Received 22/12/14

GENERALIZATIONS OF BUZANO INEQUALITY FOR n-TUPLES OF VECTORS IN INNER PRODUCT SPACES WITH APPLICATIONS

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Abstract. In this paper some generalizations of Buzano inequality for n-tuples of vectors in inner product spaces are given. Applications for norm and numerical radius inequalities for n-tuples of bounded linear operators and for functions of normal operators defined by power series with nonnegative coefficients are also provided.

1. Introduction

Let $(H,\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. The following inequality is well known in literature as the *Schwarz inequality*

$$\|x\|\,\|y\|\geq|\langle x,y\rangle| \text{ for any } x,y\in H. \tag{1.1}$$

The equality case holds in (1.1) if and only if there exists a constant $\lambda\in\mathbb{K}$ such that $x=\lambda y$.

In 1985 the author [5] (see also [18]) established the following refinement of (1.1):

$$\|x\|\,\|y\|\geq|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|+|\langle x,e\rangle\langle e,y\rangle|\geq|\langle x,y\rangle| \tag{1.2}$$

for any $x,y,e\in H$ with $\|e\|=1$.

Using the triangle inequality for modulus we have

$$|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|\geq|\langle x,e\rangle\langle e,y\rangle|-|\langle x,y\rangle|$$

and by (1.2) we get

$$\begin{aligned}\|x\|\,\|y\| &\geq |\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|+|\langle x,e\rangle\langle e,y\rangle| \\ &\geq 2|\langle x,e\rangle\langle e,y\rangle|-|\langle x,y\rangle|,\end{aligned}$$

which implies the Buzano inequality [3]

$$\frac{1}{2}\left[\|x\|\,\|y\|+|\langle x,y\rangle|\right]\geq|\langle x,e\rangle\langle e,y\rangle| \tag{1.3}$$

that holds for any $x,y,e\in H$ with $\|e\|=1$.

For other Schwarz related inequalities in inner product spaces, see [1], [6]-[10], [16], [17], [21], [22], [23], [24], [25], [26], [27], [28] and the monographs [13] and [14].

Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [20, p. 1]:

$$W(T)=\{\langle Tx,x\rangle,\ x\in H,\ \|x\|=1\}. \tag{1.4}$$

It is well known (see for instance [20]) that:

1991 *Mathematics Subject Classification.* 47A05, 47A63; 47A99.

(i) The numerical range of an operator is convex (the Toeplitz-Hausdorff theorem);
(ii) The spectrum of an operator is contained in the closure of its numerical range;
(iii) T is self-adjoint if and only if W is real.

The *numerical radius* $w(T)$ of an operator T on H is defined by [20, p. 8]:

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{1.5}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ and the following inequality holds true

$$w(T) \le \|T\| \le 2w(T), \text{ for any } T \in B(H). \tag{1.6}$$

Utilising Buzano's inequality (1.3) we obtained the following inequality for the numerical radius [11] or [12]:

Theorem 1. *Let $(H; \langle\cdot,\cdot\rangle)$ be a Hilbert space and $T : H \to H$ a bounded linear operator on H. Then*

$$w^2(T) \le \frac{1}{2}\left[w\left(T^2\right) + \|T\|^2\right]. \tag{1.7}$$

The constant $\frac{1}{2}$ is best possible in (1.7).

From the above result (1.7) we obviously have

$$w(T) \le \left\{\frac{1}{2}\left[w\left(T^2\right) + \|T\|^2\right]\right\}^{1/2} \le \left\{\frac{1}{2}\left(\left\|T^2\right\| + \|T\|^2\right)\right\}^{1/2} \le \|T\| \tag{1.8}$$

and

$$w(T) \le \left\{\frac{1}{2}\left[w\left(T^2\right) + \|T\|^2\right]\right\}^{1/2} \le \left\{\frac{1}{2}\left(w^2(T) + \|T\|^2\right)\right\}^{1/2} \le \|T\|, \tag{1.9}$$

that provide refinements for the first inequality in (1.6).

For numerical radius inequalities see the recent monograph [15] and the references therein.

Motivated by the above results, we establish some generalizations of Buzano inequality for n-tuples of vectors in inner product spaces. Applications for norm and numerical radius inequalities for n-tuples of bounded linear operators and for functions of normal operators defined by power series with nonnegative coefficients are also provided.

2. Main Results

We have the following generalization of Buzano's inequality:

Theorem 2. *Let $\mathbf{p} = (p_1, ..., p_n) \in \mathcal{P}_n$ be a probability distribution, i.e. we recall that $p_i > 0$ for any $i \in \{1, ..., n\}$ and $\sum_{i=1}^n p_i = 1$. For any $\mathbf{x} = (x_1, ..., x_n), \mathbf{y} =$*

$(y_1, ..., y_n) \in H^n$ *we have*

$$\left(\sum_{i=1}^{n} p_i \|x_i\|^2\right)^{1/2} \left(\sum_{i=1}^{n} p_i \|y_i\|^2\right)^{1/2} \tag{2.1}$$
$$\geq \left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle - \left\langle \sum_{i=1}^{n} p_i x_i, e\right\rangle \left\langle e, \sum_{i=1}^{n} p_i y_i\right\rangle\right| + \left|\left\langle \sum_{i=1}^{n} p_i x_i, e\right\rangle \left\langle e, \sum_{i=1}^{n} p_i y_i\right\rangle\right|$$
$$\geq \left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle\right|$$

and

$$\frac{1}{2}\left[\left(\sum_{i=1}^{n} p_i \|x_i\|^2\right)^{1/2} \left(\sum_{i=1}^{n} p_i \|y_i\|^2\right)^{1/2} + \left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle\right|\right] \tag{2.2}$$
$$\geq \left|\left\langle \sum_{i=1}^{n} p_i x_i, e\right\rangle \left\langle e, \sum_{i=1}^{n} p_i y_i\right\rangle\right|,$$

for any $e \in H$ *with* $\|e\| = 1$.

Proof. For a probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathcal{P}_n$, we define the inner product

$$\langle \mathbf{x}, \mathbf{y}\rangle_p := \sum_{i=1}^{n} p_i \langle x_i, y_i\rangle$$

for n-tuples $\mathbf{x} = (x_1, ..., x_n), \mathbf{y} = (y_1, ..., y_n) \in H^n$.

The attached norm is given by

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} p_i \|x_i\|^2\right)^{1/2}$$

for $\mathbf{x} = (x_1, ..., x_n) \in H^n$.

Let $\mathbf{e} = (e_1, ..., e_n) \in H^n$ with $\sum_{i=1}^{n} p_i \|e_i\|^2 = 1$. Making use of (1.2) and (1.3) for the inner product $\langle \cdot, \cdot\rangle_p$ we have the inequalities

$$\left(\sum_{i=1}^{n} p_i \|x_i\|^2\right)^{1/2} \left(\sum_{i=1}^{n} p_i \|y_i\|^2\right)^{1/2} \tag{2.3}$$
$$\geq \left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle - \sum_{i=1}^{n} p_i \langle x_i, e_i\rangle \sum_{i=1}^{n} p_i \langle e_i, y_i\rangle\right| + \left|\sum_{i=1}^{n} p_i \langle x_i, e_i\rangle \sum_{i=1}^{n} p_i \langle e_i, y_i\rangle\right|$$
$$\geq \left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle\right|$$

and

$$\frac{1}{2}\left[\left(\sum_{i=1}^{n} p_i \|x_i\|^2\right)^{1/2} \left(\sum_{i=1}^{n} p_i \|y_i\|^2\right)^{1/2} + \left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle\right|\right] \tag{2.4}$$
$$\geq \left|\sum_{i=1}^{n} p_i \langle x_i, e_i\rangle \sum_{i=1}^{n} p_i \langle e_i, y_i\rangle\right|,$$

for any $\mathbf{x}=(x_1,...,x_n)\,,\mathbf{y}=(y_1,...,y_n)\in H^n$.

If we take $\mathbf{e}=(e,...,e)\in H^n$ with $\|e\|=1$, then we get from (2.3) and (2.4) the desired inequalities (2.1) and (2.2). □

Remark 1. *If we take in (2.1) and (2.2) $n=1$ and $p_1=1$, then we get the inequalities (1.2) and (1.3).*

We observe that, if we take $H=\mathbb{C}$ with the inner product $\langle z,w\rangle = z\overline{w}$ then by taking above $x_i=a_i\in\mathbb{C}$, $y_i=\overline{b}_i, i\in\{1,...,n\}$ and $e=1$ then from (2.1) and (2.2) we get the inequalities

$$\begin{aligned}(2.5)\qquad &\left(\sum_{i=1}^n p_i\left|a_i\right|^2\right)^{1/2}\left(\sum_{i=1}^n p_i\left|b_i\right|^2\right)^{1/2}\\ &\geq\left|\sum_{i=1}^n p_i a_i b_i-\sum_{i=1}^n p_i a_i\sum_{i=1}^n p_i b_i\right|+\left|\sum_{i=1}^n p_i a_i\sum_{i=1}^n p_i b_i\right|\\ &\geq\left|\sum_{i=1}^n p_i a_i b_i\right|\end{aligned}$$

and

$$\begin{aligned}(2.6)\qquad &\frac{1}{2}\left[\left(\sum_{i=1}^n p_i\left|a_i\right|^2\right)^{1/2}\left(\sum_{i=1}^n p_i\left|b_i\right|^2\right)^{1/2}+\left|\sum_{i=1}^n p_i a_i b_i\right|\right]\\ &\geq\left|\sum_{i=1}^n p_i a_i\sum_{i=1}^n p_i b_i\right|.\end{aligned}$$

We have the following norm inequality:

Theorem 3. *Let $\mathbf{p}=(p_1,...,p_n)\in\mathcal{P}_n$ be a probability distribution and $(A_1,...,A_n)$, $(B_1,...,B_n)$ two n-tuples of bounded linear operators on H. Then we have*

$$\begin{aligned}(2.7)\qquad &\frac{1}{2}\left[\left\|\sum_{i=1}^n p_i\left|A_i\right|^2\right\|^{1/2}\left\|\sum_{i=1}^n p_i\left|B_i\right|^2\right\|^{1/2}+\left\|\sum_{i=1}^n p_i B_i^*A_i\right\|\right]\\ &\geq\left\|\sum_{i=1}^n p_i A_i e\right\|\left\|\sum_{i=1}^n p_i B_i e\right\|\geq\left|\left\langle\left(\sum_{i=1}^n p_i B_i^*\right)\sum_{i=1}^n p_i A_i e,e\right\rangle\right|\end{aligned}$$

or any $e\in H$ with $\|e\|=1$.

Proof. If we write the inequality (2.2) for $x_i=A_i x$, $y_i=B_i y$, then we get

$$\begin{aligned}(2.8)\qquad &\frac{1}{2}\left[\left(\sum_{i=1}^n p_i\left\|A_i x\right\|^2\right)^{1/2}\left(\sum_{i=1}^n p_i\left\|B_i y\right\|^2\right)^{1/2}+\left|\sum_{i=1}^n p_i\left\langle A_i x,B_i y\right\rangle\right|\right]\\ &\geq\left|\left\langle\sum_{i=1}^n p_i A_i x,e\right\rangle\left\langle e,\sum_{i=1}^n p_i B_i y\right\rangle\right|,\end{aligned}$$

for any $x,y,e\in H$ with $\|e\|=1$.

Observe that

$$\begin{aligned}\sum_{i=1}^{n} p_i \|A_i x\|^2 &= \sum_{i=1}^{n} p_i \langle A_i x, A_i x\rangle = \sum_{i=1}^{n} p_i \langle A_i x, A_i x\rangle \\ &= \sum_{i=1}^{n} p_i \langle A_i^* A_i x, x\rangle = \sum_{i=1}^{n} p_i \left\langle |A_i|^2 x, x\right\rangle \\ &= \left\langle \sum_{i=1}^{n} p_i |A_i|^2 x, x\right\rangle,\end{aligned}$$

$$\sum_{i=1}^{n} p_i \|B_i y\|^2 = \left\langle \sum_{i=1}^{n} p_i |B_i|^2 y, y\right\rangle$$

and

$$\sum_{i=1}^{n} p_i \langle A_i x, B_i y\rangle = \left\langle \sum_{i=1}^{n} p_i B_i^* A_i x, y\right\rangle$$

for any $x, y \in H$.

Then by (2.8) we get the inequality

$$\begin{aligned}&\frac{1}{2}\left[\left\langle \sum_{i=1}^{n} p_i |A_i|^2 x, x\right\rangle^{1/2} \left\langle \sum_{i=1}^{n} p_i |B_i|^2 y, y\right\rangle^{1/2} + \left|\left\langle \sum_{i=1}^{n} p_i B_i^* A_i x, y\right\rangle\right|\right] \\ &\geq \left|\left\langle \sum_{i=1}^{n} p_i A_i e, x\right\rangle \left\langle \sum_{i=1}^{n} p_i B_i e, y\right\rangle\right|,\end{aligned} \tag{2.9}$$

for any $x, y, e \in H$ with $\|e\| = 1$, which is an inequality of interest in itself.

Taking the supremum over $\|x\| = \|y\| = 1$ in (2.9) we get

$$\begin{aligned}&\sup_{\|x\|=1}\left|\left\langle \sum_{i=1}^{n} p_i A_i e, x\right\rangle\right| \sup_{\|y\|=1}\left|\left\langle \sum_{i=1}^{n} p_i B_i e, y\right\rangle\right| \\ &= \sup_{\|x\|=\|y\|=1}\left|\left\langle \sum_{i=1}^{n} p_i A_i e, x\right\rangle \left\langle \sum_{i=1}^{n} p_i B_i e, y\right\rangle\right| \\ &\leq \frac{1}{2}\sup_{\|x\|=\|y\|=1}\left[\left\langle \sum_{i=1}^{n} p_i |A_i|^2 x, x\right\rangle^{1/2} \left\langle \sum_{i=1}^{n} p_i |B_i|^2 y, y\right\rangle^{1/2}\right. \\ &\left. + \left|\left\langle \sum_{i=1}^{n} p_i B_i^* A_i x, y\right\rangle\right|\right] \\ &\leq \frac{1}{2}\left[\sup_{\|x\|=1}\left\langle \sum_{i=1}^{n} p_i |A_i|^2 x, x\right\rangle^{1/2} \sup_{\|y\|=1}\left\langle \sum_{i=1}^{n} p_i |B_i|^2 y, y\right\rangle^{1/2}\right. \\ &\left. + \sup_{\|x\|=\|y\|=1}\left|\left\langle \sum_{i=1}^{n} p_i B_i^* A_i x, y\right\rangle\right|\right]\end{aligned} \tag{2.10}$$

for any $e \in H$ with $\|e\| = 1$.

Since

$$\sup_{\|x\|=1}\left|\left\langle\sum_{i=1}^{n}p_iA_ie,x\right\rangle\right|=\left\|\sum_{i=1}^{n}p_iA_ie\right\|,$$

$$\sup_{\|y\|=1}\left|\left\langle\sum_{i=1}^{n}p_iB_ie,y\right\rangle\right|=\left\|\sum_{i=1}^{n}p_iB_ie\right\|,$$

$$\sup_{\|x\|=1}\left\langle\sum_{i=1}^{n}p_i\left|A_i\right|^2x,x\right\rangle^{1/2}=\left\|\sum_{i=1}^{n}p_i\left|A_i\right|^2\right\|^{1/2},$$

$$\sup_{\|y\|=1}\left\langle\sum_{i=1}^{n}p_i\left|B_i\right|^2y,y\right\rangle^{1/2}=\left\|\sum_{i=1}^{n}p_i\left|B_i\right|^2\right\|^{1/2}$$

and

$$\sup_{\|x\|=\|y\|=1}\left|\left\langle\sum_{i=1}^{n}p_iB_i^*A_ix,y\right\rangle\right|=\left\|\sum_{i=1}^{n}p_iB_i^*A_i\right\|.$$

Making use of (2.10) we get the first inequality in (2.7).

Using Schwarz inequality in $(H;\langle\cdot,\cdot\rangle)$ we have

$$\begin{aligned}\left\|\sum_{i=1}^{n}p_iA_ie\right\|\left\|\sum_{i=1}^{n}p_iB_ie\right\|&\geq\left|\left\langle\sum_{i=1}^{n}p_iA_ie,\sum_{i=1}^{n}p_iB_ie\right\rangle\right|\\&=\left|\left\langle\left(\sum_{i=1}^{n}p_iB_i^*\right)\sum_{i=1}^{n}p_iA_ie,e\right\rangle\right|\end{aligned}$$

for any $e\in H$ with $\|e\|=1$, and the second inequality in (2.7) is proved. □

Corollary 1. *Let* $\mathbf{p}=(p_1,...,p_n)\in\mathcal{P}_n$ *be a probability distribution and* $(A_1,...,A_n)$ *an* n*-tuple of bounded linear operators on* H*. Then we have*

$$\begin{aligned}&\frac{1}{2}\left[\left\|\sum_{i=1}^{n}p_i\left|A_i\right|^2\right\|^{1/2}\left\|\sum_{i=1}^{n}p_i\left|A_i^*\right|^2\right\|^{1/2}+\left\|\sum_{i=1}^{n}p_iA_i^2\right\|\right]\\&\geq\left\|\sum_{i=1}^{n}p_iA_ie\right\|\left\|\sum_{i=1}^{n}p_iA_i^*e\right\|\geq\left|\left\langle\left(\sum_{i=1}^{n}p_iA_i\right)^2e,e\right\rangle\right|\end{aligned}\tag{2.11}$$

for any $e\in H$ *with* $\|e\|=1$.

It follows from (2.7) by taking $B_i=A_i^*$, $i\in\{1,...,n\}$.

Remark 2. *Taking the supremum over* $\|e\|=1$ *in (2.7) and (2.11), then we get the numerical radius inequalities*

$$\begin{aligned}&w\left(\sum_{i=1}^{n}p_iB_i^*\sum_{i=1}^{n}p_iA_i\right)\\&\leq\sup_{\|e\|=1}\left\|\sum_{i=1}^{n}p_iA_ie\right\|\left\|\sum_{i=1}^{n}p_iB_ie\right\|\\&\leq\frac{1}{2}\left[\left\|\sum_{i=1}^{n}p_i\left|A_i\right|^2\right\|^{1/2}\left\|\sum_{i=1}^{n}p_i\left|B_i\right|^2\right\|^{1/2}+\left\|\sum_{i=1}^{n}p_iB_i^*A_i\right\|\right]\end{aligned}\tag{2.12}$$

and

$$w\left(\left(\sum_{i=1}^{n} p_i A_i\right)^2\right) \tag{2.13}$$
$$\leq \sup_{\|e\|=1}\left(\left\|\sum_{i=1}^{n} p_i A_i e\right\|\left\|\sum_{i=1}^{n} p_i A_i^* e\right\|\right)$$
$$\leq \frac{1}{2}\left[\left\|\sum_{i=1}^{n} p_i |A_i|^2\right\|^{1/2}\left\|\sum_{i=1}^{n} p_i |A_i^*|^2\right\|^{1/2}+\left\|\sum_{i=1}^{n} p_i A_i^2\right\|\right],$$

where $(A_1,...,A_n)$, $(B_1,...,B_n)$ *are two* n*-tuples of bounded linear operators on* H.

We recall that a bounded linear operator T is *normal* if $TT^* = T^*T$. This is equivalent to the fact that $\|Tx\| = \|T^*x\|$ for any $x \in H$.

If $(A_1,...,A_n)$ is an n-tuple of normal operators on H, then from the first inequality in (2.11) we get

$$\frac{1}{2}\left[\left\|\sum_{i=1}^{n} p_i |A_i|^2\right\|+\left\|\sum_{i=1}^{n} p_i A_i^2\right\|\right] \geq \left\|\sum_{i=1}^{n} p_i A_i e\right\|^2 \tag{2.14}$$

for any $e \in H$ with $\|e\| = 1$.

Taking the supremum over $\|e\| = 1$ in (2.14) we also have

$$\frac{1}{2}\left[\left\|\sum_{i=1}^{n} p_i |A_i|^2\right\|+\left\|\sum_{i=1}^{n} p_i A_i^2\right\|\right] \geq \left\|\sum_{i=1}^{n} p_i A_i\right\|^2, \tag{2.15}$$

where $(A_1,...,A_n)$ is an n-tuple of normal operators on H.

If we take $p_i = \frac{q_i}{\sum_{k=1}^{n} q_k}$ with $q_i \geq 0, i \in \{1,...,n\}$ and $\sum_{k=1}^{n} q_k > 0$, then we get from (2.15)

$$\frac{1}{2}\left[\left\|\sum_{i=1}^{n} q_i |A_i|^2\right\|+\left\|\sum_{i=1}^{n} q_i A_i^2\right\|\right]\sum_{k=1}^{n} q_k \geq \left\|\sum_{i=1}^{n} q_i A_i\right\|^2, \tag{2.16}$$

for any $(A_1,...,A_n)$ an n-tuple of normal operators on H.

If we take $q_i = r_i^2$, $A_i = \frac{1}{r_i}T_i$, where r_i are nonzero real numbers, $i \in \{1,...,n\}$ and $(T_1,...,T_n)$ is an n-tuple of normal operators on H, then from (2.16) we get

$$\frac{1}{2}\left[\left\|\sum_{i=1}^{n} |T_i|^2\right\|+\left\|\sum_{i=1}^{n} T_i^2\right\|\right]\sum_{k=1}^{n} r_k^2 \geq \left\|\sum_{i=1}^{n} r_i T_i\right\|^2, \tag{2.17}$$

which for $T_i = z_i 1_H$, $i \in \{1,...,n\}$ produces the *de Bruijn inequality*

$$\frac{1}{2}\left[\sum_{i=1}^{n} |z_i|^2+\left|\sum_{i=1}^{n} z_i^2\right|\right]\sum_{k=1}^{n} r_k^2 \geq \left|\sum_{i=1}^{n} r_i z_i\right|^2. \tag{2.18}$$

We have the following numerical radius inequality:

Theorem 4. *Let* $\mathbf{p} = (p_1, ..., p_n) \in \mathcal{P}_n$ *be a probability distribution and* $(A_1, ..., A_n)$ *ann-tuples of bounded linear operators on* H. *Then we have*

$$w^2\left(\sum_{i=1}^{n} p_i A_i\right) \leq \frac{1}{2}\left[\left\|\sum_{i=1}^{n} p_i |A_i|^2\right\|^{1/2} \left\|\sum_{i=1}^{n} p_i |A_i^*|^2\right\|^{1/2} + w\left(\sum_{i=1}^{n} p_i A_i^2\right)\right]. \tag{2.19}$$

Proof. If in (2.9) we take $B_i = A_i^*, i \in \{1, ..., n\}$ and $x = y = e$, then we get the inequality

$$\frac{1}{2}\left[\left\langle\sum_{i=1}^{n} p_i |A_i|^2 e, e\right\rangle^{1/2} \left\langle\sum_{i=1}^{n} p_i |A_i^*|^2 e, e\right\rangle^{1/2} + \left|\left\langle\sum_{i=1}^{n} p_i A_i^2 e, e\right\rangle\right|\right] \geq \left|\left\langle\sum_{i=1}^{n} p_i A_i e, e\right\rangle \left\langle\sum_{i=1}^{n} p_i A_i^* e, e\right\rangle\right| = \left|\left\langle\sum_{i=1}^{n} p_i A_i e, e\right\rangle\right|^2, \tag{2.20}$$

for any $e \in H$ with $\|e\| = 1$.

By taking the supremum over $\|e\| = 1$ in (2.20), we get

$$\begin{aligned}
& w^2\left(\sum_{i=1}^{n} p_i A_i\right) \\
&= \sup_{\|e\|=1} \left|\left\langle\sum_{i=1}^{n} p_i A_i e, e\right\rangle\right|^2 \\
&\leq \frac{1}{2} \sup_{\|e\|=1} \left[\left\langle\sum_{i=1}^{n} p_i |A_i|^2 e, e\right\rangle^{1/2} \left\langle\sum_{i=1}^{n} p_i |A_i^*|^2 e, e\right\rangle^{1/2} + \left|\left\langle\sum_{i=1}^{n} p_i A_i^2 e, e\right\rangle\right|\right] \\
&\leq \frac{1}{2}\left[\sup_{\|e\|=1}\left(\left\langle\sum_{i=1}^{n} p_i |A_i|^2 e, e\right\rangle^{1/2} \left\langle\sum_{i=1}^{n} p_i |A_i^*|^2 e, e\right\rangle^{1/2}\right) \right. \\
&\left. + \sup_{\|e\|=1} \left|\left\langle\sum_{i=1}^{n} p_i A_i^2 e, e\right\rangle\right|\right] \\
&\leq \frac{1}{2}\left[\sup_{\|e\|=1}\left(\left\langle\sum_{i=1}^{n} p_i |A_i|^2 e, e\right\rangle^{1/2} \sup_{\|e\|=1} \left\langle\sum_{i=1}^{n} p_i |A_i^*|^2 e, e\right\rangle^{1/2}\right) \right. \\
&\left. + \sup_{\|e\|=1} \left|\left\langle\sum_{i=1}^{n} p_i A_i^2 e, e\right\rangle\right| \right. \\
&= \frac{1}{2}\left[\left\|\sum_{i=1}^{n} p_i |A_i|^2\right\|^{1/2} \left\|\sum_{i=1}^{n} p_i |A_i^*|^2\right\|^{1/2} + w\left(\sum_{i=1}^{n} p_i A_i^2\right)\right]
\end{aligned}$$

and the inequality (2.19) is proved. □

Remark 3. *If we take in (2.19) $n=1$ and $A_1=T$, then we recapture the inequality (1.7), namely*

$$w^2(T) \leq \frac{1}{2}\left[w\left(T^2\right)+\|T\|^2\right].$$

The case $n=2$ is important since it allows to apply the above inequalities for the Cartesian decomposition of an operator.

If we take in (2.19) $n=2$ and $q_1=q_2=\frac{1}{2}$, then we have

$$w^2(A_1+A_2) \leq \left\|\left|A_1\right|^2+\left|A_2\right|^2\right\|^{1/2}\left\|\left|A_1^*\right|^2+\left|A_2^*\right|^2\right\|^{1/2}+w\left(A_1^2+A_2^2\right). \tag{2.21}$$

Assume that T is a bounded linear operator and consider the Cartesian decomposition

$$T=A+iB,$$

with the selfadjoint operators A, B given by

$$A=\frac{1}{2}\left(T^*+T\right), \ B=\frac{1}{2i}\left(T-T^*\right).$$

Take $A_1=A$, $A_2=iB$. Then $A_1+A_2=T$,

$$\left|A_1\right|^2+\left|A_2\right|^2=A^2+B^2=\frac{1}{2}\left(|T|^2+\left|T^*\right|^2\right),$$

$$\left|A_1^*\right|^2+\left|A_2^*\right|^2=A^2+B^2=\frac{1}{2}\left(|T|^2+\left|T^*\right|^2\right)$$

and

$$A_1^2+A_2^2=\frac{1}{4}\left(T^*+T\right)^2+\frac{1}{4}\left(T-T^*\right)^2=\frac{1}{2}\left(T^2+\left(T^*\right)^2\right).$$

Using (2.21) we get

$$w^2(T) \leq \frac{1}{2}\left[\left\||T|^2+\left|T^*\right|^2\right\|+w\left(T^2+\left(T^*\right)^2\right)\right] \tag{2.22}$$

for any T a bounded linear operator.

3. Applications for Functions of Normal Operators

Recall some examples of power series with nonnegative coefficients

$$\frac{1}{1-\lambda}=\sum_{n=0}^{\infty} \lambda^n, \ \lambda \in D(0,1); \tag{3.1}$$

$$\ln \frac{1}{1-\lambda}=\sum_{n=1}^{\infty} \frac{1}{n} \lambda^n, \ \lambda \in D(0,1);$$

$$\exp(\lambda)=\sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n, \ \lambda \in \mathbb{C};$$

$$\sinh \lambda=\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1}, \ \lambda \in \mathbb{C};$$

$$\cosh \lambda=\sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n}, \ \lambda \in \mathbb{C}.$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1},\ \lambda\in D(0,1); \tag{3.2}$$
$$\sin^{-1}(\lambda)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}\lambda^{2n+1},\ \lambda\in D(0,1);$$
$$\tanh^{-1}(\lambda)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1},\ \lambda\in D(0,1);$$
$${}_2F_1(\alpha,\beta,\gamma,\lambda)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^{n},\alpha,\beta,\gamma>0,$$
$$\lambda\in D(0,1)$$

where Γ is *Gamma function.*

The following inequality for power series with nonenegative coefficients holds:

Theorem 5. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series with nonnegative coefficients $a_n\geq 0$ for $n\in\mathbb{N}$ and having the radius of convergence $R>0$ or $R=\infty$. If U,V are normal operators on the Hilbert space $(H;\langle\cdot,\cdot\rangle)$, $U^*V=VU^*$ and $\alpha>0$ such that $\alpha<R$ and $\|U\|,\|V\|\leq 1$, then*

$$\begin{aligned}&|\langle f(\alpha V)e,x\rangle\langle f(\alpha U)e,y\rangle|\\&\leq\frac{1}{2}\left[\left\langle f\left(\alpha|V|^2\right)x,x\right\rangle\left\langle f\left(\alpha|U|^2\right)y,y\right\rangle+|\langle f(U^*V)x,y\rangle|\right]f(\alpha)\end{aligned} \tag{3.3}$$

for any $x,y,e\in H$ with $\|e\|=1$.

Proof. Using the inequality (2.9) we have for $n\geq 1$

$$\begin{aligned}&\frac{1}{2}\left[\left\langle\sum_{i=0}^{n}a_i\alpha^i\left|V^i\right|^2x,x\right\rangle^{1/2}\left\langle\sum_{i=0}^{n}a_i\alpha^i\left|U^i\right|^2y,y\right\rangle^{1/2}\right.\\&\left.+\left|\left\langle\sum_{i=0}^{n}a_i\alpha^i(U^*)^iV^ix,y\right\rangle\right|\right]\sum_{i=0}^{n}a_i\alpha^i,\\&\geq\left|\left\langle\sum_{i=0}^{n}a_i\alpha^iV^ie,x\right\rangle\left\langle\sum_{i=0}^{n}a_i\alpha^iU^ie,y\right\rangle\right|\end{aligned} \tag{3.4}$$

for any $x,y,e\in H$ with $\|e\|=1$.

Since U,V are normal operators, then for $i\geq 1$

$$\left|V^i\right|^2=\left(V^i\right)^*V^i=(V^*)^iV^i=(V^*V)^i=|V|^{2i}$$

and

$$\left|U^i\right|^2=|U|^{2i}.$$

Also, since $U^*V=VU^*$, then

$$(U^*)^iV^i=(U^*V)^i$$

for any $i\geq 1$.

Then from (3.4) we have

$$\begin{aligned}\frac{1}{2}&\left[\left\langle\sum_{i=0}^{n}a_i\alpha^i\left|V\right|^{2i}x,x\right\rangle^{1/2}\left\langle\sum_{i=0}^{n}a_i\alpha^i\left|U\right|^{2i}y,y\right\rangle^{1/2}\right.\\&\left.+\left|\left\langle\sum_{i=0}^{n}a_i\alpha^i\left(U^*V\right)^i x,y\right\rangle\right|\right]\sum_{i=0}^{n}a_i\alpha^i\\&\geq\left|\left\langle\sum_{i=0}^{n}a_i\alpha^iV^ie,x\right\rangle\left\langle\sum_{i=0}^{n}a_i\alpha^iU^ie,y\right\rangle\right|,\end{aligned}\tag{3.5}$$

for any $x,y,e\in H$ with $\|e\|=1$.

Since $\left\|\alpha\left|V\right|^2\right\|=\alpha\|V\|^2<R$, $\left\|\alpha\left|U\right|^2\right\|=\alpha\|U\|^2<R$, $\|\alpha U^*V\|\leq\alpha\|U\|\|V\|<R$, $\|\alpha U\|<R$ and $\|\alpha V\|<R$, then the series

$$\sum_{i=0}^{\infty}a_i\alpha^iU^i,\ \sum_{i=0}^{\infty}a_i\alpha^iV^i,\ \sum_{i=0}^{n}a_i\alpha^i\left|V\right|^{2i},\ \sum_{i=0}^{n}a_i\alpha^i\left|U\right|^{2i},\ \sum_{i=0}^{\infty}a_i\alpha^i\left(U^*V\right)^i$$

are convergent in $B(H)$ and $\sum_{i=0}^{\infty}a_i\alpha^i$ is convergent in $\mathbb{R}$.

Taking the limit over $n\to\infty$ in (3.5) we get the desired result (3.3). □

Remark 4. *If we take the supremum over* $\|x\|=\|y\|=1$ *in (3.3) then we get the norm inequality*

$$\|f(\alpha V)e\|\|f(\alpha U)e\|\leq\frac{1}{2}\left[\left\|f\left(\alpha\left|V\right|^2\right)\right\|\left\|f\left(\alpha\left|U\right|^2\right)\right\|+\|f(U^*V)\|\right]f(\alpha)\tag{3.6}$$

for any $e\in H$, $\|e\|=1$.

By Schwarz inequality in H *we have*

$$\begin{aligned}\|f(\alpha V)e\|\|f(\alpha U)e\|&\geq|\langle f(\alpha V)e,f(\alpha U)e\rangle|\\&=\left|\left\langle(f(\alpha U))^*f(\alpha V)e,e\right\rangle\right|\\&=|\langle f(\alpha U^*)f(\alpha V)e,e\rangle|\end{aligned}$$

giving the inequality

$$|\langle f(\alpha U^*)f(\alpha V)e,e\rangle|\leq\frac{1}{2}\left[\left\|f\left(\alpha\left|V\right|^2\right)\right\|\left\|f\left(\alpha\left|U\right|^2\right)\right\|+\|f(U^*V)\|\right]f(\alpha)\tag{3.7}$$

for any $e\in H$, $\|e\|=1$.

Since U *and* V *are normal and* $U^*V=VU^*$, *then* $f(\alpha U^*)$ *and* $f(\alpha V)$ *are normal and commute implying that* $f(\alpha U^*)f(\alpha V)$ *is normal. Taking the supremum over* $\|e\|=1$ *we get*

$$\|f(\alpha U^*)f(\alpha V)\|\leq\frac{1}{2}\left[\left\|f\left(\alpha\left|V\right|^2\right)\right\|\left\|f\left(\alpha\left|U\right|^2\right)\right\|+\|f(U^*V)\|\right]f(\alpha).\tag{3.8}$$

Example 1. *a. If* U,V *are normal operators on the Hilbert space* $(H;\langle\cdot,\cdot\rangle)$, $U^*V=VU^*$ *and* $\alpha>0$ *then*

$$\begin{aligned}&|\langle\exp(\alpha V)e,x\rangle\langle\exp(\alpha U)e,y\rangle|\\&\leq\frac{1}{2}\left[\left\langle\exp\left(\alpha\left|V\right|^2\right)x,x\right\rangle\left\langle\exp\left(\alpha\left|U\right|^2\right)y,y\right\rangle+|\langle\exp(U^*V)x,y\rangle|\right]\exp(\alpha)\end{aligned}\tag{3.9}$$

for any $x,y,e\in H$ *with* $\|e\|=1$.

b. If U,V are normal operators on the Hilbert space $(H;\langle\cdot,\cdot\rangle)$, $U^*V=VU^*$ and $\|U\|,\|V\|<1$, $\alpha\in(0,1)$ then

$$\left|\left\langle\left(1_H-\alpha V\right)^{-1}e,x\right\rangle\left\langle\left(1_H-\alpha U\right)^{-1}e,y\right\rangle\right| \leq \frac{1}{2}\left[\left\langle\left(1_H-\alpha|V|^2\right)^{-1}x,x\right\rangle\left\langle\left(1_H-\alpha|U|^2\right)^{-1}y,y\right\rangle + \left|\left\langle\left(1_H-U^*V\right)^{-1}x,y\right\rangle\right|\right](1-\alpha)^{-1} \tag{3.10}$$

for any $x,y,e\in H$ with $\|e\|=1$.

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