}{2}(b-a)^2 \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} m\frac{(1-2\lambda)^2}{8}(b-a)^2 &\le \frac{f(\lambda a+(1-\lambda)b)+f((1-\lambda)a+\lambda b)}{2}-f\left(\frac{a+b}{2}\right) \\ &\le M\frac{(1-2\lambda)^2}{8}(b-a)^2 \end{aligned} \tag{1.6}$$

for $\lambda\in[0,1]$.

And using the Theorem 3, some inequalities of Hermite-Hadamard-Fejer type for differentiable mappings were proved as follows:

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be a twice differentiable mapping such that there exists real constants m and M so that $m\le f''\le M$. Assume $g:[a,b]\to\mathbb{R}$ is nonnegative, integrable, and symmetric about $x=\frac{a+b}{2}$. Then, the following inequalities hold:*

$$\begin{aligned} \frac{m}{2}\int_a^b (t-a)(b-t)g(t)dt &\le \frac{f(a)+f(b)}{2}\int_a^b g(t)dt-\int_a^b f(t)g(t)dt \\ &\le \frac{M}{2}\int_a^b (t-a)(b-t)g(t)dt \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} \frac{m}{8}\int_a^b (2t-a-b)^2 g(t)dt &\le \int_a^b f(t)g(t)dt-f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt \\ &\le \frac{M}{8}\int_a^b (2t-a-b)^2 g(t)dt. \end{aligned} \tag{1.8}$$

Definition 2. *Let $f\in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha>0$ with $a\ge 0$ are defined by*

$$J_{a^+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\quad x>a$$

and

$$J_{b^-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt,\quad x<b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a^+}^{0}f(x)=J_{b^-}^{0}f(x)=f(x)$.

Meanwhile, in [9], Sarikaya et al. first presented the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.9}$$

with $\alpha > 0$.

In recently, Iscan in [6] established the following Hermite-Hadamard-Fejer type inequalities for fractional integrals:

Theorem 6. *Let $f : [a,b] \to \mathbb{R}$ be a convex function with $0 \leq a < b$ and $f \in L[a,b]$. If $g : [a,b] \to \mathbb{R}$ is nonnegative, integrable, and symmetric about $x = \frac{a+b}{2}$ (i.e. $w(x) = w(a+b-x)$), then the following inequalities for fractional integrals hold:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right] &\leq \left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right] \\ &\leq \frac{f(a)+f(b)}{2}\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right] \end{aligned} \tag{1.10}$$

with $\alpha > 0$. Here, $J_{a+}^{\alpha}g(b) = J_{b-}^{\alpha}g(a) = \frac{1}{2}\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]$.

For some recent results connected with fractional integral inequalities see [1], [2], [6], [14], [15], [18].

In this article, using functions whose derivatives absolute values are convex, we obtained new inequalities of Hermite-Hadamard-Fejer type and Hermite-Hadamard type involving fractional integrals. The results presented here would provide extensions of those given in earlier works.

2. Main Results

By using the Theorem 3, we will establish some new results connected with the Hermite-Hadamard type and Fejer type inequalities for fractional integrals. Now, we give the following our results:

Theorem 7. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping such that there exists real constants m and M so that $m \leq f'' \leq M$. Then, the following inequalities for fractional integrals hold:*

$$\begin{aligned} &\frac{m\alpha(b-a)^2}{2(\alpha+1)(\alpha+2)} \\ \leq\ &\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right] \\ \leq\ &\frac{M(b-a)^2}{2(\alpha+1)(\alpha+2)} \end{aligned} \tag{2.1}$$

and

$$m(b-a)^2\frac{4\alpha-\alpha^2+\alpha^3}{16\alpha(\alpha+1)(\alpha+2)} \tag{2.2}$$
$$\leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]-f\left(\frac{a+b}{2}\right)$$
$$\leq M(b-a)^2\frac{4\alpha-\alpha^2+\alpha^3}{16\alpha(\alpha+1)(\alpha+2)}$$

where $\alpha>0$.

Proof. Multiplying both sides of (1.5) by $\lambda^{\alpha-1}$, then integrating the resulting inequality with respect to λ over $[0,1]$, we have

$$\frac{m(b-a)^2}{2(\alpha+1)(\alpha+2)}$$
$$\leq \frac{f(a)}{\alpha+1}+\frac{f(b)}{\alpha(\alpha+1)}-\int_0^1\lambda^{\alpha-1}f(\lambda a+(1-\lambda)b)\,d\lambda$$
$$\leq \frac{M(b-a)^2}{2(\alpha+1)(\alpha+2)}$$

and using the change of the variable $\lambda a+(1-\lambda)b=t$, we get

$$\frac{m(b-a)^2}{2(\alpha+1)(\alpha+2)} \tag{2.3}$$
$$\leq \frac{f(a)}{\alpha+1}+\frac{f(b)}{\alpha(\alpha+1)}-\frac{1}{(b-a)^\alpha}\int_a^b(b-t)^{\alpha-1}f(t)\,dt$$
$$\leq \frac{M(b-a)^2}{2(\alpha+1)(\alpha+2)}.$$

Similarly, multiplying both sides of (1.5) by $(1-\lambda)^{\alpha-1}$, then integrating the resulting inequality with respect to λ over $[0,1]$ and using the change of the variable $\lambda a+(1-\lambda)b=t$, we have

$$\frac{m(b-a)^2}{2(\alpha+1)(\alpha+2)} \tag{2.4}$$
$$\leq \frac{f(a)}{\alpha(\alpha+1)}+\frac{f(b)}{\alpha+1}-\frac{1}{(b-a)^\alpha}\int_a^b(t-a)^{\alpha-1}f(t)\,dt$$
$$\leq \frac{M(b-a)^2}{2(\alpha+1)(\alpha+2)}.$$

Thus, summing (2.3) and (2.4), we obtain the inequality (2.1).

On the other hand, multiplying both sides of (1.6) by $\lambda^{\alpha-1}$ and $(1-\lambda)^{\alpha-1}$, respectively, then integrating the resulting inequality with respect to λ over $[0,1]$ and using the change of the variable, we obtain inequalities (2.2). This completes the proof. □

Remark 1. *If we take $\alpha = 1$ in (2.1) and (2.2), the inequalities (2.1) and (2.2) reduce to (1.2) and (1.3), respectively.*

Theorem 8. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping such that there exists real constants m and M so that $m \leq f'' \leq M$. Assume $g : [a,b] \to \mathbb{R}$ is non-negative, integrable, and symmetric about $x = \frac{a+b}{2}$. Then, the following inequalities for fractional integrals hold:*

$$
\begin{aligned}
(2.5) \qquad & \frac{m}{2\Gamma(\alpha)} \int_a^b (b-t)(t-a)\left[(b-t)^{\alpha-1} + (t-a)^{\alpha-1}\right] g(t)\,dt \\
\leq\ & \frac{f(a)+f(b)}{2}\left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)\right] - \left[J_{a+}^{\alpha}(fg)(b) + J_{b-}^{\alpha}(fg)(a)\right] \\
\leq\ & \frac{M}{2\Gamma(\alpha)} \int_a^b (b-t)(t-a)\left[(b-t)^{\alpha-1} + (t-a)^{\alpha-1}\right] g(t)\,dt
\end{aligned}
$$

and

$$
\begin{aligned}
(2.6) \qquad & \frac{m}{8\Gamma(\alpha)} \int_a^b \left[(t-a)^{\alpha-1} + (b-t)^{\alpha-1}\right](2t-a-b)^2 g(t)\,dt \\
\leq\ & \left[J_{a+}^{\alpha}(fg)(b) + J_{b-}^{\alpha}(fg)(a)\right] - f\left(\frac{a+b}{2}\right)\left[J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)\right] \\
\leq\ & \frac{M}{8\Gamma(\alpha)} \int_a^b \left[(t-a)^{\alpha-1} + (b-t)^{\alpha-1}\right](2t-a-b)^2 g(t)\,dt
\end{aligned}
$$

where $\alpha > 0$.

Proof. Multiplying both sides of (1.5) by $\lambda^{\alpha-1} g(\lambda a + (1-\lambda) b)$ and $(1-\lambda)^{\alpha-1} g((1-\lambda) a + b\lambda)$, respectively, then integrating the resulting inequality with respect to λ over $[0,1]$ and using the change of the variable, we have the following inequalities, respectively,

$$
\begin{aligned}
(2.7) \qquad & \frac{m}{2(b-a)^{\alpha}} \int_a^b (b-t)^{\alpha}(t-a) g(t)\,dt \\
\leq\ & \frac{f(a)}{(b-a)^{\alpha+1}} \int_a^b (b-t)^{\alpha} g(t)\,dt + \frac{f(b)}{(b-a)^{\alpha+1}} \int_a^b (b-t)^{\alpha-1}(t-a) g(t)\,dt \\
& - \frac{1}{(b-a)^{\alpha}} \int_a^b (b-t)^{\alpha-1} f(t) g(t)\,dt \\
\leq\ & \frac{M}{2(b-a)^{\alpha}} \int_a^b (b-t)^{\alpha}(t-a) g(t)\,dt
\end{aligned}
$$

and

$$\begin{aligned} &\frac{m}{2(b-a)^{\alpha}}\int_a^b (b-t)^{\alpha}(t-a)\,g(t)\,dt \qquad (2.8)\\ &\leq \frac{f(a)}{(b-a)^{\alpha+1}}\int_a^b (b-t)^{\alpha-1}(t-a)\,g(t)\,dt+\frac{f(b)}{(b-a)^{\alpha+1}}\int_a^b (b-t)^{\alpha}\,g(t)\,dt\\ &\quad -\frac{1}{(b-a)^{\alpha}}\int_a^b (t-a)^{\alpha-1}f(t)\,g(t)\,dt\\ &\leq \frac{M}{2(b-a)^{\alpha}}\int_a^b (b-t)^{\alpha}(t-a)\,g(t)\,dt. \end{aligned}$$

Summing (2.7) and (2.8), we have

$$\begin{aligned} &\frac{m}{(b-a)^{\alpha}}\int_a^b (b-t)^{\alpha}(t-a)\,g(t)\,dt \qquad (2.9)\\ &\leq \frac{f(a)}{(b-a)^{\alpha}}\int_a^b (b-t)^{\alpha-1}\,g(t)\,dt+\frac{f(b)}{(b-a)^{\alpha}}\int_a^b (b-t)^{\alpha-1}\,g(t)\,dt\\ &\quad -\frac{1}{(b-a)^{\alpha}}\int_a^b (b-t)^{\alpha-1}f(t)\,g(t)\,dt-\frac{1}{(b-a)^{\alpha}}\int_a^b (t-a)^{\alpha-1}f(t)\,g(t)\,dt\\ &\leq \frac{M}{(b-a)^{\alpha}}\int_a^b (b-t)^{\alpha}(t-a)\,g(t)\,dt. \end{aligned}$$

Multiplying both sides of (1.5) by $\lambda^{\alpha-1}g((1-\lambda)a+b\lambda)$ and $(1-\lambda)^{\alpha-1}g(\lambda a+(1-\lambda)b)$, respectively, then integrating the resulting inequality with respect to λ over $[0,1]$ and using the change of the variable, we get the following inequalities, respectively,

$$\begin{aligned} &\frac{m}{2(b-a)^{\alpha}}\int_a^b (t-a)^{\alpha}(b-t)\,g(t)\,dt \qquad (2.10)\\ &\leq \frac{f(a)}{(b-a)^{\alpha+1}}\int_a^b (t-a)^{\alpha}\,g(t)\,dt+\frac{f(b)}{(b-a)^{\alpha+1}}\int_a^b (t-a)^{\alpha-1}(b-t)\,g(t)\,dt\\ &\quad -\frac{1}{(b-a)^{\alpha}}\int_a^b (b-t)^{\alpha-1}f(t)\,g(t)\,dt\\ &\leq \frac{M}{2(b-a)^{\alpha}}\int_a^b (t-a)^{\alpha}(b-t)\,g(t)\,dt \end{aligned}$$

and

$$
\begin{aligned}
(2.11)\qquad & \frac{m}{2(b-a)^{\alpha}}\int_a^b (t-a)^{\alpha}(b-t)\,g(t)\,dt \\
\leq\; & \frac{f(a)}{(b-a)^{\alpha+1}}\int_a^b (b-t)(t-a)^{\alpha-1}g(t)\,dt+\frac{f(b)}{(b-a)^{\alpha+1}}\int_a^b (t-a)^{\alpha}g(t)\,dt \\
& -\frac{1}{(b-a)^{\alpha}}\int_a^b (t-a)^{\alpha-1}f(t)\,g(t)\,dt \\
\leq\; & \frac{M}{2(b-a)^{\alpha}}\int_a^b (t-a)^{\alpha}(b-t)\,g(t)\,dt.
\end{aligned}
$$

Summing (2.10) and (2.11), we have

$$
\begin{aligned}
(2.12)\qquad & \frac{m}{(b-a)^{\alpha}}\int_a^b (t-a)^{\alpha}(b-t)\,g(t)\,dt \\
\leq\; & \frac{f(a)}{(b-a)^{\alpha}}\int_a^b (t-a)^{\alpha-1}g(t)\,dt+\frac{f(b)}{(b-a)^{\alpha}}\int_a^b (t-a)^{\alpha-1}g(t)\,dt \\
& -\frac{1}{(b-a)^{\alpha}}\int_a^b (t-a)^{\alpha-1}f(t)\,g(t)\,dt-\frac{1}{(b-a)^{\alpha}}\int_a^b (b-t)^{\alpha-1}f(t)\,g(t)\,dt \\
\leq\; & \frac{M}{(b-a)^{\alpha}}\int_a^b (t-a)^{\alpha}(b-t)\,g(t)\,dt.
\end{aligned}
$$

Hence, by summing (2.9) and (2.12), we get the inequality (2.5).

Now, multiplying both sides of (1.6) by $\lambda^{\alpha-1}g(\lambda a+(1-\lambda)b)$ and $(1-\lambda)^{\alpha-1}g((1-\lambda)a+b\lambda)$, respectively, then integrating the resulting inequality with respect to λ over $[0,1]$ and using the change of the variable, we have the following inequalities, respectively,

$$
\begin{aligned}
(2.13)\qquad & \frac{m}{8(b-a)^{\alpha}}\int_a^b (b-t)^{\alpha-1}(2t-a-b)^2\,g(t)\,dt \\
\leq\; & \frac{1}{2(b-a)^{\alpha}}\int_a^b (b-t)^{\alpha-1}f(t)g(t)\,dt+\frac{1}{2(b-a)^{\alpha}}\int_a^b (t-a)^{\alpha-1}f(t)g(t)\,dt \\
& -f\left(\frac{a+b}{2}\right)\frac{1}{(b-a)^{\alpha}}\int_a^b (b-t)^{\alpha-1}g(t)\,dt \\
\leq\; & \frac{M}{8(b-a)^{\alpha}}\int_a^b (b-t)^{\alpha-1}(2t-a-b)^2\,g(t)\,dt
\end{aligned}
$$

and

$$\begin{aligned} (2.14) \quad & \frac{m}{8(b-a)^{\alpha}}\int_{a}^{b}(b-t)^{\alpha-1}(2t-a-b)^{2}g(t)\,dt \\ \leq \; & \frac{1}{2(b-a)^{\alpha}}\int_{a}^{b}(t-a)^{\alpha-1}f(t)g(t)\,dt+\frac{1}{2(b-a)^{\alpha}}\int_{a}^{b}(b-t)^{\alpha-1}f(t)g(t)\,dt \\ & -f\left(\frac{a+b}{2}\right)\frac{1}{(b-a)^{\alpha}}\int_{a}^{b}(b-t)^{\alpha-1}g(t)\,dt \\ \leq \; & \frac{M}{8(b-a)^{\alpha}}\int_{a}^{b}(b-t)^{\alpha-1}(2t-a-b)^{2}g(t)\,dt. \end{aligned}$$

Summing (2.13) and (2.14), we have

$$\begin{aligned} (2.15) \quad & \frac{m}{4(b-a)^{\alpha}}\int_{a}^{b}(b-t)^{\alpha-1}(2t-a-b)^{2}g(t)\,dt \\ \leq \; & \frac{1}{(b-a)^{\alpha}}\int_{a}^{b}(b-t)^{\alpha-1}f(t)g(t)\,dt+\frac{1}{(b-a)^{\alpha}}\int_{a}^{b}(t-a)^{\alpha-1}f(t)g(t)\,dt \\ & -f\left(\frac{a+b}{2}\right)\frac{2}{(b-a)^{\alpha}}\int_{a}^{b}(b-t)^{\alpha-1}g(t)\,dt \\ \leq \; & \frac{M}{4(b-a)^{\alpha}}\int_{a}^{b}(b-t)^{\alpha-1}(2t-a-b)^{2}g(t)\,dt. \end{aligned}$$

Similarly, multiplying both sides of (1.6) by $\lambda^{\alpha-1}g((1-\lambda)a+b\lambda)$ and $(1-\lambda)^{\alpha-1}g(\lambda a+(1-\lambda)b)$, respectively, then integrating the resulting inequality with respect to λ over $[0,1]$ and using the change of the variable, we get the following inequalities, respectively,

$$\begin{aligned} (2.16) \quad & \frac{m}{8(b-a)^{\alpha}}\int_{a}^{b}(t-a)^{\alpha-1}(2t-a-b)^{2}g(t)\,dt \\ \leq \; & \frac{1}{2(b-a)^{\alpha}}\int_{a}^{b}(b-t)^{\alpha-1}f(t)g(t)\,dt+\frac{1}{2(b-a)^{\alpha}}\int_{a}^{b}(t-a)^{\alpha-1}f(t)g(t)\,dt \\ & -f\left(\frac{a+b}{2}\right)\frac{1}{(b-a)^{\alpha}}\int_{a}^{b}(t-a)^{\alpha-1}g(t)\,dt \\ \leq \; & \frac{M}{8(b-a)^{\alpha}}\int_{a}^{b}(t-a)^{\alpha-1}(2t-a-b)^{2}g(t)\,dt \end{aligned}$$

and

$$\begin{aligned}
(2.17) \quad & \frac{m}{8(b-a)^{\alpha}} \int_a^b (t-a)^{\alpha-1} (2t-a-b)^2 g(t)\, dt \\
\leq\ & \frac{1}{2(b-a)^{\alpha}} \int_a^b (t-a)^{\alpha-1} f(t) g(t)\, dt + \frac{1}{2(b-a)^{\alpha}} \int_a^b (b-t)^{\alpha-1} f(t) g(t)\, dt \\
& - f\left(\frac{a+b}{2}\right) \frac{1}{(b-a)^{\alpha}} \int_a^b (t-a)^{\alpha-1} g(t)\, dt \\
\leq\ & \frac{M}{8(b-a)^{\alpha}} \int_a^b (t-a)^{\alpha-1} (2t-a-b)^2 g(t)\, dt.
\end{aligned}$$

Summing (2.16) and (2.17), we have

$$\begin{aligned}
(2.18) \quad & \frac{m}{4(b-a)^{\alpha}} \int_a^b (t-a)^{\alpha-1} (2t-a-b)^2 g(t)\, dt \\
\leq\ & \frac{1}{(b-a)^{\alpha}} \int_a^b (t-a)^{\alpha-1} f(t) g(t)\, dt + \frac{1}{(b-a)^{\alpha}} \int_a^b (b-t)^{\alpha-1} f(t) g(t)\, dt \\
& - f\left(\frac{a+b}{2}\right) \frac{2}{(b-a)^{\alpha}} \int_a^b (t-a)^{\alpha-1} g(t)\, dt \\
\leq\ & \frac{M}{4(b-a)^{\alpha}} \int_a^b (t-a)^{\alpha-1} (2t-a-b)^2 g(t)\, dt.
\end{aligned}$$

Therefore, by summing (2.15) and (2.18), we get the inequality (2.6) which the proof of theorem is completed. □

Remark 2. *If we take $\alpha = 1$ in (2.5) and (2.6), the inequalities (2.5) and (2.6) reduce to (1.7) and (1.8), respectively.*

Corollary 1. *Under the same assumptions of Theorem 8 with $g(t) = 1$, then the following inequality holds:*

$$\begin{aligned}
& \frac{\alpha m}{4(b-a)^{\alpha}} \int_a^b (b-t)(t-a) \left[(b-t)^{\alpha-1} + (t-a)^{\alpha-1}\right] dt \\
\leq\ & \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \\
\leq\ & \frac{\alpha M}{4(b-a)^{\alpha}} \int_a^b (b-t)(t-a) \left[(b-t)^{\alpha-1} + (t-a)^{\alpha-1}\right] dt
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\alpha m}{16\left(b-a\right)^{\alpha}}\int\limits_{a}^{b}\left[\left(t-a\right)^{\alpha-1}+\left(b-t\right)^{\alpha-1}\right]\left(2t-a-b\right)^{2}dt \\
&\leq \frac{\Gamma\left(\alpha+1\right)}{2\left(b-a\right)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]-f\left(\frac{a+b}{2}\right) \\
&\leq \frac{\alpha M}{16\left(b-a\right)^{\alpha}}\int\limits_{a}^{b}\left[\left(t-a\right)^{\alpha-1}+\left(b-t\right)^{\alpha-1}\right]\left(2t-a-b\right)^{2}dt.
\end{aligned}$$

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