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ON SOME NEW INEQUALITIES OF HERMITE-HADAMARD TYPE INVOLVING HARMONICALLY CONVEX FUNCTIONS VIA FRACTIONAL INTEGRALS

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Abstract. In this paper,some new results related to the left-hand side of the Hermite-Hadamard type inequality for harmonically convex functions using Riemann Liouville fractioal integrals are obtained.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality holds

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

In [4], the author introduced the class of harmonically convex functions, defined as follows.

Definition 1. *Let $I \subseteq \mathbb{R}\setminus\{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x)$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1.1) is reversed, then f is said to be harmonically concave.

We recall the following special functions and inequality

(1) The Beta function:

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x, y > 0,$$

(2).The hypergeometric function:

$${}_2F_1(a,b;c;z) = \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt, \ c > b > 0, \ |z| < 1.$$

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult ([1],[2],[7]).

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Definition 2. *Let $f \in L[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of oder $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}f(t)dt, \ x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}f(t)dt, \ x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_{0}^{\infty} e^{-t}t^{\alpha-1}dt$ and $J_{a^+}^{0}f(x) = J_{b^-}^{0}f(x) = f(x)$.

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see ([5],[6],[9],[10],[11]).

In [3], Iscan proved a variant of Hermite-Hadamard inequality which holds for the harmonically convex functions in fractional integral forms as follows:

Theorem 1. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $f \in L[a,b]$, where $a,b \in I$ with $a < b$. If f is a harmonically convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J_{1/a-}^{\alpha}(f\circ g)(1/b) + J_{1/b+}^{\alpha}(f\circ g)(1/a)\right\} \leq \frac{f(a)+f(b)}{2}$$

with $\alpha > 0$.

Lemma 1. *([8],[12]).For $0 < \alpha \leq 1$ and $0 \leq a < b$, we have*

$$|a^{\alpha} - b^{\alpha}| \leq (b-a)^{\alpha}.$$

Lemma 2. *([3])Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. Then the following equality for fractional integrals holds:*

$$\begin{aligned} & I_f(g;\alpha,a,b) \\ = & \frac{ab(b-a)}{2}\int_{0}^{1}\frac{[t^{\alpha}-(1-t)^{\alpha}]}{(ta+(1-t)b)^{2}}f'\left(\frac{ab}{ta+(1-t)b}\right)dt. \end{aligned}$$

where

$$\begin{aligned} & I_f(g;\alpha,a,b) \\ = & \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J_{1/a-}^{\alpha}(f\circ g)(1/b) + J_{1/b+}^{\alpha}(f\circ g)(1/a)\right\}. \end{aligned}$$

with $\alpha > 0, g(x) = 1/x$ and Γ is Euler Gamma function.

In [3], Iscan proved the following theorems using the above Lemma 1 and Lemma 2.

Theorem 2. *Let $f: I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I^\circ$ with $a<b$. If $|f'|^q$ is harmonically convex on $[a,b]$ for some fixed $q \geq 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left|I_f\left(g;\alpha,a,b\right)\right| \\ \leq\ & \frac{ab\left(b-a\right)}{2} C_1^{1-1/q}(\alpha;a,b)\left(C_2(\alpha;a,b)\left|f'(b)\right|^q + C_3(\alpha;a,b)\left|f'(a)\right|^q\right)^{1/q} \end{aligned}$$

where

$$\begin{aligned} C_1(\alpha;a,b) &= \frac{b^{-2}}{\alpha+1}\left[{}_2F_1\left(2,1;\alpha+2;1-\frac{a}{b}\right) + {}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{b}\right)\right], \\ C_2(\alpha;a,b) &= \frac{b^{-2}}{\alpha+2}\left[\frac{1}{\alpha+1}.{}_2F_1\left(2,2;\alpha+3;1-\frac{a}{b}\right) + {}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{a}{b}\right)\right], \\ C_3(\alpha;a,b) &= \frac{b^{-2}}{\alpha+1}\left[{}_2F_1\left(2,1;\alpha+3;1-\frac{a}{b}\right) + \frac{1}{\alpha+1}.{}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{a}{b}\right)\right]. \end{aligned}$$

Theorem 3. *Let $f: I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I^\circ$ with $a<b$. If $|f'|^q$ is harmonically convex on $[a,b]$ for some fixed $q \geq 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left|I_f\left(g;\alpha,a,b\right)\right| \\ \leq\ & \frac{ab\left(b-a\right)}{2} C_1^{1-1/q}(\alpha;a,b)\left(C_2(\alpha;a,b)\left|f'(b)\right|^q + C_3(\alpha;a,b)\left|f'(a)\right|^q\right)^{1/q}, \end{aligned}$$

where

$$\begin{aligned} &C_1\left(\alpha;a,b\right) \\ =\ & \frac{b^{-2}}{\alpha+1}\left[{}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{b}\right) - {}_2F_1\left(2,1;\alpha+2;1-\frac{a}{b}\right)\right. \\ &\left. + {}_2F_1\left(2,1;\alpha+2;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)\right], \end{aligned}$$

$$\begin{aligned} &C_2\left(\alpha;a,b\right) \\ =\ & \frac{b^{-2}}{\alpha+2}\left[{}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{a}{b}\right) - \frac{1}{\alpha+1}.{}_2F_1\left(2,2;\alpha+3;1-\frac{a}{b}\right)\right. \\ &\left. + \frac{1}{2\left(\alpha+1\right)}.{}_2F_1\left(2,2;\alpha+3;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)\right], \end{aligned}$$

$$\begin{aligned} &C_3\left(\alpha;a,b\right) \\ =\ & \frac{b^{-2}}{\alpha+2}\left[\frac{1}{\alpha+1}.{}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{a}{b}\right) - {}_2F_1\left(2,1;\alpha+3;1-\frac{a}{b}\right)\right. \\ &\left. + {}_2F_1\left(2,1;\alpha+3;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)\right] \end{aligned}$$

and $0<\alpha\leq 1$.

Theorem 4. *Let $f: I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I^\circ$ with $a<b$. If $|f'|^q$ is harmonically convex on $[a,b]$ for*

some fixed $q>1$, then the following inequality for fractional integrals holds:

$$\begin{aligned}&\left|I_f\left(g;\alpha,a,b\right)\right|\\ \leq\ & \frac{a\left(b-a\right)}{2b}\left(\frac{1}{\alpha p+1}\right)^{1/p}\left(\frac{\left|f'(b)\right|^q+\left|f'(a)\right|^q}{2}\right)^{1/q}\\ &\times\left[{}_2F_1^{1/p}\left(2p,1;\alpha p+2;1-\frac{a}{b}\right)+{}_2F_1^{1/p}\left(2p,\alpha p+1;\alpha p+2;1-\frac{a}{b}\right)\right],\end{aligned}$$

where $1/p+1/q=1$.

Theorem 5. *Let $f:I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$. If $|f'|^q$ is harmonically convex on $[a,b]$ for some fixed $q>1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\left|I_f\left(g;\alpha,a,b\right)\right|\\ \leq\ & \frac{b-a}{2\left(ab\right)^{1-1/p}}L_{2p-2}^{2-2/p}(a,b)\left(\frac{1}{\alpha q+1}\right)^{1/q}\left(\frac{\left|f'(b)\right|^q+\left|f'(a)\right|^q}{2}\right)^{1/q},\end{aligned}$$

where $1/p+1/q=1$ and $L_{2p-2}(a,b)=\left(\frac{b^{2p-1}-a^{2p-1}}{(2p-1)(b-a)}\right)^{1/(2p-2)}$ is $2p-2$-Logarithmic mean.

Theorem 6. *Let $f:I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I^\circ$ with $a<b$. If $|f'|^q$ is harmonically convex on $[a,b]$ for some fixed $q>1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}\left|I_f\left(g;\alpha,a,b\right)\right| \leq\ & \frac{a\left(b-a\right)}{2b}\left(\frac{1}{\alpha p+1}\right)^{1/p}\\ &\times\left(\frac{{}_2F_1\left(2q,2;3;1-\frac{a}{b}\right)\left|f'(b)\right|^q+{}_2F_1\left(2q,1;3;1-\frac{a}{b}\right)\left|f'(a)\right|^q}{2}\right)^{1/q},\end{aligned}$$

where $1/p+1/q=1$.

In this paper, new identity for fractional integrals have been defined. By using of this identity we obtained some new results related to the left-hand side of the Hermite-Hadamard type inequality for harmonically convex functions via Riemann Liouville fractional integral.

2. Main Results

Let $f:I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, the interior of I. Throughout this section we will take

$$\begin{aligned}&K_f(g;\alpha,a,b)\\ =\ & f\left(\frac{2ab}{a+b}\right)-\frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^\alpha\left\{J_{\frac{1}{a}-}^\alpha\left(f\circ g\right)(1/b)+J_{\frac{1}{b}+}^\alpha\left(f\circ g\right)(1/a)\right\}\end{aligned}$$

where $a,b\in I$, with $a<b,\ \alpha>0,\ g(x)=\frac{1}{x}$ and Γ is Euler Gamma function.

In order to prove our main results, we need the following Lemma:

Lemma 3. *Let $f:I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function on I^o such that $f'\in L\left[a,b\right]$, where $a,b\in I$ with $a<b$. Then the following equality for fractional*

integral holds :

$$\mathcal{K}_f\left(g;\alpha,a,b\right)=\frac{1}{2}\sum_{k=1}^{3}I_k \tag{2.1}$$

where $A_t=ta+(1-t)\,b$ *and*

$$\begin{aligned}
I_1 &= ab\,(b-a)\int\limits_0^{1/2} f'\left(\tfrac{ab}{A_t}\right)\tfrac{dt}{A_{t^2}}\\
I_2 &= -ab\,(b-a)\int\limits_{1/2}^{1} f'\left(\tfrac{ab}{A_t}\right)\tfrac{dt}{A_{t^2}}\\
I_3 &= -ab\,(b-a)\int\limits_0^{1}\left[(1-t)^{\alpha}-t^{\alpha}\right] f'\left(\tfrac{ab}{A_t}\right)\tfrac{dt}{A_{t^2}}.
\end{aligned}$$

Proof. Calculating I_1, I_2 and I_3 we have

$$\begin{aligned}
I_1 &= ab\,(b-a)\int\limits_0^{1/2} f'\left(\frac{ab}{A_t}\right)\frac{dt}{A_{t^2}}\\
&= \int\limits_0^{1/2} df\left(\frac{ab}{A_t}\right)\\
&= f\left(\frac{ab}{A_t}\right)\Bigg|_0^{1/2}\\
&= f\left(\frac{2ab}{a+b}\right)-f\left(a\right),
\end{aligned}$$

$$\begin{aligned}
I_2 &= -ab\,(b-a)\int\limits_{1/2}^{1} f'\left(\frac{ab}{A_t}\right)\frac{dt}{A_{t^2}}\\
&= -\int\limits_{1/2}^{1} df\left(\frac{ab}{A_t}\right)\\
&= -f\left(\frac{ab}{A_t}\right)\Bigg|_{1/2}^{1}\\
&= f\left(\frac{2ab}{a+b}\right)-f\left(b\right)
\end{aligned}$$

and

$$\begin{aligned} I_3 &= -ab\,(b-a)\int\limits_0^1 \left[(1-t)^{\alpha}-t^{\alpha}\right] f'\left(\frac{ab}{A_t}\right)\frac{dt}{A_{t^2}} \\ &= -\int\limits_0^1 \left[(1-t)^{\alpha}-t^{\alpha}\right] df\left(\frac{ab}{A_t}\right) \\ &= -\int\limits_0^1 (1-t)^{\alpha}\, df\left(\frac{ab}{A_t}\right)+\int\limits_0^1 t^{\alpha} df\left(\frac{ab}{A_t}\right) \\ &= I_3^{*}+I_3^{**}. \end{aligned}$$

By integrating by part in I_3^{*}, we get

$$\begin{aligned} I_3^{*} &= -\int\limits_0^1 (1-t)^{\alpha}\, df\left(\frac{ab}{A_t}\right) \\ &= (1-t)^{\alpha} f\left(\frac{ab}{A_t}\right)\Big|_0^1-\alpha\int\limits_0^1 (1-t)^{\alpha-1} f\left(\frac{ab}{A_t}\right)dt \\ &= f(a)-\alpha\int\limits_0^1 (1-t)^{\alpha-1} f\left(\frac{ab}{A_t}\right)dt. \end{aligned}$$

Here, by the changes of variables $u=\frac{ab}{A_t}$, we get

$$\begin{aligned} I_3^{*} &= f(a)-\alpha\int\limits_a^b \left(\frac{ab}{b-a}\right)^{\alpha-1}\left(\frac{1}{u}-\frac{1}{b}\right)^{\alpha-1} f(u)\frac{ab}{b-a}\frac{1}{u^2}du \\ &= f(a)-\alpha\left(\frac{ab}{b-a}\right)^{\alpha}\int\limits_a^b \left(\frac{1}{u}-\frac{1}{b}\right)^{\alpha-1}\frac{1}{u^2}f(u)du \end{aligned}$$

and again by the changes of variables $u=\frac{1}{t}$, we get

$$\begin{aligned} I_3^{*} &= f(a)-\alpha\left(\frac{ab}{b-a}\right)^{\alpha}\int_{1/a}^{1/b}\left(t-\frac{1}{b}\right)^{\alpha-1} t^2\left(-\frac{1}{t^2}\right) f(\frac{1}{t})dt \\ &= f(a)+\alpha\left(\frac{ab}{b-a}\right)^{\alpha}\int_{1/a}^{1/b}\left(t-\frac{1}{b}\right)^{\alpha-1} f\circ g(t)dt \\ &= f(a)-\alpha\left(\frac{ab}{b-a}\right)^{\alpha}\Gamma(\alpha)\frac{1}{\Gamma(\alpha)}\int_{1/b}^{1/a}\left(t-\frac{1}{b}\right)^{\alpha-1} f\circ g(t)dt \\ &= f(a)-\Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha} J_{1/a-}^{\alpha} f\circ g(1/b). \end{aligned}$$

Similarly, we get $I_3^{**}=\int\limits_0^1 t^{\alpha} df\left(\frac{ab}{A_t}\right)=f(b)-\Gamma(\alpha+1)\left(\frac{ab}{b-a}\right)^{\alpha} J_{1/b+}^{\alpha} f\circ g(1/a)$.

By adding I_1, I_2 and I_3, the desired result is obtained.

Using this Lemma, we can obtain the following inequalities. ∎

Theorem 7. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ a differentiable increasing function on I° such that $f' \in L[a,b]$, where $a,b \in I^\circ$ with $a<b$. If $(f')^q$ is harmonically convex on $[a,b]$ for some fixed $q \geq 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}&\left|\mathcal{K}_f\left(g;\alpha,a,b\right)\right| \\ \leq\ & \frac{f(b)-f(a)}{2}+\frac{ab\left(b-a\right)}{2}C_1^{1-1/q}(\alpha;a,b)\left(C_2(\alpha;a,b)\left(f'(b)\right)^q+C_3(\alpha;a,b)\left(f'(a)\right)^q\right)^{1/q},\end{aligned}$$

where

$$\begin{aligned}C_1(\alpha;a,b) &= \frac{b^{-2}}{\alpha+1}\left[{}_2F_1\left(2,1;\alpha+2;1-\frac{a}{b}\right)+{}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{b}\right)\right],\\ C_2(\alpha;a,b) &= \frac{b^{-2}}{\alpha+2}\left[\frac{1}{\alpha+1}\cdot{}_2F_1\left(2,2;\alpha+3;1-\frac{a}{b}\right)+{}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{a}{b}\right)\right],\\ C_3(\alpha;a,b) &= \frac{b^{-2}}{\alpha+1}\left[{}_2F_1\left(2,1;\alpha+3;1-\frac{a}{b}\right)+\frac{1}{\alpha+1}\cdot{}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{a}{b}\right)\right].\end{aligned}$$

Proof. Let $A_t = ta+(1-t)\,b$. From Lemma 3, using the property of the modulus , the power mean inequality and harmonically convexity of $(f')^q$,we find

$$\begin{aligned}\left|\mathcal{K}_f\left(g;\alpha,a,b\right)\right| &\leq \frac{1}{2}\left\{|I_1|+|I_2|+|I_3|\right\}\\ &= \frac{ab(b-a)}{2}\left[\int_0^{1/2} f'\left(\frac{ab}{A_t}\right)\frac{dt}{A_t^2}+\int_{1/2}^{1} f'\left(\frac{ab}{A_t}\right)\frac{dt}{A_t^2}\right]+\frac{1}{2}\left|I_3\right|\\ &= \frac{f(b)-f(a)}{2}+\frac{1}{2}\left|I_3\right|.\end{aligned}$$

∎

As in the proof of the Theorem 2, we have

$$\begin{aligned}\frac{1}{2}\left|I_3\right| &= \frac{ab(b-a)}{2}\left|\int_0^1\left[(1-t)^\alpha-t^\alpha\right]f'\left(\frac{ab}{A_t}\right)\frac{dt}{A_t^2}\right|\\ &\leq \frac{ab\left(b-a\right)}{2}C_1^{1-1/q}(\alpha;a,b)\left(C_2(\alpha;a,b)\left(f'(b)\right)^q+C_3(\alpha;a,b)\left(f'(a)\right)^q\right)^{1/q},\end{aligned}$$

where

$$\begin{aligned}C_1(\alpha;a,b) &= \frac{b^{-2}}{\alpha+1}\left[{}_2F_1\left(2,1;\alpha+2;1-\frac{a}{b}\right)+{}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{b}\right)\right],\\ C_2(\alpha;a,b) &= \frac{b^{-2}}{\alpha+2}\left[\frac{1}{\alpha+1}\cdot{}_2F_1\left(2,2;\alpha+3;1-\frac{a}{b}\right)+{}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{a}{b}\right)\right],\\ C_3(\alpha;a,b) &= \frac{b^{-2}}{\alpha+1}\left[{}_2F_1\left(2,1;\alpha+3;1-\frac{a}{b}\right)+\frac{1}{\alpha+1}\cdot{}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{a}{b}\right)\right].\end{aligned}$$

The proof is competed.

Theorem 8. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable increasing function on I° such that $f' \in L[a,b]$, where $a,b \in I^\circ$ with $a < b$. If $(f')^q$ is harmonically convex on $[a,b]$ for some fixed $q \geq 1$, then the following inequality for fractional integrals holds:*

$$\begin{aligned} & \left|\mathcal{K}_f\left(g;\alpha,a,b\right)\right| \\ \leq\ & \frac{f(b)-f(a)}{2} + \frac{ab\left(b-a\right)}{2} C_1^{1-1/q}(\alpha;a,b)\left(C_2(\alpha;a,b)(f'(b))^q + C_3(\alpha;a,b)(f'(a))^q\right)^{1/q}, \end{aligned}$$

where

$$\begin{aligned} & C_1(\alpha;a,b) \\ =\ & \frac{b^{-2}}{\alpha+1}\left[{}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{b}\right) - {}_2F_1\left(2,1;\alpha+2;1-\frac{a}{b}\right)\right. \\ & \left. + {}_2F_1\left(2,1;\alpha+2;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)\right], \end{aligned}$$

$$\begin{aligned} & C_2(\alpha;a,b) \\ =\ & \frac{b^{-2}}{\alpha+2}\left[{}_2F_1\left(2,\alpha+2;\alpha+3;1-\frac{a}{b}\right) - \frac{1}{\alpha+1}.{}_2F_1\left(2,2;\alpha+3;1-\frac{a}{b}\right)\right. \\ & \left. + \frac{1}{2\left(\alpha+1\right)}.{}_2F_1\left(2,2;\alpha+3;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)\right], \end{aligned}$$

$$\begin{aligned} & C_3(\alpha;a,b) \\ =\ & \frac{b^{-2}}{\alpha+2}\left[\frac{1}{\alpha+1}.{}_2F_1\left(2,\alpha+1;\alpha+3;1-\frac{a}{b}\right) - {}_2F_1\left(2,1;\alpha+3;1-\frac{a}{b}\right)\right. \\ & \left. + {}_2F_1\left(2,1;\alpha+3;\frac{1}{2}\left(1-\frac{a}{b}\right)\right)\right] \end{aligned}$$

and $0 < \alpha \leq 1$.

Proof. Let $A_t = ta + (1-t)\,b$. From Lemma 3, using the property of the modulus , the power mean inequality and the harmonically convexity of $(f')^q$, we find

$$\begin{aligned} \left|\mathcal{K}_f\left(g;\alpha,a,b\right)\right| &\leq \frac{1}{2}\left(|I_1| + |I_2| + |I_3|\right) \\ &\leq \frac{f(b)-f(a)}{2} + \frac{1}{2}\left|I_3\right| \end{aligned}$$

$$
\begin{aligned}
(2.2)\quad & \frac{1}{2}\left|I_3\right| \\
\leq\ & \frac{ab\left(b-a\right)}{2}\int_0^1 \frac{\left|(1-t)^\alpha - t^\alpha\right|}{A_t^2} f'\left(\frac{ab}{A_t}\right)dt \\
\leq\ & \frac{ab\left(b-a\right)}{2}\left(\int_0^1 \frac{\left|(1-t)^\alpha - t^\alpha\right|}{A_t^2}dt\right)^{1-1/q}\left(\int_0^1 \frac{\left|(1-t)^\alpha - t^\alpha\right|}{A_t^2}\left(f'\left(\frac{ab}{A_t}\right)\right)^q dt\right)^{1/q} \\
\leq\ & \frac{ab\left(b-a\right)}{2}K_1^{1-1/q}\left(\int_0^1 \frac{\left|(1-t)^\alpha - t^\alpha\right|}{A_t^2}\left[t(f'(b))^q + (1-t)(f'(a))^q\right]dt\right)^{1/q} \\
\leq\ & \frac{ab\left(b-a\right)}{2}K_1^{1-1/q}\left(K_2(f'(b))^q + K_3(f'(a))^q\right)^{1/q},
\end{aligned}
$$

where

$$K_1 = \int_0^1 \frac{\left|(1-t)^\alpha - t^\alpha\right|}{A_t^2}dt,$$

$$K_2 = \int_0^1 \frac{\left|(1-t)^\alpha - t^\alpha\right|}{A_t^2}tdt,$$

$$K_3 = \int_0^1 \frac{\left|(1-t)^\alpha - t^\alpha\right|}{A_t^2}\left(1-t\right)dt.$$

If K_1, K_2 and K_3 are calculated as in the proof of the Theorem 3, and used in the inequality (2.2), the desired result is obtained.

Theorem 9. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable increasing function on I° such that $f' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$. If $(f')^q$ is harmonically convex on $[a,b]$ for some fixed $q > 1$, then the following inequality for fractional integrals holds:*

$$
\begin{aligned}
(2.3)\quad & \left|\mathcal{K}_f\left(g;\alpha,a,b\right)\right| \\
\leq\ & \frac{f(b)-f(a)}{2} + \frac{a\left(b-a\right)}{2b}\left(\frac{1}{\alpha p+1}\right)^{1/p}\left(\frac{(f'(b))^q + (f'(a))^q}{2}\right)^{1/q} \\
& \times\left[{}_2F_1^{1/p}\left(2p,1;\alpha p+2;1-\frac{a}{b}\right) + {}_2F_1^{1/p}\left(2p,\alpha p+1;\alpha p+2;1-\frac{a}{b}\right)\right],
\end{aligned}
$$

where $1/p + 1/q = 1$.

□

Proof. Let $A_t = ta + (1-t)b$. From Lemma 3, using the Hölder inequality and the harmonically convexity of $(f')^q$, we get

$$
\begin{aligned}
&\left|\mathcal{K}_f\left(g;\alpha,a,b\right)\right| \\
\leq\ & \frac{f(b)-f(a)}{2}+\frac{ab\left(b-a\right)}{2}\left\{\int_0^1 \frac{(1-t)^{\alpha}}{A_t^2} f'\left(\frac{ab}{A_t}\right)dt+\int_0^1 \frac{t^{\alpha}}{A_t^2} f'\left(\frac{ab}{A_t}\right)dt\right\} \\
\leq\ & \frac{f(b)-f(a)}{2}+\frac{ab\left(b-a\right)}{2}\left\{\left(\int_0^1 \frac{(1-t)^{\alpha p}}{A_t^{2p}}dt\right)^{1/p}\left(\int_0^1 \left(f'\left(\frac{ab}{A_t}\right)\right)^q dt\right)^{1/q}\right. \\
&\left.+\left(\int_0^1 \frac{t^{\alpha p}}{A_t^{2p}}dt\right)^{1/p}\left(\int_0^1 \left(f'\left(\frac{ab}{A_t}\right)\right)^q dt\right)^{1/q}\right\} \\
\leq\ & \frac{f(b)-f(a)}{2}+\frac{ab\left(b-a\right)}{2}\left(K_4^{1/p}+K_5^{1/p}\right)\left(\int_0^1 \left[t(f'(b))^q+(1-t)(f'(a))^q\right]dt\right)^{1/q}
\end{aligned}
$$

$$
\leq \frac{f(b)-f(a)}{2}+\frac{ab\left(b-a\right)}{2}\left(K_4^{1/p}+K_5^{1/p}\right)\left(\frac{f'(b)^q+f'(a)^q}{2}\right)^{1/q}. \tag{2.4}
$$

As in the proof of Theorem 4, calculating K_4 and K_5, we have

$$
\begin{aligned}
K_4 &= \int_0^1 \frac{(1-t)^{\alpha p}}{A_t^{2p}}dt \\
&= \frac{b^{-2p}}{\alpha p+1}\cdot {}_2F_1\left(2p,1;\alpha p+2;1-\frac{a}{b}\right),
\end{aligned} \tag{2.5}
$$

$$
\begin{aligned}
K_5 &= \int_0^1 \frac{t^{\alpha p}}{A_t^{2p}}dt \\
&= \frac{b^{-2p}}{\alpha p+1}\cdot {}_2F_1\left(2p,\alpha p+1;\alpha p+2;1-\frac{a}{b}\right).
\end{aligned} \tag{2.6}
$$

Thus , if we use (2.5) and (2.6) in (2.4), we obtain the inequality of (2.3).This completes the proof. □

Theorem 10. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable increasing function on I° such that $f' \in L[a,b]$, where $a, b \in I^{\circ}$ with $a < b$. If $(f')^q$ is harmonically convex on $[a,b]$ for some fixed $q > 1$, then the following inequality for fractional integrals holds:*

$$
\begin{aligned}
&\left|\mathcal{K}_f\left(g;\alpha,a,b\right)\right| \\
\leq\ & \frac{f(b)-f(a)}{2}+\frac{b-a}{2\left(ab\right)^{1-1/p}}L_{2p-2}^{2-2/p}(a,b)\left(\frac{1}{\alpha q+1}\right)^{1/q}\left(\frac{(f'(b))^q+(f'(a))^q}{2}\right)^{1/q},
\end{aligned} \tag{2.7}
$$

where $1/p+1/q=1$ and $L_{2p-2}(a,b)=\left(\frac{b^{2p-1}-a^{2p-1}}{(2p-1)(b-a)}\right)^{1/(2p-2)}$ is $2p-2$-Logarithmic mean.

Proof. Let $A_t = ta+(1-t)\,b$. From Lemma 3 and Lemma 1, using the Hölder inequality and the Harmonically convexity of $(f')^q$, we get

$$\begin{aligned}
&\left|\mathcal{K}_f\left(g;\alpha,a,b\right)\right| \\
\leq\ & \frac{f(b)-f(a)}{2}+\frac{ab\,(b-a)}{2}\int_0^1 \frac{|(1-t)^\alpha - t^\alpha|}{A_t^2}\left(f'\left(\frac{ab}{A_t}\right)\right)dt \\
\leq\ & \frac{f(b)-f(a)}{2}+\frac{ab\,(b-a)}{2}\left(\int_0^1 \frac{1}{A_t^{2p}}dt\right)^{1/p}\left(\int_0^1 |(1-t)^\alpha - t^\alpha|^q\left(f'\left(\frac{ab}{A_t}\right)\right)^q dt\right)^{1/q} \\
\leq\ & \frac{f(b)-f(a)}{2}+\frac{ab\,(b-a)}{2}\left(\int_0^1 \frac{1}{A_t^{2p}}dt\right)^{1/p}\left(\int_0^1 |1-2t|^{\alpha q}\left[t\left(f'(b)\right)^q+(1-t)\left(f'(a)\right)^q\right]dt\right)^{1/q}
\end{aligned}$$

$$\leq \frac{f(b)-f(a)}{2}+\frac{ab\,(b-a)}{2}K_6^{1/p}\left(K_7\left|f'(b)\right|^q+K_8\left|f'(a)\right|^q\right)^{1/q}, \tag{2.8}$$

where as in the proof of Theorem 5

$$\begin{aligned}
K_6 &= \int_0^1 \frac{1}{A_t^{2p}}dt = b^{-2p}\int_0^1 \left(1-t\left(1-\frac{a}{b}\right)\right)^{-2p}dt \\
&= b^{-2p}.{}_2F_1\left(2p,1;2;1-\frac{a}{b}\right)=\frac{L_{2p-2}^{2p-2}(a,b)}{(ab)^{2p-1}},
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
K_7 &= \int_0^1 |1-2t|^{\alpha q}\,tdt \\
&= \int_0^{1/2} (1-2t)^{\alpha q}\,tdt+\int_{1/2}^1 (2t-1)^{\alpha q}\,tdt \\
&= \frac{1}{2\left(\alpha q+1\right)},
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
K_8 &= \int_0^1 |1-2t|^{\alpha q}\,(1-t)dt \\
&= \frac{1}{2\left(\alpha q+1\right)}.
\end{aligned} \tag{2.11}$$

If we use (2.9),(2.10) and (2.11) in (2.8), we obtain the inequality of (2.7). This completes the proof.

Theorem 11. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable increasing function on I° such that $f' \in L[a,b]$, where $a,b \in I^\circ$ with $a<b$. If $(f')^q$ is harmonically convex on $[a,b]$ for some fixed $q>1$, then the following inequality for fractional integrals holds:*

$$
\begin{aligned}
&\left|\mathcal{K}_f\left(g;\alpha,a,b\right)\right| \\
\leq\ & \frac{f(b)-f(a)}{2}+\frac{a\left(b-a\right)}{2b}\left(\frac{1}{\alpha p+1}\right)^{1/p} \\
&\times\left(\frac{{}_2F_1\left(2q,2;3;1-\frac{a}{b}\right)\left(f'(b)\right)^q+{}_2F_1\left(2q,1;3;1-\frac{a}{b}\right)\left(f'(a)\right)^q}{2}\right)^{1/q},
\end{aligned}
\tag{2.12}
$$

where $1/p+1/q=1$.

□

Proof. Let $A_t = ta+(1-t)b$. From Lemma 3 and Lemma 1, using the Hölder inequality and Harmonically convexity of $(f')^q$, we find

$$
\begin{aligned}
&\left|\mathcal{K}_f\left(g;\alpha,a,b\right)\right| \\
\leq\ & \frac{f(b)-f(a)}{2}+\frac{ab\left(b-a\right)}{2}\int_0^1\frac{\left|(1-t)^\alpha-t^\alpha\right|}{A_t^2}\left(f'\left(\frac{ab}{A_t}\right)\right)dt \\
\leq\ & \frac{f(b)-f(a)}{2}+\frac{ab\left(b-a\right)}{2}\left(\int_0^1\left|(1-t)^\alpha-t^\alpha\right|^p dt\right)^{1/p}\left(\int_0^1\frac{1}{A_t^{2q}}\left(f'\left(\frac{ab}{A_t}\right)\right)^q dt\right)^{1/q} \\
\leq\ & \frac{f(b)-f(a)}{2}+\frac{ab\left(b-a\right)}{2}\left(\int_0^1\left|1-2t\right|^{\alpha p} dt\right)^{1/p}\left(\int_0^1\frac{1}{A_t^{2q}}\left[t\left(f'(b)\right)^q+(1-t)\left(f'(a)\right)^q\right]dt\right)^{1/q}
\end{aligned}
$$

$$
\leq \frac{f(b)-f(a)}{2}+\frac{ab\left(b-a\right)}{2}K_9^{1/p}\left(K_{10}\left(f'(b)\right)^q+K_{11}\left(f'(a)\right)^q\right)^{1/q},
\tag{2.13}
$$

where as in the proof of Theorem 6,

$$
K_9=\int_0^1\left|1-2t\right|^{\alpha p}dt=\frac{1}{\alpha p+1}
\tag{2.14}
$$

$$
\begin{aligned}
K_{10} &= \int_0^1 tA_t^{-2q}dt=b^{-2q}\int_0^1 t\left(1-t\left(1-\frac{a}{b}\right)\right)^{-2q}dt \\
&= \frac{1}{2b^{2q}}\cdot{}_2F_1\left(2q,2;3;1-\frac{a}{b}\right)
\end{aligned}
\tag{2.15}
$$

and

$$
K_{11}=\int_0^1(1-t)A_t^{-2q}dt=\frac{1}{2b^{2q}}\cdot{}_2F_1\left(2q,1;3;1-\frac{a}{b}\right)
\tag{2.16}
$$

Thus, if we use (2.14),(2.15) and (2.16) in (2.13), we obtain the inequality of (2.12). This completes the proof. □

References

[1] R.Gorenflo, F.Mainardi, "Fractionalcalculus;integral and differentiable equations of fractional order", Springer Verlag, Wien,1997,p.223-276

[2] S.G.Samko, A.A.Kilbas, O.I.Marichev, "Fractional Integrals and Derivates Theory and Application ,Gordon and Breach Science", New York,1993

[3] İ. İşcan and S. Wu, "Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals",Appl. Math. Compt., 238 (2014), 237-244.

[4] İ. İşcan, "Hermite-Hadamard type inequalities for harmonically convex functions", *Hacet. J. Math. Stat.*, Accepted.

[5] İ. İşcan, Generalization of different type integral inequalitiesfor s-convex functions via fractional integrals, Applicable Analysis: An Int. J., 93 (9) (2014), 1846–1862.

[6] İ. İşcan, "New general integral inequalities for quasi-geometrically convex functions via fractional integrals", *J. Inequal. Appl.*, 2013(491), 15 pages, 2013.

[7] A.A. Kilbas, H.M. Srivastava HM and J.J. Trujillo, *Theory and applications of fractional differential equations*, Amsterdam, Elsevier, 2006.

[8] A.P. Prudnikov, Y.A. Brychkov and O.I. Marichev, *Integral and series. In: Elementary Functions*, vol. 1. Nauka, Moscow, 1981.

[9] M.Z. Sarikaya, E. Set, H. Yaldiz and N. Başak, "Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities", *Math. Comput. Model.,* 57, 2403-2407, 2013.

[10] E. Set, "New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals", *Comput. Math. Appl.*, 63, 1147-1154, 2012.

[11] J. Wang, X. Li, M. Fečkan and Y. Zhou, "Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity", *Appl. Anal.,* 92(11), 2241-2253, 2012.

[12] J. Wang, C. Zhu and Y. Zhou, "New generalized Hermite-Hadamard type inequalities and applications to special means", *J. Inequal. Appl.,* 2013(325), 15 pages, 2013.

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