Received 12/01/15

INEQUALITIES OF SCHWARZ TYPE FOR n-TUPLES OF VECTORS IN INNER PRODUCT SPACES WITH APPLICATIONS

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Abstract. In this paper some new inequalities of Schwarz and Buzano type for n-tuples of vectors in inner product spaces are given. Applications for norm and numerical radius inequalities for n-tuples of bounded linear operators and for functions of normal operators defined by power series with nonnegative coefficients are also provided.

1. Introduction

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. The following inequality is well known in literature as the *Schwarz inequality*

$$\|x\| \|y\| \geq |\langle x, y \rangle| \text{ for any } x, y \in H. \tag{1.1}$$

The equality case holds in (1.1) if and only if there exists a constant $\lambda \in \mathbb{K}$ such that $x = \lambda y$.

In 1985 the author [5] (see also [15]) established the following refinement of (1.1):

$$\|x\| \|y\| \geq |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| + |\langle x, e \rangle \langle e, y \rangle| \geq |\langle x, y \rangle| \tag{1.2}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

Using the triangle inequality for modulus we have

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \geq |\langle x, e \rangle \langle e, y \rangle| - |\langle x, y \rangle|$$

and by (1.2) we get

$$\begin{aligned} \|x\| \|y\| &\geq |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| + |\langle x, e \rangle \langle e, y \rangle| \\ &\geq 2 |\langle x, e \rangle \langle e, y \rangle| - |\langle x, y \rangle|, \end{aligned}$$

which implies the Buzano inequality [3]

$$\frac{1}{2} [\|x\| \|y\| + |\langle x, y \rangle|] \geq |\langle x, e \rangle \langle e, y \rangle| \tag{1.3}$$

that holds for any $x, y, e \in H$ with $\|e\| = 1$.

For a probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathcal{P}_n$, i.e. we recall that $p_i > 0$ for any $i \in \{1, ..., n\}$ and $\sum_{i=1}^n p_i = 1$ we define the inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle_p := \sum_{i=1}^n p_i \langle x_i, y_i \rangle$$

1991 *Mathematics Subject Classification.* 47A05, 47A63; 47A99.

for n-tuples $\mathbf{x}=(x_1,...,x_n), \mathbf{y}=(y_1,...,y_n)\in H^n$. The attached norm is given by

$$\|\mathbf{x}\|_p=\left(\sum_{i=1}^n p_i\|x_i\|^2\right)^{1/2}$$

for $\mathbf{x}=(x_1,...,x_n)\in H^n$.

Let $\mathbf{e}=(e_1,...,e_n)\in H^n$ with $\sum_{i=1}^n p_i\|e_i\|^2=1$. Making use of (1.2) and (1.3) for the inner product $\langle\cdot,\cdot\rangle_p$ we have the inequalities

$$\begin{aligned}(1.4)\qquad &\left(\sum_{i=1}^n p_i\|x_i\|^2\right)^{1/2}\left(\sum_{i=1}^n p_i\|y_i\|^2\right)^{1/2}\\ &\geq\left|\sum_{i=1}^n p_i\langle x_i,y_i\rangle-\sum_{i=1}^n p_i\langle x_i,e_i\rangle\sum_{i=1}^n p_i\langle e_i,y_i\rangle\right|+\left|\sum_{i=1}^n p_i\langle x_i,e_i\rangle\sum_{i=1}^n p_i\langle e_i,y_i\rangle\right|\\ &\geq\left|\sum_{i=1}^n p_i\langle x_i,y_i\rangle\right|\end{aligned}$$

and

$$\begin{aligned}(1.5)\qquad &\frac{1}{2}\left[\left(\sum_{i=1}^n p_i\|x_i\|^2\right)^{1/2}\left(\sum_{i=1}^n p_i\|y_i\|^2\right)^{1/2}+\left|\sum_{i=1}^n p_i\langle x_i,y_i\rangle\right|\right]\\ &\geq\left|\sum_{i=1}^n p_i\langle x_i,e_i\rangle\sum_{i=1}^n p_i\langle e_i,y_i\rangle\right|,\end{aligned}$$

for any $\mathbf{x}=(x_1,...,x_n), \mathbf{y}=(y_1,...,y_n)\in H^n$.

If we take $\mathbf{e}=(e,...,e)\in H^n$ with $\|e\|=1$, then we get from (1.4) and (1.5) the inequalities

$$\begin{aligned}(1.6)\qquad &\left(\sum_{i=1}^n p_i\|x_i\|^2\right)^{1/2}\left(\sum_{i=1}^n p_i\|y_i\|^2\right)^{1/2}\\ &\geq\left|\sum_{i=1}^n p_i\langle x_i,y_i\rangle-\left\langle\sum_{i=1}^n p_ix_i,e\right\rangle\left\langle e,\sum_{i=1}^n p_iy_i\right\rangle\right|+\left|\left\langle\sum_{i=1}^n p_ix_i,e\right\rangle\left\langle e,\sum_{i=1}^n p_iy_i\right\rangle\right|\\ &\geq\left|\sum_{i=1}^n p_i\langle x_i,y_i\rangle\right|\end{aligned}$$

and

$$\begin{aligned}(1.7)\qquad &\frac{1}{2}\left[\left(\sum_{i=1}^n p_i\|x_i\|^2\right)^{1/2}\left(\sum_{i=1}^n p_i\|y_i\|^2\right)^{1/2}+\left|\sum_{i=1}^n p_i\langle x_i,y_i\rangle\right|\right]\\ &\geq\left|\left\langle\sum_{i=1}^n p_ix_i,e\right\rangle\left\langle e,\sum_{i=1}^n p_iy_i\right\rangle\right|,\end{aligned}$$

for any $\mathbf{x}=(x_1,...,x_n), \mathbf{y}=(y_1,...,y_n)\in H^n$.

For other Schwarz related inequalities in inner product spaces, see [1], [6]-[10], [13], [14], [18], [19], [20], [21], [22], [23], [24], [26] and the monographs [11] and [12].

Motivated by the above results, we establish in this paper some new inequalities of Schwarz and Buzano type for n-tuples of vectors in inner product spaces.

Applications for norm and numerical radius inequalities for n-tuples of bounded linear operators and for functions of normal operators defined by power series with nonnegative coefficients are also provided.

2. Main Results

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. For an n-tuple $\mathbf{x} = (x_1, ..., x_n) \in H^n$ and a probability distribution $\mathbf{p} = (p_1, ..., p_n) \in \mathcal{P}_n$, we define the *average vector*

$$\overline{x}_p := \sum_{j=1}^{n} p_j x_j \in H.$$

In particular, for the *uniform probability* $\mathbf{p} = \left(\frac{1}{n}, ..., \frac{1}{n}\right)$ we define

$$\overline{x} := \frac{1}{n} \sum_{j=1}^{n} x_j \in H.$$

We have the following result:

Theorem 1. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$, $\mathbf{x} = (x_1, ..., x_n), \mathbf{y} = (y_1, ..., y_n) \in H^n$ and $\mathbf{p} = (p_1, ..., p_n) \in \mathcal{P}_n$. Then we have the inequalities*

$$\begin{aligned}
(2.1)\qquad & \left(\sum_{i=1}^{n} p_i \|x_i\|^2\right)^{1/2} \left(\sum_{i=1}^{n} p_i \|y_i\|^2\right)^{1/2} \\
& \geq \max\left\{\sum_{i=1}^{n} p_i \left|\langle x_i, y_i\rangle - 2\langle \overline{x}_p, y_i\rangle\right|, \sum_{i=1}^{n} p_i \left|\langle x_i, y_i\rangle - 2\langle x_i, \overline{y}_p\rangle\right|\right\} \\
& \geq \sum_{i=1}^{n} p_i \left[\left|\frac{1}{2}\langle x_i, y_i\rangle - \langle \overline{x}_p, y_i\rangle\right| + \left|\frac{1}{2}\langle x_i, y_i\rangle - \langle x_i, \overline{y}_p\rangle\right|\right] \\
& \geq \begin{cases} \sum_{i=1}^{n} p_i \left|\langle x_i, y_i\rangle - \langle \overline{x}_p, y_i\rangle - \langle x_i, \overline{y}_p\rangle\right|, \\ \\ \sum_{i=1}^{n} p_i \left|\langle x_i, \overline{y}_p\rangle - \langle \overline{x}_p, y_i\rangle\right|. \end{cases}
\end{aligned}$$

In particular, we have

$$\begin{aligned}
(2.2)\qquad & \left(\sum_{i=1}^{n} \|x_i\|^2\right)^{1/2} \left(\sum_{i=1}^{n} \|y_i\|^2\right)^{1/2} \\
& \geq \max\left\{\sum_{i=1}^{n} \left|\langle x_i, y_i\rangle - 2\langle \overline{x}, y_i\rangle\right|, \sum_{i=1}^{n} \left|\langle x_i, y_i\rangle - 2\langle x_i, \overline{y}\rangle\right|\right\} \\
& \geq \sum_{i=1}^{n} \left[\left|\frac{1}{2}\langle x_i, y_i\rangle - \langle \overline{x}, y_i\rangle\right| + \left|\frac{1}{2}\langle x_i, y_i\rangle - \langle x_i, \overline{y}\rangle\right|\right] \\
& \geq \begin{cases} \sum_{i=1}^{n} \left|\langle x_i, y_i\rangle - \langle \overline{x}, y_i\rangle - \langle x_i, \overline{y}\rangle\right|, \\ \\ \sum_{i=1}^{n} \left|\langle x_i, \overline{y}\rangle - \langle \overline{x}, y_i\rangle\right|. \end{cases}
\end{aligned}$$

Proof. Observe that, by the properties of inner product, we have

$$\begin{aligned}\sum_{i=1}^{n} p_i \left\|x_i - 2\overline{x}_p\right\|^2 &= \sum_{i=1}^{n} p_i \left[\left\|x_i\right\|^2 - 4\operatorname{Re}\left\langle x_i, \overline{x}_p\right\rangle + 4\left\|\overline{x}_p\right\|^2\right] \\ &= \sum_{i=1}^{n} p_i \left\|x_i\right\|^2 - 4\operatorname{Re}\left\langle \sum_{i=1}^{n} p_i x_i, \overline{x}_p\right\rangle + 4\left\|\overline{x}_p\right\|^2 \\ &= \sum_{i=1}^{n} p_i \left\|x_i\right\|^2 - 4\left\|\overline{x}_p\right\|^2 + 4\left\|\overline{x}_p\right\|^2 = \sum_{i=1}^{n} p_i \left\|x_i\right\|^2.\end{aligned}$$

By the Cauchy-Buniyakovsky-Schwarz inequality for sequences of real numbers, we have

$$\begin{aligned}\left(\sum_{i=1}^{n} p_i \left\|x_i\right\|^2 \sum_{i=1}^{n} p_i \left\|y_i\right\|^2\right)^{1/2} &= \left(\sum_{i=1}^{n} p_i \left\|x_i - 2\overline{x}_p\right\|^2 \sum_{i=1}^{n} p_i \left\|y_i\right\|^2\right)^{1/2} \\ &\geq \sum_{i=1}^{n} p_i \left\|x_i - 2\overline{x}_p\right\| \left\|y_i\right\|.\end{aligned}$$

By the Schwarz inequality in $(H, \langle\cdot,\cdot\rangle)$ we have for each $i \in \{1, ..., n\}$

$$\left\|x_i - 2\overline{x}_p\right\| \left\|y_i\right\| \geq \left|\left\langle x_i - 2\overline{x}_p, y_i\right\rangle\right| = \left|\left\langle x_i, y_i\right\rangle - 2\left\langle \overline{x}_p, y_i\right\rangle\right|$$

and then

$$\sum_{i=1}^{n} p_i \left\|x_i - 2\overline{x}_p\right\| \left\|y_i\right\| \geq \sum_{i=1}^{n} p_i \left|\left\langle x_i, y_i\right\rangle - 2\left\langle \overline{x}_p, y_i\right\rangle\right|.$$

Therefore

$$\left(\sum_{i=1}^{n} p_i \left\|x_i\right\|^2 \sum_{i=1}^{n} p_i \left\|y_i\right\|^2\right)^{1/2} \geq \sum_{i=1}^{n} p_i \left|\left\langle x_i, y_i\right\rangle - 2\left\langle \overline{x}_p, y_i\right\rangle\right| \tag{2.3}$$

and, similarly

$$\left(\sum_{i=1}^{n} p_i \left\|x_i\right\|^2 \sum_{i=1}^{n} p_i \left\|y_i\right\|^2\right)^{1/2} \geq \sum_{i=1}^{n} p_i \left|\left\langle x_i, y_i\right\rangle - 2\left\langle x_i, \overline{y}_p\right\rangle\right|. \tag{2.4}$$

Making use of (2.3) and (2.4) we get the first inequality in (2.1).

Since $\max\{a, b\} \geq \frac{1}{2}(a + b)$ for positive numbers a and b,we get the second inequality in (2.1). The last part of (2.1) follows by the triangle inequality. □

Remark 1. *Using the generalized triangle inequality we have*

$$\sum_{i=1}^{n} p_i \left|\left\langle x_i, y_i\right\rangle - \left\langle \overline{x}_p, y_i\right\rangle - \left\langle x_i, \overline{y}_p\right\rangle\right| \geq \left|\sum_{i=1}^{n} p_i \left\langle x_i, y_i\right\rangle - 2\left\langle \overline{x}_p, \overline{y}_p\right\rangle\right|$$

and by (2.1) we get the following string of inequalities

$$\begin{aligned}
&\left(\sum_{i=1}^{n} p_i \left\|x_i\right\|^2\right)^{1/2} \left(\sum_{i=1}^{n} p_i \left\|y_i\right\|^2\right)^{1/2} \\
&\geq \max\left\{\sum_{i=1}^{n} p_i \left|\langle x_i, y_i\rangle - 2\langle \overline{x}_p, y_i\rangle\right|, \sum_{i=1}^{n} p_i \left|\langle x_i, y_i\rangle - 2\langle x_i, \overline{y}_p\rangle\right|\right\} \\
&\geq \sum_{i=1}^{n} p_i \left[\left|\frac{1}{2}\langle x_i, y_i\rangle - \langle \overline{x}_p, y_i\rangle\right| + \left|\frac{1}{2}\langle x_i, y_i\rangle - \langle x_i, \overline{y}_p\rangle\right|\right] \\
&\geq \sum_{i=1}^{n} p_i \left|\langle x_i, y_i\rangle - \langle \overline{x}_p, y_i\rangle - \langle x_i, \overline{y}_p\rangle\right| \\
&\geq \left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle - 2\langle \overline{x}_p, \overline{y}_p\rangle\right|.
\end{aligned} \tag{2.5}$$

The unweighted case is as follows

$$\begin{aligned}
&\left(\sum_{i=1}^{n} \left\|x_i\right\|^2\right)^{1/2} \left(\sum_{i=1}^{n} \left\|y_i\right\|^2\right)^{1/2} \\
&\geq \max\left\{\sum_{i=1}^{n} \left|\langle x_i, y_i\rangle - 2\langle \overline{x}, y_i\rangle\right|, \sum_{i=1}^{n} \left|\langle x_i, y_i\rangle - 2\langle x_i, \overline{y}\rangle\right|\right\} \\
&\geq \sum_{i=1}^{n} \left[\left|\frac{1}{2}\langle x_i, y_i\rangle - \langle \overline{x}, y_i\rangle\right| + \left|\frac{1}{2}\langle x_i, y_i\rangle - \langle x_i, \overline{y}\rangle\right|\right] \\
&\geq \sum_{i=1}^{n} \left|\langle x_i, y_i\rangle - \langle \overline{x}, y_i\rangle - \langle x_i, \overline{y}\rangle\right| \geq \left|\sum_{i=1}^{n} \langle x_i, y_i\rangle - 2\langle \overline{x}, \overline{y}\rangle\right|.
\end{aligned} \tag{2.6}$$

The inequality between the first and 4th and 5th terms in (2.6) was obtained in [25].

Corollary 1. *With the assumptions of Theorem 1 we have*

$$\frac{1}{2}\left[\left(\sum_{i=1}^{n} p_i \left\|x_i\right\|^2\right)^{1/2} \left(\sum_{i=1}^{n} p_i \left\|y_i\right\|^2\right)^{1/2} + \left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle\right|\right] \geq \left|\langle \overline{x}_p, \overline{y}_p\rangle\right| \tag{2.7}$$

and, in particular

$$\frac{1}{2n}\left[\left(\sum_{i=1}^{n} \left\|x_i\right\|^2\right)^{1/2} \left(\sum_{i=1}^{n} \left\|y_i\right\|^2\right)^{1/2} + \left|\sum_{i=1}^{n} \langle x_i, y_i\rangle\right|\right] \geq \left|\langle \overline{x}, \overline{y}\rangle\right|. \tag{2.8}$$

Proof. By the triangle inequality we have

$$\left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle - 2\langle \overline{x}_p, \overline{y}_p\rangle\right| \geq 2\left|\langle \overline{x}_p, \overline{y}_p\rangle\right| - \left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle\right|$$

and from (2.5) we get

$$\left(\sum_{i=1}^{n} p_i \left\|x_i\right\|^2\right)^{1/2} \left(\sum_{i=1}^{n} p_i \left\|y_i\right\|^2\right)^{1/2} \geq 2\left|\langle \overline{x}_p, \overline{y}_p\rangle\right| - \left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle\right|,$$

i.e.

$$\left(\sum_{i=1}^{n} p_i \|x_i\|^2\right)^{1/2} \left(\sum_{i=1}^{n} p_i \|y_i\|^2\right)^{1/2} + \left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle\right| \geq 2\left|\left\langle \overline{x}_p, \overline{y}_p\right\rangle\right|,$$

and the inequality (2.7) is proved. □

The following result also holds:

Theorem 2. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$, $\mathbf{x} = (x_1, ..., x_n), \mathbf{y} = (y_1, ..., y_n) \in H^n$ and $\mathbf{p} = (p_1, ..., p_n) \in \mathcal{P}_n$. Then we have the inequalities*

$$\begin{aligned} &\left(\sum_{i=1}^{n} p_i \|x_i\|^2 \sum_{i=1}^{n} p_i \|y_i\|^2\right)^{1/2} \\ &\geq \sum_{i=1}^{n} p_i \left|\langle x_i, y_i\rangle - 2\langle \overline{x}_p, y_i\rangle - 2\left\langle x_i, \overline{y}_p\right\rangle + 4\left\langle \overline{x}_p, \overline{y}_p\right\rangle\right| \\ &\geq \sum_{i=1}^{n} p_i \left|\langle x_i, y_i\rangle\right| \geq \left|\sum_{i=1}^{n} p_i \langle x_i, y_i\rangle\right| \end{aligned} \tag{2.9}$$

and, in particular

$$\begin{aligned} &\left(\sum_{i=1}^{n} \|x_i\|^2 \sum_{i=1}^{n} \|y_i\|^2\right)^{1/2} \\ &\geq \sum_{i=1}^{n} \left|\langle x_i, y_i\rangle - 2\langle \overline{x}, y_i\rangle - 2\langle x_i, \overline{y}\rangle + 4\langle \overline{x}, \overline{y}\rangle\right| \\ &\geq \sum_{i=1}^{n} \left|\langle x_i, y_i\rangle\right| \geq \left|\sum_{i=1}^{n} \langle x_i, y_i\rangle\right|. \end{aligned} \tag{2.10}$$

Proof. By the Cauchy-Buniyakovsky-Schwarz inequality for sequences of real numbers, we have

$$\begin{aligned} &\left(\sum_{i=1}^{n} p_i \|x_i\|^2 \sum_{i=1}^{n} p_i \|y_i\|^2\right) \\ &= \left(\sum_{i=1}^{n} p_i \|x_i - 2\overline{x}_p\|^2 \sum_{i=1}^{n} p_i \left\|y_i - 2\overline{y}_p\right\|^2\right)^{1/2} \\ &\geq \sum_{i=1}^{n} p_i \|x_i - 2\overline{x}_p\| \left\|y_i - 2\overline{y}_p\right\|. \end{aligned} \tag{2.11}$$

By the Schwarz inequality in $(H, \langle\cdot,\cdot\rangle)$ we have

$$\begin{aligned} \|x_i - 2\overline{x}_p\| \left\|y_i - 2\overline{y}_p\right\| &\geq \left|\left\langle x_i - 2\overline{x}_p, y_i - 2\overline{y}_p\right\rangle\right| \\ &= \left|\langle x_i, y_i\rangle - 2\langle \overline{x}_p, y_i\rangle - 2\left\langle x_i, \overline{y}_p\right\rangle + 4\left\langle \overline{x}_p, \overline{y}_p\right\rangle\right| \end{aligned}$$

for any $i \in \{1, ..., n\}$.

Therefore

$$\begin{aligned}
(2.12)\qquad & \sum_{i=1}^{n} p_i \left\| x_i - 2\overline{x}_p \right\| \left\| y_i - 2\overline{y}_p \right\| \\
&\geq \sum_{i=1}^{n} p_i \left| \langle x_i, y_i \rangle - 2 \langle \overline{x}_p, y_i \rangle - 2 \langle x_i, \overline{y}_p \rangle + 4 \langle \overline{x}_p, \overline{y}_p \rangle \right| \\
&\geq \left| \sum_{i=1}^{n} p_i \left[\langle x_i, y_i \rangle - 2 \langle \overline{x}_p, y_i \rangle - 2 \langle x_i, \overline{y}_p \rangle + 4 \langle \overline{x}_p, \overline{y}_p \rangle \right] \right| \\
&= \left| \sum_{i=1}^{n} p_i \langle x_i, y_i \rangle - 2 \left\langle \overline{x}_p, \sum_{i=1}^{n} p_i y_i \right\rangle - 2 \left\langle \sum_{i=1}^{n} p_i x_i, \overline{y}_p \right\rangle + 4 \langle \overline{x}_p, \overline{y}_p \rangle \right| \\
&= \sum_{i=1}^{n} p_i \left| \langle x_i, y_i \rangle \right|.
\end{aligned}$$

Making use of (2.11) and (2.12) we get the desired result (2.9). □

The following corollary holds:

Corollary 2. *With the assumptions of Theorem 2 we have*

$$\begin{aligned}
(2.13)\qquad & \frac{1}{4} \left[\left(\sum_{i=1}^{n} p_i \left\| x_i \right\|^2 \sum_{i=1}^{n} p_i \left\| y_i \right\|^2 \right)^{1/2} + \sum_{i=1}^{n} p_i \left| \langle x_i, y_i \rangle \right| \right] \\
&\geq \sum_{i=1}^{n} p_i \left| \langle \overline{x}_p, \overline{y}_p \rangle - \frac{1}{2} \left[\langle \overline{x}_p, y_i \rangle + \langle x_i, \overline{y}_p \rangle \right] \right|.
\end{aligned}$$

Proof. If we add to the first inequality in (2.9) the quantity $\sum_{i=1}^{n} p_i \left| \langle x_i, y_i \rangle \right|$ we have

$$\begin{aligned}
(2.14)\qquad & \left(\sum_{i=1}^{n} p_i \left\| x_i \right\|^2 \sum_{i=1}^{n} p_i \left\| y_i \right\|^2 \right)^{1/2} + \sum_{i=1}^{n} p_i \left| \langle x_i, y_i \rangle \right| \\
&\geq \sum_{i=1}^{n} p_i \left[\left| \langle x_i, y_i \rangle - 2 \langle \overline{x}_p, y_i \rangle - 2 \langle x_i, \overline{y}_p \rangle + 4 \langle \overline{x}_p, \overline{y}_p \rangle \right| + \left| \langle x_i, y_i \rangle \right| \right] \\
&\geq \sum_{i=1}^{n} p_i \left| 4 \langle \overline{x}_p, \overline{y}_p \rangle - 2 \langle \overline{x}_p, y_i \rangle - 2 \langle x_i, \overline{y}_p \rangle \right| \\
&= 4 \sum_{i=1}^{n} p_i \left| \langle \overline{x}_p, \overline{y}_p \rangle - \frac{1}{2} \left[\langle \overline{x}_p, y_i \rangle + \langle x_i, \overline{y}_p \rangle \right] \right|.
\end{aligned}$$

Dividing by 4 we get the desired result (2.13). □

We observe that, if we take $H = \mathbb{C}$ with the inner product $\langle z, w \rangle = z\overline{w}$ then by taking above $x_i = a_i \in \mathbb{C}$ and $y_i = \overline{b}_i, i \in \{1, ..., n\}$, then from (2.1) we get the inequality

$$\left(\sum_{i=1}^{n} p_i |a_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} p_i |b_i|^2\right)^{1/2} \tag{2.15}$$
$$\geq \max\left\{\sum_{i=1}^{n} p_i |a_i b_i - 2b_i \overline{a}_p|, \sum_{i=1}^{n} p_i \left|a_i b_i - 2a_i \overline{b}_p\right|\right\}$$
$$\geq \sum_{i=1}^{n} p_i \left[\left|\frac{1}{2} a_i b_i - b_i \overline{a}_p\right| + \left|\frac{1}{2} a_i b_i - a_i \overline{b}_p\right|\right]$$
$$\geq \begin{cases} \sum_{i=1}^{n} p_i \left|a_i b_i - b_i \overline{a}_p - a_i \overline{b}_p\right|, \\ \\ \sum_{i=1}^{n} p_i \left|a_i \overline{b}_p - b_i \overline{a}_p\right|, \end{cases}$$

where $\mathbf{p} = (p_1, ..., p_n) \in \mathcal{P}_n$ and

$$\overline{a}_p = \sum_{i=1}^{n} p_i a_i.$$

Utilising the inequality (2.7) we also have

$$\frac{1}{2}\left[\left(\sum_{i=1}^{n} p_i |a_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} p_i |b_i|^2\right)^{1/2} + \left|\sum_{i=1}^{n} p_i a_i b_i\right|\right] \tag{2.16}$$
$$\geq \left|\sum_{i=1}^{n} p_i a_i\right| \left|\sum_{i=1}^{n} p_i b_i\right|$$

for any $\mathbf{p} = (p_1, ..., p_n) \in \mathcal{P}_n$ and $a_i, b_i \in \mathbb{C}$, $i \in \{1, ..., n\}$.

If in (2.16) we take $p_i = \frac{q_i}{\sum_{k=1}^{n} q_k}$ with $q_i \geq 0, i \in \{1, ..., n\}$ and $\sum_{k=1}^{n} q_k > 0$, then we get

$$\frac{1}{2}\left[\left(\sum_{i=1}^{n} q_i |a_i|^2\right)^{1/2} \left(\sum_{i=1}^{n} q_i |b_i|^2\right)^{1/2} + \left|\sum_{i=1}^{n} q_i a_i b_i\right|\right] \sum_{k=1}^{n} q_k \tag{2.17}$$
$$\geq \left|\sum_{i=1}^{n} q_i a_i\right| \left|\sum_{i=1}^{n} q_i b_i\right|.$$

Moreover, by taking $b_i = a_i$, $i \in \{1, ..., n\}$ in (2.17) we get

$$\frac{1}{2}\left[\sum_{i=1}^{n} q_i |a_i|^2 + \left|\sum_{i=1}^{n} q_i a_i^2\right|\right] \sum_{k=1}^{n} q_k \geq \left|\sum_{i=1}^{n} q_i a_i\right|^2. \tag{2.18}$$

If $r_i \in \mathbb{R} \backslash \{0\}$ and $z_i \in \mathbb{C}$, $i \in \{1, ..., n\}$, then by taking $q_i = r_i^2$ and $a_i = \frac{z_i}{r_i}$, $i \in \{1, ..., n\}$ we get the well known *de Bruijn inequality* [2]

$$\frac{1}{2}\left[\sum_{i=1}^{n} |z_i|^2 + \left|\sum_{i=1}^{n} z_i^2\right|\right] \sum_{i=1}^{n} r_i^2 \geq \left|\sum_{i=1}^{n} r_i a_i\right|^2. \tag{2.19}$$

The other vector inequalities from above have similar versions for complex numbers. However the details are not presented here.

3. Applications for n-Tuples of Operators

If in (2.7) we take $p_i = \frac{q_i}{\sum_{k=1}^n q_k}$ with $q_i \geq 0, i \in \{1, ..., n\}$ and $\sum_{k=1}^n q_k > 0$, then we get

$$\frac{1}{2}\left[\left(\sum_{i=1}^n q_i \|x_i\|^2\right)^{1/2}\left(\sum_{i=1}^n q_i \|y_i\|^2\right)^{1/2} + \left|\sum_{i=1}^n q_i \langle x_i, y_i\rangle\right|\right]\sum_{k=1}^n q_k \geq \left|\left\langle \sum_{i=1}^n q_i x_i, \sum_{i=1}^n q_i y_i \right\rangle\right| \tag{3.1}$$

for any $\mathbf{x} = (x_1, ..., x_n), \mathbf{y} = (y_1, ..., y_n) \in H^n$.

Let $(H; \langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [17, p. 1]:

$$W(T) = \{\langle Tx, x\rangle,\ x \in H,\ \|x\| = 1\}.$$

The *numerical radius* $w(T)$ of an operator T on H is given by [17, p. 8]:

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{3.2}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators on the Hilbert space H. This norm is equivalent with the operator norm. In fact, the following more precise result holds [17, p. 9]:

Theorem 3 (Equivalent norm)**.** *For any $T \in B(H)$ one has*

$$w(T) \leq \|T\| \leq 2w(T). \tag{3.3}$$

The following result holds:

Theorem 4. *Let $(A_1, ..., A_n), (B_1, ..., B_n)$ be two n-tuple of bounded linear operators on the Hilbert space $(H; \langle\cdot,\cdot\rangle)$ and $q_i \geq 0, i \in \{1, ..., n\}$ with $\sum_{k=1}^n q_k > 0$. Then*

$$\left\|\sum_{i=1}^n q_i B_i^* \sum_{i=1}^n q_i A_i\right\| \leq \frac{1}{2}\left[\left\|\sum_{i=1}^n q_i |A_i|^2\right\|^{1/2}\left\|\sum_{i=1}^n q_i |B_i|^2\right\|^{1/2} + \left\|\sum_{i=1}^n q_i B_i^* A_i\right\|\right]\sum_{k=1}^n q_k \tag{3.4}$$

and

$$w\left(\sum_{i=1}^n q_i B_i^* \sum_{i=1}^n q_i A_i\right) \leq \frac{1}{2}\left[\left\|\sum_{i=1}^n q_i |A_i|^2\right\|^{1/2}\left\|\sum_{i=1}^n q_i |B_i|^2\right\|^{1/2} + w\left(\sum_{i=1}^n q_i B_i^* A_i\right)\right]\sum_{k=1}^n q_k. \tag{3.5}$$

Proof. We take in inequality (3.1) $x_i = A_i x$, $y_i = B_i y$, $i \in \{1, ..., n\}$ to get

$$\frac{1}{2}\left[\left(\sum_{i=1}^{n} q_i \|A_i x\|^2\right)^{1/2}\left(\sum_{i=1}^{n} q_i \|B_i y\|^2\right)^{1/2} + \left|\sum_{i=1}^{n} q_i \langle A_i x, B_i y\rangle\right|\right]\sum_{k=1}^{n} q_k$$
$$\geq \left|\left\langle \sum_{i=1}^{n} q_i A_i x, \sum_{i=1}^{n} q_i B_i y\right\rangle\right|,$$

for any $x, y \in H$.

Since

$$\sum_{i=1}^{n} q_i \|A_i x\|^2 = \sum_{i=1}^{n} q_i \langle A_i x, A_i x\rangle = \sum_{i=1}^{n} q_i \langle A_i^* A_i x, x\rangle$$
$$= \sum_{i=1}^{n} q_i \left\langle |A_i|^2 x, x\right\rangle = \left\langle \sum_{i=1}^{n} q_i |A_i|^2 x, x\right\rangle,$$

$$\sum_{i=1}^{n} q_i \|B_i y\|^2 = \left\langle \sum_{i=1}^{n} q_i |B_i|^2 y, y\right\rangle,$$

$$\sum_{i=1}^{n} q_i \langle A_i x, B_i y\rangle = \left\langle \sum_{i=1}^{n} q_i B_i^* A_i x, y\right\rangle$$

and

$$\left\langle \sum_{i=1}^{n} q_i A_i x, \sum_{i=1}^{n} q_i B_i y\right\rangle = \left\langle \sum_{i=1}^{n} q_i B_i^* \sum_{i=1}^{n} q_i A_i x, y\right\rangle,$$

then we have

$$\left|\left\langle \sum_{i=1}^{n} q_i B_i^* \sum_{i=1}^{n} q_i A_i x, y\right\rangle\right| \tag{3.6}$$
$$\leq \frac{1}{2}\left[\left\langle \sum_{i=1}^{n} q_i |A_i|^2 x, x\right\rangle^{1/2}\left\langle \sum_{i=1}^{n} q_i |B_i|^2 y, y\right\rangle^{1/2}\right.$$
$$\left. + \left|\left\langle \sum_{i=1}^{n} q_i B_i^* A_i x, y\right\rangle\right|\right]\sum_{k=1}^{n} q_k$$

for any $x, y \in H$.

This inequality is of interest in itself.

We know that for any bounded operator T we have

$$\|T\| = \sup_{\|x\|=\|y\|=1} |\langle Tx, y\rangle|.$$

Taking the supremum in (3.6) over $\|x\| = \|y\| = 1$ we have

$$
\begin{aligned}
(3.7)\qquad & \left\|\sum_{i=1}^{n} q_i B_i^* \sum_{i=1}^{n} q_i A_i\right\| \\
&= \sup_{\|x\|=\|y\|=1} \left|\left\langle \sum_{i=1}^{n} q_i B_i^* \sum_{i=1}^{n} q_i A_i x, y\right\rangle\right| \sum_{k=1}^{n} q_k \\
&\le \frac{1}{2} \sup_{\|x\|=\|y\|=1} \left[\left\langle \sum_{i=1}^{n} q_i \left|A_i\right|^2 x, x\right\rangle^{1/2} \left\langle \sum_{i=1}^{n} q_i \left|B_i\right|^2 y, y\right\rangle^{1/2} \right. \\
&\left. + \left|\left\langle \sum_{i=1}^{n} q_i B_i^* A_i x, y\right\rangle\right|\right] \sum_{k=1}^{n} q_k \\
&\le \frac{1}{2} \left[\sup_{\|x\|=1} \left[\left\langle \sum_{i=1}^{n} q_i \left|A_i\right|^2 x, x\right\rangle^{1/2} \sup_{\|y\|=1} \left\langle \sum_{i=1}^{n} q_i \left|B_i\right|^2 x, x\right\rangle^{1/2}\right] \right. \\
&\left. + \sup_{\|x\|=\|y\|=1} \left|\left\langle \sum_{i=1}^{n} q_i B_i^* A_i x, y\right\rangle\right|\right] \sum_{k=1}^{n} q_k \\
&= \frac{1}{2} \left[\left\|\sum_{i=1}^{n} q_i \left|A_i\right|^2\right\|^{1/2} \left\|\sum_{i=1}^{n} q_i \left|B_i\right|^2\right\|^{1/2} + \left\|\sum_{i=1}^{n} q_i B_i^* A_i\right\|\right] \sum_{k=1}^{n} q_k,
\end{aligned}
$$

because $\sum_{i=1}^{n} q_i \left|A_i\right|^2$ and $\sum_{i=1}^{n} q_i \left|B_i\right|^2$ are positive selfadjoint operators.

This proves (3.4).

If we put in (3.6) $y = x$ and then take the supremum over $\|x\| = 1$, we get

$$
\begin{aligned}
(3.8)\qquad & w\left(\sum_{i=1}^{n} q_i B_i^* \sum_{i=1}^{n} q_i A_i\right) \\
&= \sup_{\|x\|=1} \left|\left\langle \sum_{i=1}^{n} q_i B_i^* \sum_{i=1}^{n} q_i A_i x, x\right\rangle\right| \\
&\le \frac{1}{2} \left[\sup_{\|x\|=1} \left[\left(\left\langle \sum_{i=1}^{n} q_i \left|A_i\right|^2 x, x\right\rangle\right)^{1/2} \left(\left\langle \sum_{i=1}^{n} q_i \left|B_i\right|^2 x, x\right\rangle\right)^{1/2}\right] \right. \\
&\left. + \sup_{\|x\|=1} \left|\left\langle \sum_{i=1}^{n} q_i B_i^* A_i x, x\right\rangle\right|\right] \sum_{k=1}^{n} q_k.
\end{aligned}
$$

Since

$$\sup_{\|x\|=1}\left[\left\langle \sum_{i=1}^{n} q_i |A_i|^2 x, x\right\rangle^{1/2} \left\langle \sum_{i=1}^{n} q_i |B_i|^2 x, x\right\rangle^{1/2}\right] \tag{3.9}$$
$$\leq \sup_{\|x\|=1}\left\langle \sum_{i=1}^{n} q_i |A_i|^2 x, x\right\rangle^{1/2} \sup_{\|x\|=1}\left\langle \sum_{i=1}^{n} q_i |B_i|^2 x, x\right\rangle^{1/2}$$
$$= \left\|\sum_{i=1}^{n} q_i |A_i|^2\right\|^{1/2} \left\|\sum_{i=1}^{n} q_i |B_i|^2\right\|^{1/2},$$

we get from (3.8) and (3.9) the desired result (3.5). □

Corollary 3. *Let $(A_1, ..., A_n)$ be an n-tuple of bounded linear operators on the Hilbert space $(H; \langle\cdot,\cdot\rangle)$ and $q_i \geq 0, i \in \{1, ..., n\}$ with $\sum_{k=1}^{n} q_k > 0$. Then*

$$\left\|\left(\sum_{i=1}^{n} q_i A_i\right)^2\right\| \tag{3.10}$$
$$\leq \frac{1}{2}\left[\left\|\sum_{i=1}^{n} q_i |A_i|^2\right\|^{1/2} \left\|\sum_{i=1}^{n} q_i |A_i^*|^2\right\|^{1/2} + \left\|\sum_{i=1}^{n} q_i A_i^2\right\|\right]\sum_{k=1}^{n} q_k$$

and

$$w\left(\left(\sum_{i=1}^{n} q_i A_i\right)^2\right) \tag{3.11}$$
$$\leq \frac{1}{2}\left[\left\|\sum_{i=1}^{n} q_i |A_i|^2\right\|^{1/2} \left\|\sum_{i=1}^{n} q_i |A_i^*|^2\right\|^{1/2} + w\left(\sum_{i=1}^{n} q_i A_i^2\right)\right]\sum_{k=1}^{n} q_k.$$

Remark 2. *If we take $n = 2$ and $q_1 = q_2 = 1$ in (3.4), then we get*

$$\|(B_1^* + B_2^*)(A_1 + A_2)\| \tag{3.12}$$
$$\leq \left\||A_1|^2 + |A_2|^2\right\|^{1/2} \left\||B_1|^2 + |B_2|^2\right\|^{1/2} + \|B_1^* A_1 + B_2^* A_2\|$$

for any operators A_1, A_2, B_1, B_2.

Assume that T, V are bounded linear operators and consider the Cartesian decomposition

$$T = A + iB,\ V = C + iD$$

with the selfadjoint operators A, B, C, D given by

$$A = \frac{1}{2}(T^* + T),\ B = \frac{1}{2i}(T - T^*)$$

and

$$C = \frac{1}{2}(V^* + V),\ D = \frac{1}{2i}(V - V^*).$$

Take $A_1 = A$, $A_2 = iB$, $B_1 = C$ and $B_2 = -iD$. Then

$$(B_1^* + B_2^*)(A_1 + A_2) = VT,$$

$$|A_1|^2 + |A_2|^2 = A^2 + B^2 = \frac{1}{2}\left(|T|^2 + |T^*|^2\right),$$

$$|B_1|^2 + |B_2|^2 = C^2 + D^2 = \frac{1}{2}\left(|V|^2 + |V^*|^2\right)$$

and

$$\begin{aligned} B_1^* A_1 + B_2^* A_2 &= CA + (-iD)^* (iB) = CA - DB \\ &= \frac{1}{2}(V^* + V)\frac{1}{2}(T^* + T) - \frac{1}{2i}(V - V^*)\frac{1}{2i}(T - T^*) \\ &= \frac{1}{4}\left[(V^* + V)(T^* + T) + (V - V^*)(T - T^*)\right] \\ &= \frac{1}{2}(VT + V^*T^*). \end{aligned}$$

Then by (3.12) we get

$$\|VT\| \le \frac{1}{2}\left[\left\||V|^2 + |V^*|^2\right\|^{1/2}\left\||T|^2 + |T^*|^2\right\|^{1/2} + \|VT + V^*T^*\|\right]. \tag{3.13}$$

If we replace in this inequality V with V^* and T with T^*, then we get the dual inequality

$$\|V^*T^*\| \le \frac{1}{2}\left[\left\||V|^2 + |V^*|^2\right\|^{1/2}\left\||T|^2 + |T^*|^2\right\|^{1/2} + \|VT + V^*T^*\|\right]. \tag{3.14}$$

Adding these inequalities give us

$$\|VT\| + \|V^*T^*\| \le \left\||V|^2 + |V^*|^2\right\|^{1/2}\left\||T|^2 + |T^*|^2\right\|^{1/2} + \|VT + V^*T^*\|, \tag{3.15}$$

which implies that

$$\begin{aligned} 0 &\le \|VT\| + \|V^*T^*\| - \|VT + V^*T^*\| \\ &\le \left\||V|^2 + |V^*|^2\right\|^{1/2}\left\||T|^2 + |T^*|^2\right\|^{1/2}, \end{aligned} \tag{3.16}$$

for any bounded linear operators V, T.

Using the inequality (3.5) for $n = 2$ we can derive in a similar way the numerical radius inequality

$$\begin{aligned} 0 &\le w(VT) + w(V^*T^*) - w(VT + V^*T^*) \\ &\le \left\||V|^2 + |V^*|^2\right\|^{1/2}\left\||T|^2 + |T^*|^2\right\|^{1/2}, \end{aligned} \tag{3.17}$$

for any bounded linear operators V, T.

If we take in (3.16) and (3.17) $V = T$, then we get

$$0 \le \|V^2\| + \left\|(V^*)^2\right\| - \left\|V^2 + (V^*)^2\right\| \le \left\||V|^2 + |V^*|^2\right\|, \tag{3.18}$$

and

$$0 \le w(V^2) + w\left((V^*)^2\right) - w\left(V^2 + (V^*)^2\right) \le \left\||V|^2 + |V^*|^2\right\| \tag{3.19}$$

for any bounded linear operators V.

4. Applications for Functions of Normal Operators

Recall some examples of power series with nonnegative coefficients

$$\frac{1}{1-\lambda}=\sum_{n=0}^{\infty}\lambda^{n},\ \lambda\in D(0,1);\tag{4.1}$$
$$\ln\frac{1}{1-\lambda}=\sum_{n=1}^{\infty}\frac{1}{n}\lambda^{n},\ \lambda\in D(0,1);$$
$$\exp(\lambda)=\sum_{n=0}^{\infty}\frac{1}{n!}\lambda^{n},\ \lambda\in\mathbb{C};$$
$$\sinh\lambda=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\lambda^{2n+1},\ \lambda\in\mathbb{C};$$
$$\cosh\lambda=\sum_{n=0}^{\infty}\frac{1}{(2n)!}\lambda^{2n},\ \lambda\in\mathbb{C}.$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\frac{1}{2}\ln\left(\frac{1+\lambda}{1-\lambda}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1},\ \lambda\in D(0,1);\tag{4.2}$$
$$\sin^{-1}(\lambda)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}\lambda^{2n+1},\ \lambda\in D(0,1);$$
$$\tanh^{-1}(\lambda)=\sum_{n=1}^{\infty}\frac{1}{2n-1}\lambda^{2n-1},\ \lambda\in D(0,1);$$
$${}_2F_1(\alpha,\beta,\gamma,\lambda)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^{n},\alpha,\beta,\gamma>0,$$
$$\lambda\in D(0,1)$$

where Γ is *Gamma function*.

The following result for power series with nonnegative coefficients holds:

Theorem 5. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a power series with nonnegative coefficients $a_n\geq 0$ for $n\in\mathbb{N}$ and having the radius of convergence $R>0$ or $R=\infty$. If U,V are normal operators on the Hilbert space $(H;\langle\cdot,\cdot\rangle)$, $U^*V=VU^*$ and $\alpha>0$ such that $\alpha<R$ and $\|U\|,\|V\|\leq 1$, then*

$$\begin{aligned}&|\langle f(\alpha U^*)f(\alpha V)x,y\rangle|\\&\leq\frac{1}{2}f(\alpha)\left[\left\langle f\left(\alpha|V|^2\right)x,x\right\rangle^{1/2}\left\langle f\left(\alpha|U|^2\right)y,y\right\rangle^{1/2}+|\langle f(\alpha U^*V)x,y\rangle|\right]\end{aligned}\tag{4.3}$$

for any $x,y\in H$.

Proof. Using the inequality (3.6) we can state that

$$\begin{aligned}(4.4)\quad & \left|\left\langle \sum_{i=0}^{n} a_i\alpha^i \left(U^*\right)^i \sum_{i=0}^{n} a_i\alpha^i V^i x, y\right\rangle\right| \\ & \leq \frac{1}{2}\left[\left\langle \sum_{i=0}^{n} a_i\alpha^i \left|V^i\right|^2 x, x\right\rangle^{1/2} \left\langle \sum_{i=0}^{n} a_i\alpha^i \left|\left(U^*\right)^i\right|^2 y, y\right\rangle^{1/2} \right. \\ & \left. + \left|\left\langle \sum_{i=0}^{n} a_i\alpha^i \left(U^*\right)^i V^i x, y\right\rangle\right|\right] \sum_{i=0}^{n} a_i\alpha^i\end{aligned}$$

for any $x, y \in H$ and $n \geq 1$.

Since U, V are normal operators, then for $i \geq 1$

$$\left|V^i\right|^2 = \left(V^i\right)^* V^i = \left(V^*\right)^i V^i = \left(V^*V\right)^i = |V|^{2i}$$

and

$$\left|\left(U^*\right)^i\right|^2 = |U|^{2i}.$$

Also, since $U^*V = VU^*$, then

$$\left(U^*\right)^i V^i = \left(U^*V\right)^i$$

for any $i \geq 1$.

Therefore from (4.4) we get

$$\begin{aligned}(4.5)\quad & \left|\left\langle \sum_{i=0}^{n} a_i\alpha^i \left(U^*\right)^i \sum_{i=0}^{n} a_i\alpha^i V^i x, y\right\rangle\right| \\ & \leq \frac{1}{2}\left[\left\langle \sum_{i=0}^{n} a_i\alpha^i |V|^{2i} x, x\right\rangle^{1/2} \left\langle \sum_{i=0}^{n} a_i\alpha^i |U|^{2i} y, y\right\rangle^{1/2} \right. \\ & \left. + \left|\left\langle \sum_{i=0}^{n} a_i\alpha^i \left(U^*V\right)^i x, y\right\rangle\right|\right] \sum_{i=0}^{n} a_i\alpha^i,\end{aligned}$$

for any $x, y \in H$ and $n \geq 1$.

Since $\left\|\alpha |V|^2\right\| = \alpha \|V\|^2 < R$, $\left\|\alpha |U|^2\right\| = \alpha \|U\|^2 < R$, $\|\alpha U^*V\| \leq \alpha \|U\| \|V\| < R$, $\|\alpha U^*\| < R$ and $\|\alpha V\| < R$, then the series

$$\sum_{i=0}^{\infty} a_i\alpha^i \left(U^*\right)^i, \ \sum_{i=0}^{\infty} a_i\alpha^i V^i, \ \sum_{i=0}^{n} a_i\alpha^i |V|^{2i}, \ \sum_{i=0}^{n} a_i\alpha^i |U|^{2i}, \ \sum_{i=0}^{\infty} a_i\alpha^i \left(U^*V\right)^i$$

are convergent in $B(H)$ and $\sum_{i=0}^{\infty} a_i\alpha^i$ is convergent in $\mathbb{R}$.

Taking the limit over $n \to \infty$ in (4.5) we get the desired result (4.3). □

Corollary 4. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with nonnegative coefficients $a_n \geq 0$ for $n \in \mathbb{N}$ and having the radius of convergence $R > 0$ or $R = \infty$. If V is normal operator on the Hilbert space $(H; \langle\cdot,\cdot\rangle)$ and $\alpha > 0$ such that $\alpha < R$ and $\|V\| \leq 1$, then*

$$\begin{aligned}(4.6)\quad & \left|\left\langle f^2(\alpha V) x, y\right\rangle\right| \\ & \leq \frac{1}{2} f(\alpha) \left[\left\langle f\left(\alpha |V|^2\right) x, x\right\rangle^{1/2} \left\langle f\left(\alpha |V|^2\right) y, y\right\rangle^{1/2} + \left|\left\langle f\left(\alpha V^2\right) x, y\right\rangle\right|\right]\end{aligned}$$

and

$$\left|\left\langle |f(\alpha V)|^2 x, y\right\rangle\right| \leq \frac{1}{2} f(\alpha)\left[\left\langle f\left(\alpha |V|^2\right) x, x\right\rangle^{1/2}\left\langle f\left(\alpha |V|^2\right) y, y\right\rangle^{1/2} + \left|\left\langle f\left(\alpha |V|^2\right) x, y\right\rangle\right|\right] \tag{4.7}$$

for any $x, y \in H$.

Proof. It follows by Theorem 5 by choosing $U = V^*$ and $U = V$. □

Example 1. *a. If* U, V *are normal operators on the Hilbert space* $(H; \langle\cdot,\cdot\rangle)$, $U^*V = VU^*$ *and* $\alpha > 0$ *then*

$$\begin{aligned} &|\langle \exp(\alpha(U^* + V)) x, y\rangle| \\ &\leq \frac{1}{2}\left[\left\langle \exp\left(\alpha |V|^2\right) x, x\right\rangle^{1/2}\left\langle \exp\left(\alpha |U|^2\right) y, y\right\rangle^{1/2} \right. \\ &\left. + |\langle \exp(\alpha U^*V) x, y\rangle|\right] \exp(\alpha) \end{aligned} \tag{4.8}$$

for any $x, y \in H$.

b. If U, V *are normal operators on the Hilbert space* $(H; \langle\cdot,\cdot\rangle)$, $U^*V = VU^*$ *and* $\|U\|, \|V\| < 1$, $\alpha \in (0,1)$ *then*

$$\begin{aligned} &\left|\left\langle (1_H - \alpha U^*)^{-1}(1_H - \alpha V)^{-1} x, y\right\rangle\right| \\ &\leq \frac{1}{2}\left[\left\langle \left(1 - \alpha |V|^2\right)^{-1} x, x\right\rangle^{1/2}\left\langle \left(1 - \alpha |U|^2\right)^{-1} y, y\right\rangle^{1/2} \right. \\ &\left. + \left|\left\langle (1 - \alpha U^*V)^{-1} x, y\right\rangle\right|\right](1 - \alpha)^{-1} \end{aligned} \tag{4.9}$$

for any $x, y \in H$.

By taking the supremum in (4.3) over $\|x\| = \|y\| = 1$ we can state the following norm inequality:

Corollary 5. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a power series with nonnegative coefficients* $a_n \geq 0$ *for* $n \in \mathbb{N}$ *and having the radius of convergence* $R > 0$ *or* $R = \infty$. *If* U, V *are normal operators on the Hilbert space* $(H; \langle\cdot,\cdot\rangle)$, $U^*V = VU^*$ *and* $\alpha > 0$ *such that* $\alpha < R$ *and* $\|U\|, \|V\| \leq 1$, *then*

$$\begin{aligned} &\|f(\alpha U^*) f(\alpha V)\| \\ &\leq \frac{1}{2} f(\alpha)\left[\left\|f\left(\alpha |V|^2\right)\right\|^{1/2}\left\|f\left(\alpha |U|^2\right)\right\|^{1/2} + \|f(\alpha U^*V)\|\right]. \end{aligned} \tag{4.10}$$

The reader may state various particular inequalities of interest. However the details are omitted here.

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