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INTEGRAL INEQUALITIES FOR $n-$TIMES DIFFERENTIABLE MAPPINGS

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ABSTRACT. In this paper, using integral representations for $n-$times differentiable mappings, we establish new generalizations of certain Hermite-Hadamard type inequality for convex functions by using fairly elementary analysis. Also a parallel development is made base on concavity.

1. INRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

Both the inequalities hold in reversed direction if f is concave. On November 22, 1881, Hermite (1822-1901) sent a letter to the Journal Mathesis. This letter was published in Mathesis 3 (1883, p: 82) and in this letter an inequality presented which is well-known in the literature as Hermite-Hadamard integral inequality. Since its discovery in 1883, Hermite-Hadamard inequality has been considered the most useful inequality in mathematical analysis. Many uses of these inequalities have been discovered in a variety of settings. Moreover, many inequalities of special means can be obtained for a particular choice of the function f. Due to the rich geometrical significance of Hermite-Hadamard's inequality, there is growing literature providing its new proofs, extensions, refinements and generalizations, see for example ([2], [6]-[7], [10]-[13], [15], [18]-[22]) and the references therein.

Definition 1. *A function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex if whenever $x, y \in [a, b]$ and $t \in [0, 1]$, the following inequality holds:*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

We say that f is concave if $(-f)$ is convex. This definition has its origins in Jensen's results from [9] and has opened up the most extended, useful and multidisciplinary domain of mathematics, namely, convex analysis. Convex curves and convex bodies have appeared in mathematical literature since antiquity and there are many important results related to them.

Cerone *et. al.* (see [5]) proved the following generalization for $n-$time differentiable functions.

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Lemma 1. *Let $f:[a,b]\to\mathbb{R}$ be a mapping such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Then for all $x\in[a,b]$ we have the identity:*

$$\int_a^b f(t)dt = \sum_{k=0}^{n-1}\left[\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\right]f^{(k)}(x) +(-1)^n\int_a^b K_n(x,t)f^{(n)}(t)dt$$

where the kernel $K_n:[a,b]^2\to\mathbb{R}$ is given by

$$K_n(x,t)=\begin{cases}\frac{(t-a)^n}{n!} & if\ t\in[a,x]\\ \frac{(t-b)^n}{n!} & if\ t\in(x,b],\end{cases}$$

$x\in[a,b]$ and n is natural number, $n\geq 1$.

For other recent results concerning the $n-$time differentiable functions see [3]-[5], [8], [10], [14], [16], [21] where further references are given.

In [17], Özdemir *et. al.* proved the following Hadamard type inequalities.

Theorem 1. *Let $f:I\subset[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° such that $f''\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f''|^q$, $q\geq 1$ is s-convex on $[a,b]$, for some fixed $s\in(0,1]$, then the following inequality holds:*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ & \frac{(b-a)^2}{16}\left(\frac{1}{3}\right)^{\frac{1}{p}}\left\{\left(\frac{2}{(s+1)(s+2)(s+3)}|f''(a)|^q+\frac{1}{s+3}\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\frac{1}{s+3}\left|f''\left(\frac{a+b}{2}\right)\right|^q+\frac{2}{(s+1)(s+2)(s+3)}|f''(b)|^q\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Corollary 1. *In Theorem 1 , if we choose $s=1$ we have*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|\\ \leq\ & \frac{(b-a)^2}{48}\left(\frac{3}{4}\right)^{\frac{1}{q}}\left\{\left(\frac{|f''(a)|^q}{3}+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q+\frac{|f''(b)|^q}{3}\right)^{\frac{1}{q}}\right\}.\end{aligned}\tag{1.2}$$

In [1], Alomari and Darus obtained the following theorem and corollary.

Theorem 2. *Let $f:I\subseteq[0,\infty)\to\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'|^{\frac{p}{p-1}}$ is concave on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right|\\ \leq\ & \frac{(b-x)^2}{(b-a)(p+1)^{1/p}}\left|f'\left(\frac{b+x}{2}\right)\right|+\frac{(x-a)^2}{(b-a)(p+1)^{1/p}}\left|f'\left(\frac{a+x}{2}\right)\right|\end{aligned}$$

for each $x \in [a,b]$, *where* $p > 1$.

Corollary 2. *In Theorem 2, choose* $x = \frac{a+b}{2}$, *then*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)du \right| \leq \frac{b-a}{4(p+1)^{1/p}}\left[\left| f'\left(\frac{a+3b}{4}\right)\right| + \left| f'\left(\frac{3a+b}{4}\right)\right| \right] \tag{1.3}$$

for each $x \in [a,b]$, *where* $p > 1$

The main purpose of the present paper is to establish several new inequalities for $n-$ time differantiable mappings that are connected with the celebrated Hermite-Hadamard integral inequality.

2. MAIN RESULTS

In order to reach our aim, the following lemma are necessary:

Lemma 2. *For* $n \geq 1$, *let* $f : [a,b] \to \mathbb{R}$ *be n-time differentiable functions. If* $f^{(n)} \in L[a,b]$, *then*

(2.1)

$$\begin{aligned} \int_a^b f(t)dt &= \sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1} f^{(k)}\left(\frac{a+b}{2}\right) \\ &+(-1)^n \frac{(b-a)^{n+1}}{2^{n+1}n!}\left\{ \int_0^1 t^n f^{(n)}\left(t\frac{a+b}{2} + (1-t)a\right) dt \right. \\ &\left. + \int_0^1 (t-1)^n f^{(n)}\left(tb + (1-t)\frac{a+b}{2}\right) dt \right\}. \end{aligned}$$

Proof. The proof is by mathematical induction.

For $n = 1$, we have the equality in paper [[2], Lemma 2.1].

Assume that (2.1) holds for "$n = m$" and let us prove it for "$n = m+1$". That is, we have to obtain the equality

(2.2)

$$\begin{aligned} \int_a^b f(x)dx &= \sum_{k=0}^{m}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1} f^{(k)}\left(\frac{a+b}{2}\right) \\ &+(-1)^{m+1} \frac{(b-a)^{m+2}}{2^{m+2}(m+1)!}\left\{ \int_0^1 t^{m+1} f^{(m+1)}\left(t\frac{a+b}{2} + (1-t)a\right) dt \right. \\ &\left. + \int_0^1 (t-1)^{m+1} f^{(m+1)}\left(tb + (1-t)\frac{a+b}{2}\right) dt \right\}. \end{aligned}$$

Then, we can write

$$\begin{aligned} I &= \frac{(b-a)^{m+2}}{2^{m+2}(m+1)!}\left\{ \int_0^1 t^{m+1} f^{(m+1)}\left(t\frac{a+b}{2} + (1-t)a\right) dt \right. \\ &\left. + \int_0^1 (t-1)^{m+1} f^{(m+1)}\left(tb + (1-t)\frac{a+b}{2}\right) dt \right\} \end{aligned}$$

and integrating by parts gives

$$\begin{aligned}
I &= \frac{(b-a)^{m+2}}{2^{m+2}(m+1)!}\left\{ t^{m+1}\frac{2}{b-a}\ f^{(m)}\left(t\frac{a+b}{2}+(1-t)a\right)\Big|_0^1 \right.\\
&\qquad -\frac{2(m+1)}{b-a}\int_0^1 t^m f^{(m)}\left(t\frac{a+b}{2}+(1-t)a\right)dt\\
&\qquad +(t-1)^{m+1}\frac{2}{b-a}\ f^{(m)}\left(tb+(1-t)\frac{a+b}{2}\right)\Big|_0^1\\
&\qquad \left. -\frac{2(m+1)}{b-a}\int_0^1 (t-1)^m f^{(m)}\left(tb+(1-t)\frac{a+b}{2}\right)dt\right\}\\
&= \frac{(b-a)^{m+1}}{2^{m+1}(m+1)!}f^{(m)}\left(\frac{a+b}{2}\right)-\frac{(b-a)^{m+1}}{2^{m+1}m!}\int_0^1 t^m f^{(m)}\left(t\frac{a+b}{2}+(1-t)a\right)dt\\
&\quad +\frac{(-1)^{m+2}(b-a)^{m+1}}{2^{m+1}(m+1)!}f^{(m)}\left(\frac{a+b}{2}\right)-\frac{(b-a)^{m+1}}{2^{m+1}m!}\int_0^1 (t-1)^m f^{(m)}\left(tb+(1-t)\frac{a+b}{2}\right)dt.
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{(b-a)^{m+1}}{2^{m+1}m!}\left\{\int_0^1 t^m f^{(m)}\left(t\frac{a+b}{2}+(1-t)a\right)dt+\int_0^1 (t-1)^m f^{(m)}\left(tb+(1-t)\frac{a+b}{2}\right)dt\right\}\\
=\ &\left(\frac{1+(-1)^{m+2}}{2^{m+1}(m+1)!}\right)(b-a)^{m+1}f^{(m)}\left(\frac{a+b}{2}\right)-I
\end{aligned}$$

Now, using the mathematical induction hypothesis, upon rearrangement we obtain the following equality:

(2.3)

$$\begin{aligned}
\frac{1}{(-1)^m}\int_a^b f(x)dx &= \frac{1}{(-1)^m}\sum_{k=0}^{m-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\\
&\quad +\left(\frac{1+(-1)^{m+2}}{2^{m+1}(m+1)!}\right)(b-a)^{m+1}f^{(m)}\left(\frac{a+b}{2}\right)-I.
\end{aligned}$$

Multiplying the both sides of (2.3) by $(-1)^n$ and substituting I in the right membership of (2.3), we obtain

$$\begin{aligned}
\int_a^b f(x)dx &= \sum_{k=0}^{m-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\\
&\quad +(-1)^{m+1}\frac{(b-a)^{m+2}}{2^{m+2}(m+1)!}\left\{\int_0^1 t^{m+1}f^{(m+1)}\left(t\frac{a+b}{2}+(1-t)a\right)dt\right.\\
&\qquad \left.+\int_0^1 (t-1)^{m+1}f^{(m+1)}\left(tb+(1-t)\frac{a+b}{2}\right)dt\right\}.
\end{aligned}$$

Thus, the identity (2.2) and the lemma is proved. □

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be $n-$time differentiable function and $a<b$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^q$ $(n\geq 1)$ is convex on $[a,b]$, then we have:*

$$
\begin{aligned}
(2.4)\qquad & \left|\int_a^b f(x)dx-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{(b-a)^{n+1}}{2^{n+1}n!}\left(\frac{1}{np+1}\right)^{\frac{1}{p}} \\
& \times\left\{\left[\frac{\left|f^{(n)}(a)\right|^q+\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q}{2}\right]^{\frac{1}{q}}+\left[\frac{\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q+\left|f^{(n)}(b)\right|^q}{2}\right]^{\frac{1}{q}}\right\}
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 2 and Hölder integral inequality, it follows that

$$
\begin{aligned}
& \left|\int_a^b f(x)dx-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{(b-a)^{n+1}}{2^{n+1}n!}\left\{\left(\int_0^1 t^{np}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n)}\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
& \left.+\left(\int_0^1(1-t)^{np}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n)}\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}
$$

Since $\left|f^{(n)}\right|^q$ is convex on $[a,b]$, then we can write

$$
\begin{aligned}
& \left|\int_a^b f(x)dx-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{(b-a)^{n+1}}{2^{n+1}n!}\left\{\left(\frac{1}{np+1}\right)^{\frac{1}{p}}\left(\int_0^1\left[t\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q+(1-t)\left|f^{(n)}(a)\right|^q\right]dt\right)^{\frac{1}{q}}\right. \\
& \left.+\left(\frac{1}{np+1}\right)^{\frac{1}{p}}\left(\int_0^1\left[t\left|f^{(n)}(b)\right|^q+(1-t)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right]dt\right)^{\frac{1}{q}}\right\} \\
=\ & \frac{(b-a)^{n+1}}{2^{n+1}n!}\left(\frac{1}{np+1}\right)^{\frac{1}{p}} \\
& \times\left\{\left[\frac{\left|f^{(n)}(a)\right|^q+\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q}{2}\right]^{\frac{1}{q}}+\left[\frac{\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q+\left|f^{(n)}(b)\right|^q}{2}\right]^{\frac{1}{q}}\right\}
\end{aligned}
$$

which completes the proof. □

Corollary 3. *Under the assumptions of Theorem 3, we have*

$$\begin{aligned}(2.5)\qquad & \left|\int_a^b f(x)dx-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \\ \leq\ & \frac{(b-a)^{n+1}}{2^{n+1}n!} \\ & \times\left\{\left[\frac{\left|f^{(n)}(a)\right|^q+\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q}{2}\right]^{\frac{1}{q}}+\left[\frac{\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q+\left|f^{(n)}(b)\right|^q}{2}\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

Proof. For $p>1$, since

$$\lim_{p\to\infty}\left(\frac{1}{np+1}\right)^{\frac{1}{p}}=1 \quad\text{and}\quad \lim_{p\to1^+}\left(\frac{1}{np+1}\right)^{\frac{1}{p}}=\frac{1}{n+1},$$

we have

$$\frac{1}{n+1}<\lim_{p\to\infty}\left(\frac{1}{np+1}\right)^{\frac{1}{p}}<1,\quad p\in(1,\infty).$$

Hence we obtain the inequality (2.5). □

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be $n-$time differentiable function and $a<b$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^q$ $(n\geq1)$ is convex on $[a,b]$, then we get*

$$\begin{aligned}(2.6)\qquad & \left|\int_a^b f(x)dx-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \\ \leq\ & \frac{(b-a)^{n+1}}{2^{n+1}n!}\left(\frac{q-1}{nq+q-p-1}\right)^{1-\frac{1}{q}} \\ & \times\left\{\left[\frac{1}{(p+1)(p+2)}\left|f^{(n)}(a)\right|^q+\frac{1}{p+2}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\right. \\ & \left.+\left[\frac{1}{p+2}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{(p+1)(p+2)}\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right\}\end{aligned}$$

where $p>1$.

Proof. From Lemma 2 and using the properties of modulus, we get

$$\begin{aligned}& \left|\int_a^b f(x)dx-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \\ \leq\ & \frac{(b-a)^{n+1}}{2^{n+1}n!}\left\{\int_0^1 t^n\left|f^{(n)}\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt\right. \\ & \left.+\int_0^1(1-t)^n\left|f^{(n)}\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right\} \\ =\ & \frac{(b-a)^{n+1}}{2^{n+1}n!}\left\{\int_0^1\frac{t^nt^{\frac{p}{q}}}{t^{\frac{p}{q}}}\left|f^{(n)}\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt\right. \\ & \left.+\int_0^1\frac{(1-t)^n(1-t)^{\frac{p}{q}}}{(1-t)^{\frac{p}{q}}}\left|f^{(n)}\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right\}.\end{aligned}$$

Using the Hölder integral inequality, we can write

$$\begin{aligned}
&\left|\int_a^b f(x)dx-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{(b-a)^{n+1}}{2^{n+1}n!}\left\{\left(\int_0^1\left[\frac{t^n}{t^{\frac{p}{q}}}\right]^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^p\left|f^{(n)}\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1\left[\frac{(1-t)^n}{(1-t)^{\frac{p}{q}}}\right]^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^p\left|f^{(n)}\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Since $\left|f^{(n)}\right|^q$ is convex on $[a,b]$, we have

$$\begin{aligned}
&\left|\int_a^b f(x)dx-\sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{(b-a)^{n+1}}{2^{n+1}n!}\left(\frac{q-1}{nq+q-p-1}\right)^{1-\frac{1}{q}} \\
&\times\left\{\left[\frac{1}{(p+1)(p+2)}\left|f^{(n)}(a)\right|^q+\frac{1}{p+2}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\frac{1}{p+2}\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{(p+1)(p+2)}\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right\}
\end{aligned}$$

which completes the proof of the theorem. □

Corollary 4. *In Theorem 4, if we choose $n=1$, we have*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
\leq\ & \frac{(b-a)}{4}\left(\frac{q-1}{2q-p-1}\right)^{1-\frac{1}{q}} \\
&\times\left\{\left[\frac{1}{(p+1)(p+2)}\left|f'(a)\right|^q+\frac{1}{p+2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\frac{1}{p+2}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{(p+1)(p+2)}\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Corollary 5. *In Theorem 4, if we choose $n = 2$, then we obtain*

$$\begin{aligned}
& \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \\
\leq & \frac{(b-a)^2}{16}\left(\frac{q-1}{3q-p-1}\right)^{1-\frac{1}{q}} \\
& \times \left\{ \left[\frac{1}{(p+1)(p+2)} \left|f''(a)\right|^q + \frac{1}{p+2}\left| f''\left(\frac{a+b}{2}\right)\right|^q \right]^{\frac{1}{q}} \right. \\
& \left. + \left[\frac{1}{p+2}\left| f''\left(\frac{a+b}{2}\right)\right|^q + \frac{1}{(p+1)(p+2)}\left|f''(b)\right|^q \right]^{\frac{1}{q}} \right\}.
\end{aligned}$$

Theorem 5. *For $n \geq 1$, let $f : [a, b] \to \mathbb{R}$ be $n-$time differentiable function and $a < b$. If $f^{(n)} \in L[a, b]$ and $\left|f^{(n)}\right|^q$ is convex on $[a, b]$, for $q \geq 1$, then the following inequality holds:*

$$\begin{aligned}
(2.7) \qquad & \left| \int_a^b f(x)dx - \sum_{k=0}^{n-1} \left(\frac{1+(-1)^k}{2^{k+1}(k+1)!} \right)(b-a)^{k+1} f^{(k)}\left(\frac{a+b}{2}\right) \right| \\
\leq & \frac{(b-a)^{n+1}}{2^{n+1}(n+1)!} \left\{ \left[\frac{1}{(n+2)}\left|f^{(n)}(a)\right|^q + \left(\frac{n+1}{n+2}\right)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q \right]^{\frac{1}{q}} \right. \\
& \left. + \left[\left(\frac{n+1}{n+2}\right)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q + \frac{1}{(n+2)}\left|f^{(n)}(b)\right|^q \right]^{\frac{1}{q}} \right\}.
\end{aligned}$$

Proof. Suppose that $q = 1$. From Lemma 2, we have

$$\begin{aligned}
& \left| \int_a^b f(x)dx - \sum_{k=0}^{n-1} \left(\frac{1+(-1)^k}{2^{k+1}(k+1)!} \right)(b-a)^{k+1} f^{(k)}\left(\frac{a+b}{2}\right) \right| \\
\leq & \frac{(b-a)^{n+1}}{2^{n+1}(n+2)!}\left[\left|f^{(n)}(a)\right| + 2(n+1)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right| + \left|f^{(n)}(b)\right| \right].
\end{aligned}$$

Suppose now that $q > 1$. Using the well known Power-mean integral inequality and Lemma 2, we obtain

$$\begin{aligned}
& \left| \int_a^b f(x)dx - \sum_{k=0}^{n-1} \left(\frac{1+(-1)^k}{2^{k+1}(k+1)!} \right)(b-a)^{k+1} f^{(k)}\left(\frac{a+b}{2}\right) \right| \\
\leq & \frac{(b-a)^{n+1}}{2^{n+1}n!} \left\{ \left(\int_0^1 t^n dt\right)^{1-\frac{1}{q}} \left(\int_0^1 t^n \left| f^{(n)}\left(t\frac{a+b}{2} + (1-t)a\right)\right|^q dt \right)^{\frac{1}{q}} \right. \\
& \left. + \left(\int_0^1 (1-t)^n dt\right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)^n \left| f^{(n)}\left(tb + (1-t)\frac{a+b}{2}\right)\right|^q dt \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

Since $\left|f^{(n)}\right|^q$ is convex on $[a,b]$, for $q \geq 1$, then we obtain

$$\begin{aligned}
&\left|\int_a^b f(x)dx - \sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{(b-a)^{n+1}}{2^{n+1}n!}\left(\frac{1}{n+1}\right)^{1-\frac{1}{q}}\left\{\left(\int_0^1 t^n\left[t\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q+(1-t)\left|f^{(n)}(a)\right|^q\right]dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1(1-t)^n\left[t\left|f^{(n)}(b)\right|^q+(1-t)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right]dt\right)^{\frac{1}{q}}\right\} \\
=\ & \frac{(b-a)^{n+1}}{2^{n+1}(n+1)!}\left\{\left[\frac{1}{(n+2)}\left|f^{(n)}(a)\right|^q+\left(\frac{n+1}{n+2}\right)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\left(\frac{n+1}{n+2}\right)\left|f^{(n)}\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{(n+2)}\left|f^{(n)}(b)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Hence, the proof of the theorem is completed. □

Corollary 6. *In Theorem 5, if we choose $n=1$, we obtain*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)du\right| \\
\leq\ & \frac{(b-a)}{8}\left\{\left(\frac{|f'(a)|^q+2\left|f'\left(\frac{a+b}{2}\right)\right|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{2\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q}{3}\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Remark 1. *In Theorem 5, if we choose $n=2$, we obtain the inequality (1.2).*

Now, we give the following Hadamard type inequality for concave mappings.

Theorem 6. *Let $f:[a,b]\to\mathbb{R}$ be $n-$time differentiable function and $a<b$. If $f^{(n)}\in L[a,b]$ and $\left|f^{(n)}\right|^q$ is concave on $[a,b]$, then we have*

$$\begin{aligned}
&\left|\int_a^b f(x)dx - \sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{(b-a)^{n+1}}{2^{n+1}n!}\left(\frac{1}{np+1}\right)^{\frac{1}{p}}\left\{\left|f^{(n)}\left(\frac{3a+b}{4}\right)\right|+\left|f^{(n)}\left(\frac{a+3b}{4}\right)\right|\right\}
\end{aligned} \tag{2.8}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2 and Hölder integral inequality, we can write

$$\begin{aligned}
&\left|\int_a^b f(x)dx - \sum_{k=0}^{n-1}\left(\frac{1+(-1)^k}{2^{k+1}(k+1)!}\right)(b-a)^{k+1}f^{(k)}\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{(b-a)^{n+1}}{2^{n+1}n!}\left\{\left(\int_0^1 t^{np}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n)}\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1(1-t)^{np}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f^{(n)}\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right\}.
\end{aligned} \tag{2.9}$$

Since $\left|f^{(n)}\right|^q$ is concave on $[a,b]$, we can use the Jensen's integral inequality to obtain

(2.10)

$$\begin{aligned}\int_0^1 \left|f^{(n)}\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt &= \int_0^1 t^0 \left|f^{(n)}\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt \\ &\le \left(\int_0^1 t^0 dt\right)\left|f^{(n)}\left(\frac{\int_0^1 \left(t\frac{a+b}{2}+(1-t)a\right)dt}{\int_0^1 t^0 dt}\right)\right|^q \\ &= \left|f^{(n)}\left(\frac{3a+b}{4}\right)\right|^q\end{aligned}$$

and similarly

$$\int_0^1 \left|f^{(n)}\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt \le \left|f^{(n)}\left(\frac{a+3b}{4}\right)\right|^q. \tag{2.11}$$

Therefore, if we use (2.10) and (2.11) in the inequality (2.9), we obtain the inequality of (2.8). This completes the proof. □

Remark 2. *In Theorem 6, if we choose $n=1$, we have the inequality (1.3).*

Corollary 7. *In the inequality (2.8) if we choose $n=2$, then we can*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| \\ \le\ &\frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left\{\left|f''\left(\frac{3a+b}{4}\right)\right|+\left|f''\left(\frac{a+3b}{4}\right)\right|\right\}.\end{aligned}$$

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