Received 24/12/14

ON GENERALIZED INEQUALITIES OF HERMITE-HADAMARD TYPE FOR CONVEX FUNCTIONS

ÇETİN YILDIZ★,♠ AND M. EMİN ÖZDEMİR★

Abstract. In this paper, new integral inequalities of Hermite-Hadamard type are developed for $n-$times differentiable convex functions. Also a parallel development is made base on concavity.

1. INRODUCTION

A function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is said to be convex if whenever $x,y\in[a,b]$ and $t\in[0,1]$, the following inequality holds:

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y).$$

We say that f is concave if $(-f)$ is convex. This definition has its origins in Jensen's results from [6] and has opened up the most extended, useful and multidisciplinary domain of mathematics, namely, convex analysis. Convex curves and convex bodies have appeared in mathematical literature since antiquity and there are many important results related to them.

On November 22, 1881, Hermite (1822-1901) sent a letter to the Journal Mathesis. This letter was published in Mathesis 3 (1883, p: 82) and in this letter an inequality presented which is well-known in the literature as Hermite-Hadamard integral inequality:

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2} \tag{1.1}$$

where $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is a convex function on the interval I of a real numbers and $a,b\in I$ with $a<b$. If the function f is concave, the inequality in (1.1) is reversed.

The inequalities (1.1) have become an important cornerstone in mathematical anlysis and optimization. Many uses of these inequalities have been discovered in a variety of settings. Moreover , many inequalities of special means can be obtained for a particular choice of the function f. Due to the rich geometrical significance of Hermite-Hadamard's inequality, there is growing literature providing its new proofs, extensions, refinements and generalizations, see for example ([4], [7]-[11], [13], [15]-[19]) and the references therein.

In 2000, Cerone *et. al.* (see [3]) proved the following generalization for $n-$time differentiable functions.

2000 *Mathematics Subject Classification.* 26D15, 26D10.

Key words and phrases. Hermite-Hadamard Inequality, Hölder Inequality, Convex Functions.

♠Corresponding Author.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}$ ($n \geq 1$) is absolutely continuous on $[a,b]$. Then*

$$\int_a^b f(t)dt = \sum_{k=0}^{n-1} \frac{1}{(k+1)!}\left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)\right] + \frac{1}{n!}\int_a^b (x-t)^n f^{(n)}(t)dt,$$

for all $x \in [a,b]$.

For other recent results concerning the $n-$time differentiable functions see [1]-[3], [5], [7], [12], [14], [18] where further references are given.

In [8], Kavurmacı *et. al.* obtained the following theorems.

Theorem 2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{(x-a)f(a) + (b-x)f(b)}{(b-a)} - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{(x-a)^2}{(b-a)}\left[\frac{2|f'(a)| + |f'(x)|}{6}\right] + \frac{(b-x)^2}{(b-a)}\left[\frac{|f'(x)| + 2|f'(b)|}{6}\right] \tag{1.2}$$

for each $x \in [a,b]$.

Theorem 3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^{\frac{p}{p-1}}$ is convex on $[a,b]$ and for some fixed $q > 1$, then the following inequality holds:*

$$\left|\frac{(x-a)f(a) + (b-x)f(b)}{(b-a)} - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}} \times \left[\frac{(x-a)^2\left[|f'(a)|^q + |f'(x)|^q\right]^{\frac{1}{q}} + (b-x)^2\left[|f'(x)|^q + |f'(b)|^q\right]^{\frac{1}{q}}}{b-a}\right] \tag{1.3}$$

and $q = \frac{p}{p-1}$.

Theorem 4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a,b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$ and for some fixed $q \geq 1$, then the following inequality holds:*

$$\left|\frac{(x-a)f(a) + (b-x)f(b)}{(b-a)} - \frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{1}{2}\left(\frac{1}{3}\right)^{\frac{1}{q}} \times \left[\frac{(x-a)^2\left[|f'(x)|^q + 2|f'(a)|^q\right]^{\frac{1}{q}} + (b-x)^2\left[|f'(x)|^q + 2|f'(b)|^q\right]^{\frac{1}{q}}}{b-a}\right] \tag{1.4}$$

for each $x \in [a,b]$.

The main purpose of the present paper is to establish several new inequalities for $n-$ time differantiable mappings that are connected with the celebrated Hermite-Hadamard integral inequality.

2. MAIN RESULTS

Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be n-time differentiable functions. If $f^{(n)}\in L[a,b]$, then*

$$\begin{aligned} (2.1)\qquad \int_a^b f(t)dt &= \sum_{k=0}^{n-1}\frac{(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!} \\ &+(-1)^n\frac{(x-a)^{n+1}}{n!}\int_0^1(t-1)^n f^{(n)}(tx+(1-t)a)dt \\ &+(-1)^n\frac{(b-x)^{n+1}}{n!}\int_0^1(1-t)^n f^{(n)}(tx+(1-t)b)dt \end{aligned}$$

where $x\in[a,b]$ and n natural number, $n\geq 1$.

Proof. The proof is by mathematical induction.

The case $n=1$ is [[8], Lemma 1].

Assume that (2.1) holds for "n" and let us prove it for "$n+1$". That is, we have to prove the equality

$$\begin{aligned} (2.2)\int_a^b f(t)dt &= \sum_{k=0}^{n}\frac{(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!} \\ &+(-1)^{n+1}\frac{(x-a)^{n+2}}{(n+1)!}\int_0^1(t-1)^{n+1} f^{(n+1)}(tx+(1-t)a)dt \\ &+(-1)^{n+1}\frac{(b-x)^{n+2}}{(n+1)!}\int_0^1(1-t)^{n+1} f^{(n+1)}(tx+(1-t)b)dt \end{aligned}$$

where $x\in[a,b]$.

Then, we can write

$$\begin{aligned} I &= \frac{(x-a)^{n+2}}{(n+1)!}\int_0^1(t-1)^{n+1} f^{(n+1)}(tx+(1-t)a)dt \\ &+\frac{(b-x)^{n+2}}{(n+1)!}\int_0^1(1-t)^{n+1} f^{(n+1)}(tx+(1-t)b)dt \end{aligned}$$

and integrating by parts gives

$$\begin{aligned} I &= \frac{(x-a)^{n+2}}{(n+1)!}\left\{(t-1)^{n+1}\left.\frac{f^{(n)}(tx+(1-t)a)}{x-a}\right|_0^1\right. \\ &\qquad \left.-\frac{m+1}{x-a}\int_0^1(t-1)^n f^{(n)}(tx+(1-t)a)dt\right\} \end{aligned}$$

$$
\begin{aligned}
&+\frac{(b-x)^{n+2}}{(n+1)!}\left\{(1-t)^{n+1}\ \frac{f^{(n)}(tx+(1-t)b)}{x-b}\Big|_0^1\right.\\
&\qquad\left.+\frac{m+1}{x-b}\int_0^1(1-t)^n f^{(n)}(tx+(1-t)b)dt\right\}\\
=\;&(-1)^{n+2}\frac{(x-a)^{n+1}}{(n+1)!}f^{(n)}(a)-\frac{(x-a)^{n+1}}{n!}\int_0^1(t-1)^n f^{(n)}(tx+(1-t)a)dt\\
&+\frac{(b-x)^{n+1}}{(n+1)!}f^{(n)}(b)-\frac{(b-x)^{n+1}}{n!}\int_0^1(1-t)^n f^{(n)}(tx+(1-t)b)dt.
\end{aligned}
$$

Now, using the mathematical induction hypothesis, we get

$$
\begin{aligned}
\frac{1}{(-1)^n}\int_a^b f(t)dt \;=\;& \frac{1}{(-1)^n}\sum_{k=0}^{n-1}\frac{(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!}\\
&+(-1)^{n+2}\frac{(x-a)^{n+1}}{(n+1)!}f^{(n)}(a)+\frac{(b-x)^{n+1}}{(n+1)!}f^{(n)}(b)-I.
\end{aligned}
\tag{2.3}
$$

Multiplying the both sides of (2.3) by $(-1)^n$, we obtain

$$
\begin{aligned}
\int_a^b f(t)dt \;=\;& \sum_{k=0}^{n-1}\frac{(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!}\\
&+\frac{(x-a)^{n+1}}{(n+1)!}f^{(n)}(a)+(-1)^n\frac{(b-x)^{n+1}}{(n+1)!}f^{(n)}(b)\\
&-(-1)^n\left\{\frac{(x-a)^{n+2}}{(n+1)!}\int_0^1(t-1)^{n+1}f^{(n+1)}(tx+(1-t)a)dt\right.\\
&\qquad\left.+\frac{(b-x)^{n+2}}{(n+1)!}\int_0^1(1-t)^{n+1}f^{(n+1)}(tx+(1-t)b)dt\right\}\\
=\;& \sum_{k=0}^{n}\frac{(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!}\\
&+(-1)^{n+1}\frac{(x-a)^{n+2}}{(n+1)!}\int_0^1(t-1)^{n+1}f^{(n+1)}(tx+(1-t)a)dt\\
&+(-1)^{n+1}\frac{(b-x)^{n+2}}{(n+1)!}\int_0^1(1-t)^{n+1}f^{(n+1)}(tx+(1-t)b)dt.
\end{aligned}
$$

Thus, the identity (2.2) and the lemma is proved. □

Theorem 5. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be $n-$time differentiable function, $a,b\in I$ and $a<b$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|$ $(n\geq 1)$ is convex on $[a,b]$, then we have*

$$
\begin{aligned}
&\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\frac{(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!}\right|\\
\leq\;& \frac{(x-a)^{n+1}}{(n+2)!}\left\{(n+1)\left|f^{(n)}(a)\right|+\left|f^{(n)}(x)\right|\right\}\\
&+\frac{(b-x)^{n+1}}{(n+2)!}\left\{\left|f^{(n)}(x)\right|+(n+1)\left|f^{(n)}(b)\right|\right\}
\end{aligned}
\tag{2.4}
$$

where $x \in [a,b]$.

Proof. From Lemma 1 and using the properties of modulus, we can write

$$\begin{aligned}
&\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)}{(k+1)!} \right| \\
\leq\ & \frac{(x-a)^{n+1}}{n!} \int_0^1 (1-t)^n \left| f^{(n)}(tx+(1-t)a) \right| dt \\
&+ \frac{(b-x)^{n+1}}{n!} \int_0^1 (1-t)^n \left| f^{(n)}(tx+(1-t)b) \right| dt.
\end{aligned}$$

Since $\left|f^{(n)}\right|$ is convex on $[a,b]$, it follows that

$$\begin{aligned}
&\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)}{(k+1)!} \right| \\
\leq\ & \frac{(x-a)^{n+1}}{n!} \int_0^1 (1-t)^n \left[t\left| f^{(n)}(x) \right| + (1-t)\left| f^{(n)}(a) \right| \right] dt \\
&+ \frac{(b-x)^{n+1}}{n!} \int_0^1 (1-t)^n \left[t\left| f^{(n)}(x) \right| + (1-t)\left| f^{(n)}(b) \right| \right] dt \\
=\ & \frac{(x-a)^{n+1}}{(n+2)!} \left\{ (n+1)\left| f^{(n)}(a) \right| + \left| f^{(n)}(x) \right| \right\} \\
&+ \frac{(b-x)^{n+1}}{(n+2)!} \left\{ \left| f^{(n)}(x) \right| + (n+1)\left| f^{(n)}(b) \right| \right\}.
\end{aligned}$$

This completes the proof. □

Remark 1. *In the inequality (2.4), if we choose* $n=1$*, then we have the inequality (1.2).*

Corollary 1. *In the inequality (2.4), if we choose* $n=2$*,* $x=\frac{a+b}{2}$ *and* $f'(x) = f'(a+b-x)$ *(that is,* f' *symmetric function), then we have*

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \right| \\
\leq\ & \frac{(b-a)^2}{192} \left\{ 3\left|f''(a)\right| + 2\left| f''\left(\frac{a+b}{2}\right) \right| + 3\left|f''(b)\right| \right\} \\
\leq\ & \frac{(b-a)^2}{48} \left\{ \left|f''(a)\right| + \left|f''(b)\right| \right\}.
\end{aligned}$$

Theorem 6. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be* $n-$*time differentiable function,* $x \in [a,b]$ *and* $a < b$*. If* $f^{(n)} \in L[a,b]$ *and* $\left|f^{(n)}\right|^q$ $(n \geq 1)$ *is convex on* $[a,b]$*, then we the following* *inequality:*

$$\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\frac{(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!}\right| \tag{2.5}$$
$$\le\left(\frac{1}{np+1}\right)^{\frac{1}{p}}\left\{\frac{(x-a)^{n+1}}{n!}\left[\frac{\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(x)\right|^q}{2}\right]^{\frac{1}{q}}\right.$$
$$\left.+\frac{(b-x)^{n+1}}{n!}\left[\frac{\left|f^{(n)}(x)\right|^q+\left|f^{(n)}(b)\right|^q}{2}\right]^{\frac{1}{q}}\right\}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 1 and Hölder integral inequality, we obtain

$$\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\frac{(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!}\right|$$
$$\le\frac{(x-a)^{n+1}}{n!}\left(\int_0^1(1-t)^{np}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n)}(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^{n+1}}{n!}\left(\int_0^1(1-t)^{np}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n)}(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $\left|f^{(n)}\right|^q$ is convex on $[a,b]$, then

$$\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\frac{(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!}\right|$$
$$\le\frac{(x-a)^{n+1}}{n!}\left(\frac{1}{np+1}\right)^{\frac{1}{p}}\left(\int_0^1\left[t\left|f^{(n)}(x)\right|^q+(1-t)\left|f^{(n)}(a)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^{n+1}}{n!}\left(\frac{1}{np+1}\right)^{\frac{1}{p}}\left(\int_0^1\left[t\left|f^{(n)}(x)\right|^q+(1-t)\left|f^{(n)}(b)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$=\left(\frac{1}{np+1}\right)^{\frac{1}{p}}\left\{\frac{(x-a)^{n+1}}{n!}\left[\frac{\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(x)\right|^q}{2}\right]^{\frac{1}{q}}\right.$$
$$\left.+\frac{(b-x)^{n+1}}{n!}\left[\frac{\left|f^{(n)}(x)\right|^q+\left|f^{(n)}(b)\right|^q}{2}\right]^{\frac{1}{q}}\right\}$$

which completes the proof. □

Remark 2. *In Theorem 6, if we choose* $n=1$*, then we have the inequality (1.3).*

Corollary 2. *In Theorem 6, if we choose $n = 2$, $x = \frac{a+b}{2}$ and $f'(x) = f'(a+b-x)$ (that is, f' symmetric function), then we obtain*

$$\begin{aligned} & \left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \right| \\ \leq & \frac{(b-a)^2}{16} \left(\frac{1}{2p+1} \right)^{\frac{1}{p}} \\ & \times \left\{ \left[\frac{|f''(a)|^q + \left| f''\left(\frac{a+b}{2}\right) \right|^q}{2} \right]^{\frac{1}{q}} + \left[\frac{\left| f''\left(\frac{a+b}{2}\right) \right|^q + |f''(b)|^q}{2} \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Theorem 7. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be $n-$time differentiable function and $a < b$. If $f^{(n)} \in L[a,b]$ and $\left|f^{(n)}\right|^q$ is convex on $[a,b]$, then we get*

$$\begin{aligned} (2.6) \qquad & \left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)}{(k+1)!} \right| \\ \leq & \left(\frac{q-1}{nq+q-p-1} \right)^{1-\frac{1}{q}} \\ & \times \left\{ \frac{(x-a)^{n+1}}{n!} \left[\frac{1}{p+2} \left| f^{(n)}(a) \right|^q + \frac{1}{(p+1)(p+2)} \left| f^{(n)}(x) \right|^q \right]^{\frac{1}{q}} \right. \\ & \left. + \frac{(b-x)^{n+1}}{n!} \left[\frac{1}{(p+1)(p+2)} \left| f^{(n)}(x) \right|^q + \frac{1}{p+2} \left| f^{(n)}(b) \right|^q \right]^{\frac{1}{q}} \right\} \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $x \in [a,b]$.

Proof. From Lemma 1 and using the properties of modulus, we get

$$\begin{aligned} & \left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)}{(k+1)!} \right| \\ \leq & \frac{(x-a)^{n+1}}{n!} \int_0^1 (1-t)^n \left| f^{(n)}(tx+(1-t)a) \right| dt \\ & + \frac{(b-x)^{n+1}}{n!} \int_0^1 (1-t)^n \left| f^{(n)}(tx+(1-t)b) \right| dt \\ = & \frac{(x-a)^{n+1}}{n!} \int_0^1 \frac{(1-t)^n (1-t)^{\frac{p}{q}}}{(1-t)^{\frac{p}{q}}} \left| f^{(n)}(tx+(1-t)a) \right| dt \\ & + \frac{(b-x)^{n+1}}{n!} \int_0^1 \frac{(1-t)^n (1-t)^{\frac{p}{q}}}{(1-t)^{\frac{p}{q}}} \left| f^{(n)}(tx+(1-t)b) \right| dt. \end{aligned}$$

Using the Hölder integral inequality, we can write

$$\begin{aligned}
&\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\frac{(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!}\right|\\
\le\ &\frac{(x-a)^{n+1}}{n!}\left(\int_0^1\left[\frac{(1-t)^n}{(1-t)^{\frac{p}{q}}}\right]^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^p\left|f^{(n)}(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\\
&+\frac{(b-x)^{n+1}}{n!}\left(\int_0^1\left[\frac{(1-t)^n}{(1-t)^{\frac{p}{q}}}\right]^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^p\left|f^{(n)}(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\left|f^{(n)}\right|^q$ is convex on $[a,b]$, then

$$\begin{aligned}
&\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\frac{(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!}\right|\\
\le\ &\frac{(x-a)^{n+1}}{n!}\left(\int_0^1(1-t)^{\frac{nq-p}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^p\left[t\left|f^{(n)}(x)\right|^q+(1-t)\left|f^{(n)}(a)\right|^q\right]dt\right)^{\frac{1}{q}}\\
&+\frac{(b-x)^{n+1}}{n!}\left(\int_0^1(1-t)^{\frac{nq-p}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^p\left[t\left|f^{(n)}(x)\right|^q+(1-t)\left|f^{(n)}(b)\right|^q\right]dt\right)^{\frac{1}{q}}\\
=\ &\left(\frac{q-1}{nq+q-p-1}\right)^{1-\frac{1}{q}}\\
&\times\left\{\frac{(x-a)^{n+1}}{n!}\left[\frac{1}{p+2}\left|f^{(n)}(a)\right|^q+\frac{1}{(p+1)(p+2)}\left|f^{(n)}(x)\right|^q\right]^{\frac{1}{q}}\right.\\
&\left.+\frac{(b-x)^{n+1}}{n!}\left[\frac{1}{(p+1)(p+2)}\left|f^{(n)}(x)\right|^q+\frac{1}{p+2}\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right\}
\end{aligned}$$

which completes the proof of the theorem. □

Corollary 3. *In Theorem 7, if we choose $n=1$, we have*

$$\begin{aligned}
&\left|\frac{(x-a)f(a)+(b-x)f(b)}{(b-a)}-\frac{1}{b-a}\int_a^b f(t)dt\right|\\
\le\ &\left(\frac{q-1}{2q-p-1}\right)^{1-\frac{1}{q}}\\
&\times\left\{\frac{(x-a)^2}{(b-a)}\left[\frac{1}{p+2}\left|f'(a)\right|^q+\frac{1}{(p+1)(p+2)}\left|f'(x)\right|^q\right]^{\frac{1}{q}}\right.\\
&\left.+\frac{(b-x)^2}{(b-a)}\left[\frac{1}{(p+1)(p+2)}\left|f'(x)\right|^q+\frac{1}{p+2}\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}\tag{2.7}$$

Corollary 4. *In the inequality (2.7), if we choose $x=\frac{a+b}{2}$, then we obtain*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \\
\leq\ & \frac{b-a}{4}\left(\frac{q-1}{2q-p-1}\right)^{1-\frac{1}{q}} \\
&\times\left\{\left[\frac{1}{p+2}\left|f'(a)\right|^q+\frac{1}{(p+1)(p+2)}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\frac{1}{(p+1)(p+2)}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{p+2}\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Corollary 5. *In Theorem 7, if we choose $n=2$, $x=\frac{a+b}{2}$ and $f'(x)=f'(a+b-x)$ (that is, f' symmetric function), then we have*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \\
\leq\ & \frac{(b-a)^2}{16}\left(\frac{q-1}{3q-p-1}\right)^{1-\frac{1}{q}} \\
&\times\left\{\left[\frac{1}{p+2}\left|f''(a)\right|^q+\frac{1}{(p+1)(p+2)}\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\frac{1}{(p+1)(p+2)}\left|f''\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{p+2}\left|f''(b)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Theorem 8. *For $n\geq 1$, let $f:I\subset\mathbb{R}\to\mathbb{R}$ be $n-$time differentiable function and $a<b$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^q$ is convex on $[a,b]$, for $q\geq 1$, then the following inequality holds:*

$$
\begin{aligned}
(2.8)\qquad &\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\frac{(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!}\right| \\
\leq\ & \frac{(x-a)^{n+1}}{(n+1)!}\left[\frac{(n+1)\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(x)\right|^q}{(n+2)}\right]^{\frac{1}{q}} \\
&+\frac{(b-x)^{n+1}}{(n+1)!}\left[\frac{\left|f^{(n)}(x)\right|^q+(n+1)\left|f^{(n)}(b)\right|^q}{(n+2)}\right]^{\frac{1}{q}}.
\end{aligned}
$$

Proof. From Lemma 1 and using the well known Power-mean integral inequality, we have

$$\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\frac{(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!}\right|$$
$$\leq \frac{(x-a)^{n+1}}{n!}\left(\int_0^1(1-t)^n dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^n\left|f^{(n)}(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^{n+1}}{n!}\left(\int_0^1(1-t)^n dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^n\left|f^{(n)}(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.$$

Since $\left|f^{(n)}\right|^q$ is convex on $[a,b]$, for $q\geq 1$, then we obtain

$$\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\frac{(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!}\right|$$
$$\leq \frac{(x-a)^{n+1}}{n!}\left(\frac{1}{n+1}\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^n\left[t\left|f^{(n)}(x)\right|^q+(1-t)\left|f^{(n)}(a)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^{n+1}}{n!}\left(\frac{1}{n+1}\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^n\left[t\left|f^{(n)}(x)\right|^q+(1-t)\left|f^{(n)}(b)\right|^q\right]dt\right)^{\frac{1}{q}}$$
$$=\frac{(x-a)^{n+1}}{(n+1)!}\left[\frac{(n+1)\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(x)\right|^q}{(n+2)}\right]^{\frac{1}{q}}$$
$$+\frac{(b-x)^{n+1}}{(n+1)!}\left[\frac{\left|f^{(n)}(x)\right|^q+(n+1)\left|f^{(n)}(b)\right|^q}{(n+2)}\right]^{\frac{1}{q}}.$$

Hence, the proof of the theorem is completed. □

Remark 3. *In Theorem 8, if we choose $n=1$, we obtain the inequality (1.4).*

Corollary 6. *In the inequality (2.8) if we choose $n=2$, $x=\frac{a+b}{2}$ and $f'(x)=f'(a+b-x)$ (that is, f' symmetric function), then we get*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq \frac{(b-a)^2}{48}\left\{\left[\frac{3\left|f''(a)\right|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q}{4}\right]^{\frac{1}{q}}+\left[\frac{\left|f''\left(\frac{a+b}{2}\right)\right|^q+3\left|f''(b)\right|^q}{4}\right]^{\frac{1}{q}}\right\}.$$

Theorem 9. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be $n-$time differentiable function and $a<b$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^q$ is concave on $[a,b]$, then we obtain*

$$\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\frac{(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!}\right| \tag{2.9}$$
$$\leq \left(\frac{1}{np+1}\right)^{\frac{1}{p}}\left\{\frac{(x-a)^{n+1}}{n!}\left|f^{(n)}\left(\frac{a+x}{2}\right)\right|+\frac{(b-x)^{n+1}}{n!}\left|f^{(n)}\left(\frac{x+b}{2}\right)\right|\right\}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 1 and Hölder integral inequality, we obtain

$$\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\frac{(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!}\right| \tag{2.10}$$
$$\leq \frac{(x-a)^{n+1}}{n!}\left\{\left(\int_0^1(1-t)^{np}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n)}(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1(1-t)^{np}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n)}(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right\}.$$

Since $\left|f^{(n)}\right|^q$ is concave on $[a,b]$, we can write the following inequalities via Jensen inequality:

(2.11)
$$\begin{aligned}\int_0^1\left|f^{(n)}(tx+(1-t)a)\right|^q dt &= \int_0^1 t^0\left|f^{(n)}(tx+(1-t)a)\right|^q dt\\ &\leq \left(\int_0^1 t^0dt\right)\left|f^{(n)}\left(\frac{\int_0^1(tx+(1-t)a)dt}{\int_0^1 t^0dt}\right)\right|^q\\ &= \left|f^{(n)}\left(\frac{a+x}{2}\right)\right|^q\end{aligned}$$

and similarly

$$\int_0^1\left|f^{(n)}(tx+(1-t)b)\right|^q dt\leq\left|f^{(n)}\left(\frac{x+b}{2}\right)\right|^q. \tag{2.12}$$

Thus, if we use (2.11) and (2.12) in the inequality (2.10), we obtain the inequality of (2.9). This completes the proof. □

Corollary 7. *In the inequality (2.9) if we choose $n=2$, $x=\frac{a+b}{2}$ and $f'(x)=f'(a+b-x)$ (that is, f' symmetric function), then we obtain*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left\{\left|f''\left(\frac{3a+b}{4}\right)\right|+\left|f''\left(\frac{a+3b}{4}\right)\right|\right\}.$$

Theorem 10. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be $n-$time differentiable function and $a<b$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^q$ is concave on $[a,b]$, for $q\geq 1$, then the following inequality holds:*

(2.13)
$$\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\frac{(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!}\right|$$
$$\leq \frac{(x-a)^{n+1}}{(n+1)!}\left|f^{(n)}\left(\frac{(n+1)a+x}{n+2}\right)\right|+\frac{(b-x)^{n+1}}{(n+1)!}\left|f^{(n)}\left(\frac{x+(n+1)b}{n+2}\right)\right|.$$

Proof. From Lemma 1 and using the well known Power-mean inequality, we have

$$\left|\int_a^b f(t)dt-\sum_{k=0}^{n-1}\frac{(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!}\right| \tag{2.14}$$
$$\leq \frac{(x-a)^{n+1}}{n!}\left(\int_0^1(1-t)^n dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^n\left|f^{(n)}(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{(b-x)^{n+1}}{n!}\left(\int_0^1(1-t)^n dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^n\left|f^{(n)}(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}.$$

Using the Jensen inequality, we can write

$$\int_0^1(1-t)^n\left|f^{(n)}(tx+(1-t)a)\right|^q dt \tag{2.15}$$
$$\leq \left(\int_0^1(1-t)^n dt\right)\left|f^{(n)}\left(\frac{\int_0^1(1-t)^n(tx+(1-t)a)dt}{\int_0^1(1-t)^n dt}\right)\right|^q$$
$$=\left(\frac{1}{n+1}\right)\left|f^{(n)}\left(\frac{(n+1)a+x}{n+2}\right)\right|^q$$

and similarly

$$\int_0^1(1-t)^n\left|f^{(n)}(tx+(1-t)b)\right|^q dt\leq\left(\frac{1}{n+1}\right)\left|f^{(n)}\left(\frac{x+(n+1)b}{n+2}\right)\right|^q. \tag{2.16}$$

Thus, if we use (2.15) and (2.16) in the inequality (2.14), we obtain the inequality of (2.13). The proof of the theorem is completed. □

Corollary 8. *In Theorem 10, if we choose* $n=2$, $x=\frac{a+b}{2}$ *and* $f'(x)=f'(a+b-x)$ *(that is,* f' *symmetric function), then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{(b-a)^2}{48}\left\{\left|f''\left(\frac{7a+b}{8}\right)\right|+\left|f''\left(\frac{a+7b}{8}\right)\right|\right\}.$$

References

[1] S.-P. Bai, S.-H. Wang and F. Qi, Some Hermite-Hadamard type inequalities for n-time differentiable (α,m)-convex functions, Jour. of Ineq. and Appl., 2012, 2012:267.

[2] P. Cerone, S.S. Dragomir and J. Roumeliotis, Some Ostrowski type inequalities for n-time differentiable mappings and applications, Demonstratio Math., 32 (4) (1999), 697-712.

[3] P. Cerone, S.S. Dragomir and J. Roumeliotis and J. Šunde, A new generalization of the trapezoid formula for n-time differentiable mappings and applications, Demonstratio Math., 33 (4) (2000), 719-736.

[4] S.S. Dragomir and C.E.M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000. Online:[http://www.staxo.vu.edu.au/RGMIA/monographs/hermite hadamard.html].

[5] D.-Y. Hwang, Some Inequalities for n-time Differentiable Mappings and Applications, Kyung. Math. Jour., 43 (2003), 335-343.

[6] J. L. W. V. Jensen, On konvexe funktioner og uligheder mellem middlvaerdier, Nyt. Tidsskr. Math. B., 16, 49-69, 1905.

[7] W.-D. Jiang, D.-W. Niu, Y. Hua and F. Qi, Generalizations of Hermite-Hadamard inequality to n-time differentiable function which are s-convex in the second sense, Analysis (Munich), 32 (2012), 209-220

[8] H. Kavurmaci, M. Avci, M.E. Özdemir, New inequalities of Hermite–Hadamard type for convex functions with applications, Jour. of Ineq. and Appl., 2011, 2011:86.
[9] U.S. Kırmacı, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. and Comp., 147 (2004), 137–146.
[10] U.S. Kırmacı, M.K. Bakula, M.E. Özdemir and J. Pećarić, Hadamard-type inequalities for s-convex functions, Appl. Math. and Comp., 193(2007), 26-35.
[11] M.E. Özdemir and U.S. Kırmacı, Two new theorem on mappings uniformly continuous and convex with applications to quadrature rules and means, Appl. Math. and Comp., 143(2003), 269-274.
[12] M.E. Özdemir, Ç. Yıldız, New Inequalities for n-time differentiable functions, Arxiv:1402.4959v1.
[13] M.E. Özdemir, Ç. Yıldız, New Inequalities for Hermite-Hadamard and Simpson Type with Applications, Tamkang J. of Math., 44, 2, 209-216, 2013.
[14] M.E. Özdemir, Ç. Yıldız, A New Generalization of the Midpoint Formula for n-Time Differentiable Mappings which are Convex, RGMIA *Research Report Collection*, 17 (2014), Article 48.
[15] A. Saglam, M.Z Sarıkaya and H. Yıldırım, Some new inequalities of Hermite-Hadamard's type, Kyung. Math. Jour., 50 (2010), 399-410.
[16] M.Z. Sarıkaya and N. Aktan, On the generalization some integral inequalities and their applications, Math. and Comp. Mod., 54(2011), 2175-2182.
[17] E. Set, M.E. Özdemir and S.S. Dragomir, On Hadamard-Type Inequalities Involving Several Kinds of Convexity, Jour. of Ineq. and Appl., 2010, 286845.
[18] S.H. Wang, B.-Y. Xi and F. Qi, Some new inequalities of Hermite-Hadamard type for n-time differentiable functions which are m-convex, Analysis (Munich), 32 (2012), 247-262.
[19] B.-Y. Xi and F. Qi, Some integral inequalities of Hermite-Hadamard type for convex functions with applications to means, J. Funct. Spaces Appl., 2012 (2012), http://dx.doi.org/10.1155/2012/980438.

★ATATÜRK UNIVERSITY, K. K. EDUCATION FACULTY, DEPARTMENT OF MATHEMATICS, 25240, CAMPUS, ERZURUM, TURKEY

E-mail address: cetin@atauni.edu.tr

E-mail address: emos@atauni.edu.tr