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HERMITE HADAMARD-FEJER TYPE INEQUALITIES FOR QUASI CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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ABSTRACT. In this paper, Hermite-Hadamard-Fejer type inequalities for quasi-convex via fractional integrals are obtained.

1. INTRODUCTION

The following definition for convex functions is well know in the mathematical literature:

A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$ is said to be convex on I if inequality

$$f\left(tx + (1-t)y\right) \leq tf(x) + (1-t) f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

which holds for all convex functions $f : [a, b] \to \mathbb{R}$, is known in the literature as Hermite-Hadamard's inequality.

In [3], Fejer established the following Hermite-Hadamard Fejer inequality which is the weighted generalization of Hermite-Hadamard inequality.

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be convex function. Then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \leq \int_a^b f(x)\,g(x)\,dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx \tag{1.2}$$

holds, where $g : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $(a + b)/2$.

We recall that the notion of quasi-convex functions generalizes the notion of convex functions. More exactly, a function $f : [a, b] \to \mathbb{R}$, is said quasi-convex on $[a, b]$ if

$$f\left(\lambda x + (1-\lambda)y\right) \leq \sup\left\{f(x), f(y)\right\}, \forall x, y \in [a, b]$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$(see [10]).

Furthermore, there exist quasi-convex functions which are not convex (see [5]).

In [8] Özdemir et. al. represented Hermite-Hadamard's inequalities for quasi-convex functions in fractional integral forms as follows:

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Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|$ is quasi convex on $[a,b]$ and $\alpha>0$, then the following inequality for fractional integrals holds*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\right| \leq \frac{b-a}{(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\sup\left\{|f'(a)|,|f'(b)|\right\}. \tag{1.3}$$

In [9] Set et. al. obtained the following lemma.

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ and let $g:[a,b]\to\mathbb{R}$. If $f',g\in L[a,b]$, then the following identity for fractional integrals holds:*

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\left[J_{\left(\frac{a+b}{2}\right)^-}^{\alpha}g(a)+J_{\left(\frac{a+b}{2}\right)^+}^{\alpha}g(b)\right]-\left[J_{\left(\frac{a+b}{2}\right)^-}^{\alpha}(fg)(a)+J_{\left(\frac{a+b}{2}\right)^+}^{\alpha}(fg)(b)\right]\\ =&\ \frac{1}{\Gamma(\alpha)}\int_a^b k(t)f'(t)\,dt\end{aligned} \tag{1.4}$$

where

$$k(t)=\begin{cases}\int_a^t(s-a)^{\alpha-1}g(s)\,ds & t\in\left[a,\frac{a+b}{2}\right)\\ -\int_t^b(b-s)^{\alpha-1}g(s)\,ds & t\in\left[\frac{a+b}{2},b\right].\end{cases}$$

In [11] İşcan proved the following lemma.

Lemma 2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) and $a<b$ with $f'\in L[a,b]$. If $g:[a,b]\to\mathbb{R}$ integrable and is symmetric to $(a+b)/2$ then the following equality for fractional integrals holds*

$$\begin{aligned}&\frac{f(a)+f(b)}{2}\left[J_{a^+}^{\alpha}g(b)+J_{b^-}^{\alpha}g(a)\right]-\left[J_{a^+}^{\alpha}(fg)(b)+J_{b^-}^{\alpha}(fg)(a)\right]\\ =&\ \frac{1}{\Gamma(\alpha)}\int_a^b\left[\int_a^t(b-s)^{\alpha-1}g(s)\,ds-\int_t^b(s-a)^{\alpha-1}g(s)\,ds\right]f'(t)\,dt\end{aligned} \tag{1.5}$$

with $\alpha>0$.

We give some neccessary definitions and mathematical preliminiaries of fractional calculus theory which are used throughout this paper.

Lemma 3. *([6],[7])For $0<\alpha\leq1$ and $0\leq a<b$, we have*

$$|a^{\alpha}-b^{\alpha}|\leq(b-a)^{\alpha}.$$

Definition 1. *Let $f\in L[a,b]$. The Riemann-Liouville integrals $J_{a^+}^{\alpha}f(x)$ and $J_{b^-}^{\alpha}f(x)$ of oder $\alpha>0$ with $\alpha\geq0$ are defined by*

$$J_{a^+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)\,dt,\ x>a$$

and

$$J_{b^-}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)\,dt,\ x<b$$

respectively, where $\Gamma(\alpha)$ is the Gamma functions by $\Gamma(\alpha)=\int_0^{\infty}e^{-t}t^{\alpha-1}dt$ and $J_{a^+}^{0}f(x)=J_{b^-}^{0}f(x)=f(x)$.

In this paper, motivated by the recent results given in [11], [9] , we established Hermite-Hadamard-Fejer type inequalities for quasi convex functions via fractional integral.

2. MAIN RESULTS

Throughout this paper, let I be an interval on $\mathbb{R}$ and let $\|g\|_{[a,b],\infty}=\sup_{t\in[a,b]} g(x)$, for the continuous function $g:[a,b]\to\mathbb{R}$.

Theorem 3. *Let $f:I\to\mathbb{R}$ be a differentiable mapping on I° and $f'\in L[a,b]$ with $a<b$ and $g:[a,b]\to\mathbb{R}$ is continuous. If $|f'|^{q}$ is quasi convex on $[a,b]$, $q>1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}g(b)\right]-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)_{-}}(fg)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}(fg)(b)\right]\right| \\
\leq\; & \frac{(b-a)^{\alpha+1}\|g\|_{[a,b],\infty}}{2^{\alpha}(\alpha+1)\Gamma(\alpha+1)}\left(\sup\left\{|f'(a)|^{q},|f'(b)|^{q}\right\}\right)^{\frac{1}{q}}
\end{aligned} \tag{2.1}$$

with $\alpha>0$.

Proof. Since is $|f'|^{q}$ is quasi-convex on $[a,b]$, we know that for $t\in[a,b]$

$$|f'(t)|^{q}=\left|f'\left(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\right)\right|^{q}\leq\sup\left\{|f'(a)|^{q},|f'(b)|^{q}\right\} \tag{2.2}$$

Using Lemma 1, Power mean inequality and the quasi-convex of $|f'|^{q}$, it follows that

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}g(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}g(b)\right]-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)_{-}}(fg)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}(fg)(b)\right]\right| \\
\leq\; & \frac{1}{\Gamma(\alpha)}\left(\int_{a}^{\frac{a+b}{2}}\left|\int_{a}^{t}(s-a)^{\alpha-1}g(s)\,ds\right|dt\right)^{1-\frac{1}{q}}\left(\int_{a}^{\frac{a+b}{2}}\left|\int_{a}^{t}(s-a)^{\alpha-1}g(s)\,ds\right||f'(t)|^{q}\,dt\right)^{\frac{1}{q}} \\
&+\frac{1}{\Gamma(\alpha)}\left(\int_{\frac{a+b}{2}}^{b}\left|\int_{t}^{b}(b-s)^{\alpha-1}g(s)\,ds\right|dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{a+b}{2}}^{b}\left|\int_{t}^{b}(b-s)^{\alpha-1}g(s)\,ds\right||f'(t)|^{q}\,dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\|g\|_{[a,\frac{a+b}{2}],\infty}}{\Gamma(\alpha)}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\alpha-1}\,ds\right|dt\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\alpha-1}\,ds\right|\left|f'(t)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad+\frac{\|g\|_{[\frac{a+b}{2},b],\infty}}{\Gamma(\alpha)}\left(\int_{\frac{a+b}{2}}^{b}\left|\int_t^b (b-s)^{\alpha-1}\,ds\right|dt\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_{\frac{a+b}{2}}^{b}\left|\int_t^b (b-s)^{\alpha-1}\,ds\right|\left|f'(t)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{1}{\Gamma(\alpha+1)}\left(\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}} \\
&\quad\times\left\{\|g\|_{[a,\frac{a+b}{2}],\infty}\left(\int_a^{\frac{a+b}{2}}(t-a)^{\alpha}\,dt\right)^{\frac{1}{q}}+\|g\|_{[\frac{a+b}{2},b],\infty}\left(\int_{\frac{a+b}{2}}^{b}(b-t)^{\alpha}\,dt\right)^{\frac{1}{q}}\right\} \\
&\leq \frac{1}{\Gamma(\alpha+1)}\left(\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}\right)^{\frac{1}{q}} \\
&\quad\times\left(\|g\|_{[a,\frac{a+b}{2}],\infty}+\|g\|_{[\frac{a+b}{2},b],\infty}\right)\left(\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}} \\
&\leq \frac{(b-a)^{\alpha+1}\|g\|_{[a,b],\infty}}{\Gamma(\alpha+1)\,2^{\alpha}(\alpha+1)}\left(\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}
\end{aligned}
$$

where it is easily seen that

$$
\begin{aligned}
\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\alpha-1}\,ds\right|dt &= \int_{\frac{a+b}{2}}^{b}\left|\int_t^b (b-s)^{\alpha-1}\,ds\right|dt \\
&= \frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)\,\alpha}.
\end{aligned}
$$

Hence, the proof is completed. □

Corollary 1. *If we choose $g(x)=1$ and $\alpha=1$ in the inequality (2.1), then we have*

$$
\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{b-a}{4}\left(\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}.
$$

We can state another inequality for $q>1$ as follows:

Theorem 4. *Let $f:I\to\mathbb{R}$ be a differentiable mapping on I° and $f'\in L[a,b]$ with $a<b$ and $g:[a,b]\to\mathbb{R}$ is continuous. If $|f'|^q$ is quasi convex on $[a,b]$, $q>1$, then* *the following inequality for fractional integrals holds:*

(2.3)
$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} g(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+} g(b)\right] - \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}(fg)(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}(fg)(b)\right]\right| \\
\leq\ & \frac{(b-a)^{\alpha+1}\|g\|_{\infty}}{2^{\alpha}(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\left(\sup\left\{|f'(a)|^q, |f'(b)|^q\right\}\right)^{\frac{1}{q}}
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 1, Hölder's inequality and the quasi convexity of $|f'|^q$, it follows that

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} g(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+} g(b)\right] - \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)-}(fg)(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}(fg)(b)\right]\right| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\left\{\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\alpha-1} g(s)\,ds\right| |f'(t)|\,dt\right. \\
& \left. + \int_{\frac{a+b}{2}}^b \left|\int_t^b (b-s)^{\alpha-1} g(s)\,ds\right| |f'(t)|\,dt\right\} \\
\leq\ & \frac{1}{\Gamma(\alpha)}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\alpha-1} g(s)\,ds\right|^p dt\right)^{\frac{1}{p}}\left(\int_a^{\frac{a+b}{2}}|f'(t)|^q\,dt\right)^{\frac{1}{q}} \\
& + \frac{1}{\Gamma(\alpha)}\left(\int_{\frac{a+b}{2}}^b\left|\int_t^b (b-s)^{\alpha-1} g(s)\,ds\right|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{a+b}{2}}^b|f'(t)|^q\,dt\right)^{\frac{1}{q}} \\
=\ & \frac{1}{\Gamma(\alpha)}\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\alpha-1} g(s)\,ds\right|^p\right)^{\frac{1}{p}} \\
& \times\left[\left(\int_a^{\frac{a+b}{2}}|f'(t)|^q\,dt\right)^{\frac{1}{q}} + \left(\int_{\frac{a+b}{2}}^b|f'(t)|^q\,dt\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{\|g\|_{\infty}}{\Gamma(\alpha)}\left(\frac{(b-a)^{\alpha p+1}}{2^{\alpha p+1}(\alpha p+1)\alpha^p}\right)^{\frac{1}{p}}\left[\left(\int_a^{\frac{a+b}{2}}\sup\left\{|f'(a)|^q, |f'(b)|^q\right\}dt\right)^{\frac{1}{q}}\right. \\
& \left. + \left(\int_{\frac{a+b}{2}}^b\sup\left\{|f'(a)|^q, |f'(b)|^q\right\}dt\right)^{\frac{1}{q}}\right] \\
=\ & \frac{\|g\|_{\infty}(b-a)^{\alpha+1}}{2^{\alpha}(\alpha p+1)^{1/p}\Gamma(\alpha+1)}\left(\sup\left\{|f'(a)|^q, |f'(b)|^q\right\}\right)^{\frac{1}{q}}.
\end{aligned}$$

Here we use

$$\int_a^{\frac{a+b}{2}}\left|\int_a^t (s-a)^{\alpha-1}\,ds\right|^p dt = \frac{(b-a)^{\alpha p+1}}{2^{\alpha p+1}(\alpha p+1)\alpha^p}$$

$$\int_a^{\frac{a+b}{2}} |f'(t)|^q dt \leq \frac{b-a}{2} \sup\left\{|f'(a)|^q, |f'(b)|^q\right\}$$
$$\int_{\frac{a+b}{2}}^b |f'(t)|^q dt \leq \frac{b-a}{2} \sup\left\{|f'(a)|^q, |f'(b)|^q\right\}.$$

Hence the inequality (2.3) is proved. □

Corollary 2. *If we choose $g(x) = 1$ and $\alpha = 1$ in the inequality (2.3), then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}} \left(\sup\left\{|f'(a)|^q, |f'(b)|^q\right\}\right)^{\frac{1}{q}}.$$

Theorem 5. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a,b]$ with $a < b$. If $|f'|$ is quasi convex on $[a,b]$ and $g : [a,b] \to \mathbb{R}$ is continuous and symmetric to $\frac{(a+b)}{2}$, then the following inequality for fractional integrals holds:*

$$\left|\frac{f(a)+f(b)}{2}\left[J_{a^+}^\alpha g(b) + J_{b^-}^\alpha g(a)\right] - \left[J_{a^+}^\alpha (fg)(b) + J_{b^-}^\alpha (fg)(a)\right]\right| \tag{2.4}$$
$$\leq \frac{2(b-a)^{\alpha+1}\|g\|_\infty}{(\alpha+1)\Gamma(\alpha+1)}\left(1 - \frac{1}{2^\alpha}\right)\sup\{|f'(a)|, |f'(b)|\}$$

with $\alpha > 0$.

Proof. From Lemma 2, we have

$$\left|\frac{f(a)+f(b)}{2}\left[J_{a^+}^\alpha g(b) + J_{b^-}^\alpha g(a)\right] - \left[J_{a^+}^\alpha (fg)(b) + J_{b^-}^\alpha (fg)(a)\right]\right| \tag{2.5}$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_a^b \left|\int_a^t (b-s)^{\alpha-1} g(s)\,ds - \int_t^b (s-a)^{\alpha-1} g(s)\,ds\right| |f'(t)|\,dt.$$

Since $|f'|$ is quasi convex on $[a,b]$, we know that for $t \in [a,b]$

$$|f'(t)| = \left|f'\left(\frac{b-t}{b-a}a + \frac{t-b}{b-a}b\right)\right| \leq \sup\{|f'(a)|, |f'(b)|\} \tag{2.6}$$

and since $g : [a,b] \to \mathbb{R}$ is continuous and symmetric to $(a+b)/2$ we write

$$\int_t^b (s-a)^{\alpha-1} g(s)\,ds = \int_a^{a+b-t} (b-s)^{\alpha-1} g(a+b-s)\,ds$$
$$= \int_a^{a+b-t} (b-s)^{\alpha-1} g(s)\,ds.$$

Then we get

$$\left|\int_a^t (b-s)^{\alpha-1} g(s)\,ds - \int_t^b (s-a)^{\alpha-1} g(s)\,ds\right|$$
$$= \int_t^{a+b-t} (b-s)^{\alpha-1} g(s)\,ds$$
$$\leq \begin{cases} \int_t^{a+b-t} \left|(b-s)^{\alpha-1} g(s)\right| ds, & t \in \left[a, \frac{a+b}{2}\right] \\ \int_{a+b-t}^t \left|(b-s)^{\alpha-1} g(s)\right| ds & t \in \left[\frac{a+b}{2}, b\right]. \end{cases} \tag{2.7}$$

A combination of (2.5) (2.6) and(2.7), we get

$$\begin{aligned}
(2.8)\quad & \left|\frac{f(a)+f(b)}{2}\left[J_{a^+}^{\alpha}g(b)+J_{b^-}^{\alpha}g(a)\right]-\left[J_{a^+}^{\alpha}(fg)(b)+J_{b^-}^{\alpha}(fg)(a)\right]\right| \\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|(b-s)^{\alpha-1}g(s)\right|ds\right)\left(\sup\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}\right)dt \\
& +\frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}\left|(b-s)^{\alpha-1}g(s)\right|ds\right)\sup\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}dt \\
\leq\ & \frac{\|g\|_\infty \sup\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}}{\Gamma(\alpha)} \\
& \times\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|(b-s)^{\alpha-1}\right|ds\right)dt+\int_{\frac{a+b}{2}}^{b}\left(\int_{a+b-t}^{t}\left|(b-s)^{\alpha-1}\right|ds\right)dt\right] \\
=\ & \frac{\|g\|_\infty \sup\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}}{\Gamma(\alpha+1)} \\
& \times\left[\int_a^{\frac{a+b}{2}}\left[(b-t)^{\alpha}-(t-a)^{\alpha}\right]dt+\int_{\frac{a+b}{2}}^{b}\left[(t-a)^{\alpha}-(b-t)^{\alpha}\right]dt\right].
\end{aligned}$$

Since

$$(2.9)\qquad \int_a^{\frac{a+b}{2}}(b-t)^{\alpha}dt=\int_{\frac{a+b}{2}}^{b}(t-a)^{\alpha}dt=\frac{(b-a)^{\alpha+1}\left(2^{\alpha+1}-1\right)}{2^{\alpha+1}(\alpha+1)}$$

and

$$(2.10)\qquad \int_a^{\frac{a+b}{2}}(t-a)^{\alpha}dt=\int_{\frac{a+b}{2}}^{b}(b-t)^{\alpha}dt=\frac{(b-a)^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}.$$

Hence, if we use (2.9) and (2.10) in (2.8), we obtaion the desired result. This completes the proof. □

Remark 1. *In Theorem 1.5, if we take $g(x)=1$, then inequality (2.4), becomes inequality (1.3) of Theorem 2.*

Theorem 6. *Let $f:I\to\mathbb{R}$ be a differentiable mapping on I° and $f'\in L[a,b]$ with $a<b$. If $|f'|^q$, $q\geq 1$, is quasi convex on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ is continuous and symmetric to $\frac{(a+b)}{2}$, then the following inequality for fractional integrals holds*

$$\begin{aligned}
(2.11)\quad & \left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a^+}^{\alpha}g(b)+J_{b^-}^{\alpha}g(a)\right]-\left[J_{a^+}^{\alpha}(fg)(b)+J_{b^-}^{\alpha}(fg)(a)\right]\right| \\
\leq\ & \frac{2(b-a)^{\alpha+1}\|g\|_\infty}{(\alpha+1)\Gamma(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left(\sup\left\{\left|f'(a)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}
\end{aligned}$$

where $\alpha>0$.

Proof. Using Lemma 2, Power mean inequality, (2.7) and the quasi convexity of $|f'|^q$, it follows that

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a^+}^{\alpha}g(b)+J_{b^-}^{\alpha}g(a)\right]-\left[J_{a^+}^{\alpha}(fg)(b)+J_{b^-}^{\alpha}(fg)(a)\right]\right| \tag{2.12}$$
$$\leq \frac{1}{\Gamma(\alpha)}\left(\int_a^b\left|\int_t^{a+b-t}(b-s)^{\alpha-1}g(s)\,ds\right|dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_a^b\left|\int_t^{a+b-t}(b-s)^{\alpha-1}g(s)\,ds\right||f'(t)|^q\,dt\right)^{\frac{1}{q}}$$
$$\leq \frac{1}{\Gamma(\alpha)}\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|(b-s)^{\alpha-1}g(s)\right|ds\right)dt\right.$$
$$\left.+\int_{\frac{a+b}{2}}^b\left(\int_{a+b-t}^t\left|(b-s)^{\alpha-1}g(s)\right|ds\right)dt\right]^{1-\frac{1}{q}}$$
$$\times\left[\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|(b-s)^{\alpha-1}g(s)\right|ds\right)|f'(t)|^q\,dt\right.$$
$$\left.+\int_{\frac{a+b}{2}}^b\left(\int_{a+b-t}^t\left|(b-s)^{\alpha-1}g(s)\right|ds\right)|f'(t)|^q\,dt\right]^{\frac{1}{q}}$$
$$\leq \frac{2(b-a)^{\alpha+1}\|g\|_\infty}{(\alpha+1)\Gamma(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\left(\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}$$

where it is easily seen that

$$\int_a^{\frac{a+b}{2}}\left(\int_t^{a+b-t}\left|(b-s)^{\alpha-1}g(s)\right|ds\right)dt+\int_{\frac{a+b}{2}}^b\left(\int_{a+b-t}^t\left|(b-s)^{\alpha-1}g(s)\right|ds\right)dt$$
$$=\frac{2(b-a)^{\alpha+1}}{\alpha(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right).$$

Hence if we use (2.9) and (2.10) in (2.12), we obtain the desired result. This completes the proof. □

We can state another inequality for $q>1$ as follows:

Theorem 7. *Let $f: I\to\mathbb{R}$ be a differentiable mapping on I° and $f'\in L[a,b]$ with $a<b$.If $|f'|^q$, $q>1$, is quasi convex on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ is continuous and symmetric to $(a+b)/2$, then the following inequality for fractional integrals holds*

(i)

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a^+}^{\alpha}g(b)+J_{b^-}^{\alpha}g(a)\right]-\left[J_{a^+}^{\alpha}(fg)(b)+J_{b^-}^{\alpha}(fg)(a)\right]\right|$$
$$\leq \frac{2^{\frac{1}{p}}\|g\|_\infty(b-a)^{\alpha+1}}{(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\left(1-\frac{1}{2^{\alpha p}}\right)^{\frac{1}{p}}\left(\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}} \tag{2.13}$$

with $\alpha>0$.

(ii)

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right|$$

$$\leq\frac{\|g\|_{\infty}(b-a)^{\alpha+1}}{(\alpha p+1)^{\frac{1}{p}}\Gamma(\alpha+1)}\left(\sup\left\{|f'(a)|^{q},|f'(b)|^{q}\right\}\right)^{\frac{1}{q}} \tag{2.14}$$

for $0<\alpha\leq 1$, where $1/p+1/q=1$.

Proof. (i) Using Lemma 2, Hölder's inequality, (2.7) and the quasi convexity of $|f'|^{q}$, it follows that

$$\left|\left(\frac{f(a)+f(b)}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]-\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\right| \tag{2.15}$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)}\left(\int_{a}^{b}\left|\int_{t}^{a+b-t}(b-s)^{\alpha-1}g(s)\,ds\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_{a}^{b}|f'(t)|^{q}\,dt\right)^{\frac{1}{q}}\\
&\leq \frac{\|g\|_{\infty}}{\Gamma(\alpha+1)}\left(\int_{a}^{\frac{a+b}{2}}\left[(b-t)^{\alpha}-(t-a)^{\alpha}\right]^{p}dt+\int_{\frac{a+b}{2}}^{b}\left[(t-a)^{\alpha}-(b-t)^{\alpha}\right]^{p}dt\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_{a}^{b}\sup\left\{|f'(a)|^{q},|f'(b)|^{q}\right\}\right)^{\frac{1}{q}}\\
&= \frac{\|g\|_{\infty}(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left(\int_{0}^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]^{p}dt+\int_{\frac{1}{2}}^{1}\left[t^{\alpha}-(1-t)^{\alpha}\right]^{p}dt\right)^{\frac{1}{p}}\\
&\quad\times\left(\sup\left\{|f'(a)|^{q},|f'(b)|^{q}\right\}\right)^{\frac{1}{q}}\\
&\leq \frac{\|g\|_{\infty}(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left(\int_{0}^{\frac{1}{2}}\left[(1-t)^{\alpha p}-t^{\alpha p}\right]dt+\int_{\frac{1}{2}}^{1}\left[t^{\alpha p}-(1-t)^{\alpha p}\right]dt\right)^{\frac{1}{p}}\\
&\quad\times\left(\sup\left\{|f'(a)|^{q},|f'(b)|^{q}\right\}\right)^{\frac{1}{q}}\\
&\leq \frac{2^{\frac{1}{p}}\|g\|_{\infty}(b-a)^{\alpha+1}}{\Gamma(\alpha+1)(\alpha p+1)^{\frac{1}{p}}}\left(1-\frac{1}{2^{\alpha p}}\right)^{\frac{1}{p}}\left(\sup\left\{|f'(a)|^{q},|f'(b)|^{q}\right\}\right)^{\frac{1}{q}}.
\end{aligned}$$

Here we use

$$\left[(1-t)^{\alpha}-t^{\alpha}\right]^{p}\leq(1-t)^{\alpha p}-t^{\alpha p}$$

for $t\in\left[0,\frac{1}{2}\right]$ and

$$\left[t^{\alpha}-(1-t)^{\alpha}\right]^{p}\leq t^{\alpha p}-(1-t)^{\alpha p}$$

for $t\in\left[\frac{1}{2},1\right]$ which follows from $(A-B)^{q}\leq A^{q}-B^{q}$ for any $A\geq B\geq 0$ and $q\geq 1$. Hence the inequality (2.13) is proved.

(ii) The inequality (2.14) is easily proved using the inequality (2.15) and Lemma 3. □

References

[1] S. S. Drogamir and C.E.M. Pearce, Selected Topics on Hermite- Hadamard Inequalities and Applications, RGMİA Monographs, Victoria Universty, 2000.

[2] J. Hadamard, Etude sur les proprietes des functions entieres et en particulier d'une fonction consideree par Riemann, J. Math. Pures Appl., 58 (1893), 171-215.

[3] L. Fejer, Uberdie Fourierreihen, II, Math. Naturwise. Anz Ungar. Akad., Wiss, 24 (1906), 369-390, (in Hungarian).

[4] M. Z. Sarıkaya, E. Set, H. Yaldız and N.Başak, Hermite Hadamards inequalities for fractional integrals and related fractional inequalities, Mathematical and Computer Modelling, 57 (9), (2013), 2403-2407.

[5] D. A. Ion, Some estimates on the Hermite-Hadamard inequality through quasi-convex functions, Annals of University of Craiova, Math. Sci. Ser., 34 (2007), 82-87.

[6] A. P. Prudnikov, Y. A. Brychkov and O. I. Marichev, Integral and series. In: Elementary Functions, vol. 1. Nauka, Moscow, 1981

[7] J. Wang, C. Zhu and Y. Zhou, "New generalized Hermite-Hadamard-type inequalitiesand applications to special means", J. Inequal Appl., 2013(325) (2013), 15 pages.

[8] M. E. Özdemir and Çetin Yıldız, The Hadamard's inequality for quasi-convex functions via fractional integrals, Annals of University of Craiova, Math. and Computer Sci. Ser., 40 (2), 2013, Pages 167-173.

[9] E. Set, İ. İşcan, M. E. Özdemir and M. Z. Sarıyaka, Some new Hermite-Hadamard-Fejer type inequalities for convex functions via fractional integrals,submited.

[10] J. E. Pecaric, F. Proschan and Y. L. Tong, Convex Functions, Partial Orderings, and Statistical Applications, Academic Press, Inc, Boston/London, 1992.

[11] İ. İşcan, Hermite-Hadamard-Fejer type inequalities for convex functions via fractional integrals, arXiv:1404.7722v1.

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