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SOME RESULTS ON SINGULAR VALUE INEQUALITIES OF COMPACT OPERATORS IN HILBERT SPACE

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Abstract. We prove several singular value inequalities for sum and product of compact operators in Hilbert space. Some of our results generalize the previous inequalities for operators. Also, applications of some inequalities are given.

1. Introduction

Let $B(H)$ stand for the C^*-algebra of all bounded linear operators on a complex separable Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and let $K(H)$ denote the two-sided ideal of compact operators in $B(H)$. For $A \in B(H)$, let $\|A\| = \sup\{\|Ax\| : \|x\| = 1\}$ denote the usual operator norm of A and $|A| = (A^*A)^{1/2}$ be the absolute value of A.
An operator $A \in B(H)$ is positive and write $A \geq 0$ if $\langle Ax, x \rangle \geq 0$ for all $x \in H$. We say $A \leq B$ whenever $B - A \geq 0$.
We consider the wide class of unitarily invariant norms $||| \cdot |||$. Each of these norms is defined on an ideal in $B(H)$ and it will be implicitly understood that when we talk of $|||T|||$, then the operator T belongs to the norm ideal associated with $||| \cdot |||$. Each unitarily invariant norm $||| \cdot |||$ is characterized by the invariance property $|||UTV||| = |||T|||$ for all operators T in the norm ideal associated with $||| \cdot |||$ and for all unitary operators U and V in $B(H)$. For $1 \leq p < \infty$, the Schatten p-norm of a compact operator A is defined by $\|A\|_p = (\operatorname{tr} |A|^p)^{1/p}$, where tr is the usual trace functional. Note that for $A \in K(H)$ we have, $\|A\| = s_1(A)$, and if A is a Hilbert-Schmidt operator, then $\|A\|_2 = (\sum_{j=1}^{\infty} s_j^2(A))^{1/2}$. These norms are special examples of the more general class of the Schatten p-norms, which are unitarily invariant [2].

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The direct sum $A \oplus B$ denotes the block diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ defined on $H \oplus H$, see [1, 15]. It is easy to see that

$$\|A \oplus B\| = \max(\|A\|, \|B\|), \tag{1.1}$$

and

$$\|A \oplus B\|_p = (\|A\|_p^p + \|B\|_p^p)^{1/p}. \tag{1.2}$$

We denote the singular values of an operator $A \in K(H)$ as $s_1(A) \geq s_2(A) \geq \ldots$ are the eigenvalues of the positive operator $|A| = (A^*A)^{1/2}$ and eigenvalues of the self-adjoint operator A denote as $\lambda_1 \geq \lambda_2 \geq \ldots$ which repeated accordingly to multiplicity.

There is a one-to-one correspondence between symmetric gauge functions defined on sequences of real numbers and unitarily invariant norms defined on norm ideals of operators. More precisely, if $|||\cdot|||$ is unitarily invariant norm, then there exists a unique symmetric gauge function Φ such that

$$|||A||| = \Phi(s_1(A), s_2(A), \ldots),$$

for every operator $A \in K(H)$. Let $A \in K(H)$, and if $U, V \in B(H)$ are unitarily operators, then

$$s_j(UAV) = s_j(A),$$

for $j = 1, 2, \ldots$ and so unitarily invariant norms satisfies the invariance property

$$|||UAV||| = |||A|||.$$

In this paper, we obtain some inequalities for sum and product of operators. In section 2, we give the following inequality for A and B in $K(H)$

$$s_j(A+B) \leq s_j\left((|A|^{2\alpha} + |B|^{2\alpha}) \oplus (|A^*|^{2(1-\alpha)} + |B^*|^{2(1-\alpha)})\right),$$

for $j = 1, 2, \ldots$, where $0 \leq \alpha \leq 1$.

Moreover, for $A, B, X \in B(H)$ such that X is compact operator, then we show that

$$s_j\left(AXB^*\right) \leq s_j\left(A^* f(|X|)^2 A \oplus B^* g(|X^*|)^2 B\right),$$

for $j = 1, 2, \ldots$ which implies the following inequality when $f(t) = t^{\frac{1}{2}}$ and $g(t) = t^{\frac{1}{2}}$

$$s_j\left(AXB^*\right) \leq s_j\left(A^*|X|A \oplus B^*|X^*|B\right), \tag{1.3}$$

for $j = 1, 2, \ldots$.
As an application we obtain the following results.
Hirzallah and Kittaneh have shown the following inequality in [9], for two $n \times n$ matrices A and B

$$s_j(AB^* + BA^*) \leq s_j((A^*A + B^*B) \oplus (A^*A + B^*B)), \quad 1 \leq j \leq n.$$

Now, by inequality (1.3) the following inequality is obtained for $A, B \in K(H)$

$$s_j(AB^* - BA^*) \leq s_j((AA^* + BB^*) \oplus (AA^* + BB^*)),$$

for $j = 1, 2, \ldots$.
Our other inequality which is proved for $A, B, X \in B(H)$ such that A and B are compact is

$$s_j(AXB^*) \leq \|X\| s_j^2(A \oplus B),$$

for $j = 1, 2, \ldots$.
Moreover, for $\alpha \in \mathbb{R}$ and positive operator X, we have

$$s_j(AXB^*) \leq s_j(AX^{2\alpha}A^* \oplus BX^{2(1-\alpha)}B^*),$$

for all $j = 1, 2, \ldots$.

In section 3, we investigate another condition for the following inequality

$$\frac{1}{\sqrt{2}} s_j(A + B) \leq s_j(A + iB),$$

which has been proved by Bhatia and Kittaneh in [7] and asserts that if A and B are two $n \times n$ positive semi-definite matrices, then

$$\frac{1}{\sqrt{2}} s_j(A+B) \leq s_j(A+iB), \quad 1 \leq j \leq n. \tag{1.4}$$

In section 4, we give some inequalities for normal operators. We prove the following inequality

$$\begin{aligned} \frac{1}{\sqrt{2}} s_j\left(\oplus_{i=1}^n (\Re(A_i)+\Im(A_i))\right) &\leq s_j(\oplus_{i=1}^n A_i) \\ &\leq s_j\left(\oplus_{i=1}^n (|\Re(A_i)|+|\Im(A_i)|)\right), \end{aligned} \tag{1.5}$$

for $j = 1, 2, \ldots$ where $A_1, A_2, \ldots, A_n$ are compact normal operators. As an application of (1.5), we find the upper and lower bound for $\oplus_{i=1}^n (\Re(A_i)+\Im(A_i)$ for arbitrary operators $A_i \in K(H)$.
Also some inequalities are obtained for normal operators.

2. Some singular value inequalities for sum and product of operators

In this section we give inequalities for singular value of operators. Also, some norm inequalities are obtained as an application.
First we should remind the following inequalities. We apply inequalities (2.1) and (2.3) in our proofs.

The following inequality due to Tao [13] asserts that if $A, B, C \in K(H)$ such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$, then

$$2s_j(B) \leq s_j \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}, \tag{2.1}$$

for $j = 1, 2, \ldots$.
Here, we give another proof for above inequality.
Let $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$ then $\begin{bmatrix} A & -B \\ -B^* & C \end{bmatrix} \geq 0$ and have the same singular values (see[1, Theorem 2.1]). So, we can write

$$\begin{bmatrix} 0 & 2B \\ 2B^* & 0 \end{bmatrix} \leq \begin{bmatrix} A & B \\ B^* & C \end{bmatrix},$$

and

$$\begin{bmatrix} 0 & -2B \\ -2B^* & 0 \end{bmatrix} \leq \begin{bmatrix} A & -B \\ -B^* & C \end{bmatrix}.$$

On the other hand, we know that for every self-adjoint compact operator X we have $s_j(X) \leq \lambda_j(X \oplus -X)$, for all $j = 1, 2, \ldots$. By using of this fact we obtain

$$\begin{aligned} s_j\left(\begin{bmatrix} 0 & 2B \\ 2B^* & 0 \end{bmatrix}\right) &= \lambda_j\left(\begin{bmatrix} 0 & 2B \\ 2B^* & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & -2B \\ -2B^* & 0 \end{bmatrix}\right) \\ &\leq \lambda_j\left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \oplus \begin{bmatrix} A & -B \\ -B^* & C \end{bmatrix}\right) \\ &= s_j\left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \oplus \begin{bmatrix} A & -B \\ -B^* & C \end{bmatrix}\right). \end{aligned}$$

So, we obtain

$$s_j\left(\begin{bmatrix} 0 & 2B \\ 2B^* & 0 \end{bmatrix}\right) \leq s_j\left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \oplus \begin{bmatrix} A & -B \\ -B^* & C \end{bmatrix}\right).$$

Equivalently,

$$2s_j(B \oplus B^*) \leq \left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \oplus \begin{bmatrix} A & -B \\ -B^* & C \end{bmatrix}\right).$$

Since $s_j(B) = s_j(B^*)$ and $s_j\left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\right) = s_j\left(\begin{bmatrix} A & -B \\ -B^* & C \end{bmatrix}\right)$, we have

$$2s_j(B) \leq s_j\left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\right).$$

In [1, Remark 2.2], Audeh and Kittaneh proved that for every $A, B, C \in K(H)$ such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$, then

$$s_j\left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\right) \leq 2s_j(A \oplus C), \tag{2.2}$$

for $j = 1, 2, \ldots$. Therefore, by inequality (2.1) we have the following inequality

$$s_j(B) \leq s_j(A \oplus C), \tag{2.3}$$

for $j = 1, 2, \ldots$. Since every unitarily invariant norm is a monotone function of the singular values of an operator, we can write

$$\left|\left|\left| \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \right|\right|\right| \leq 2s_j|||A \oplus C|||. \tag{2.4}$$

We can obtain the reverse of inequality (2.4) for arbitrary operators $X, Y \in B(H)$ by pointing out the following inequality holds because of norm property

$$|||X + Y||| \leq |||X||| + |||Y|||.$$

Replace X and Y by $X - Y$ and $X + Y$, respectively. We have

$$2|||X||| \leq |||X - Y||| + |||X + Y|||,$$

for all $X, Y \in B(H)$.

Let $X = \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$ in above inequality. So,

$$\begin{aligned} 2\left|\left|\left| \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} \right|\right|\right| &\leq \left|\left|\left| \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \right|\right|\right| + \left|\left|\left| \begin{bmatrix} A & -B \\ -B^* & C \end{bmatrix} \right|\right|\right| \\ &= 2\left|\left|\left| \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \right|\right|\right|. \end{aligned}$$

Hence,

$$|||A \oplus C||| \leq \left|\left|\left| \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \right|\right|\right|,$$

for all $A, B, C \in B(H)$. $\begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}$ is called a *pinching* of $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$.

For operator norm we have

$$\max\{\|A\|, \|C\|\} \leq \left\| \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \right\|.$$

Here we give a generalization of the inequality which has been proved by Bhatia and Kittaneh in [6]. They have shown that if A and B are two $n \times n$ matrices, then

$$s_j(A+B) \le s_j\left((|A|+|B|) \oplus (|A^*|+|B^*|)\right),$$

for $1 \le j \le n$.
For giving a generalization of above inequality, we need the following lemmas.

In the rest of this section, we always assume that f and g are non-negative functions on $[0,\infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0,\infty)$.

The following lemma is due to Kittaneh [10].

Lemma 2.1. *Let $A, B,$ and C be operators in $B(H)$ such that A and B are positive and $BC = CA$. If $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}$ is positive in $B(H \oplus H)$, then $\begin{bmatrix} f(A)^2 & C^* \\ C & g(B)^2 \end{bmatrix}$ is also positive.*

Let T be an operator in $B(H)$. We know that $\begin{bmatrix} |T| & T^* \\ T & |T^*| \end{bmatrix} \ge 0$, if T is normal then we have $\begin{bmatrix} |T| & T^* \\ T & |T| \end{bmatrix} \ge 0$, (see [5]).

Lemma 2.2. *Let A be an operator in $B(H)$. Then we have*

$$\begin{bmatrix} |A|^{2\alpha} & A^* \\ A & |A^*|^{2(1-\alpha)} \end{bmatrix} \ge 0, \tag{2.5}$$

where $0 \le \alpha \le 1$.

Proof. It is easy to check that $A|A|^2 = |A^*|^2 A$, then we have $A|A| = |A^*|A$ for $A \in B(H)$. Now by making use of Lemma 2.1, for $f(t) = t^\alpha$

and $g(t) = t^{1-\alpha}$, $0 \leq \alpha \leq 1$, and positivity of $\begin{bmatrix} |A| & A^* \\ A & |A^*| \end{bmatrix}$, we obtain the result. □

Theorem 2.3. *Let A and B be two operators in $K(H)$. Then we have*

$$s_j(A+B) \leq s_j\left((|A|^{2\alpha} + |B|^{2\alpha}) \oplus (|A^*|^{2(1-\alpha)} + |B^*|^{2(1-\alpha)})\right),$$

for $j = 1, 2, \ldots$ where $0 \leq \alpha \leq 1$.

Proof. Since sum of two positive operator is positive, Lemma 2.2 implies that

$$\begin{bmatrix} |A|^{2\alpha} + |B|^{2\alpha} & A^* + B^* \\ A + B & |A^*|^{2(1-\alpha)} + |B^*|^{2(1-\alpha)} \end{bmatrix} \geq 0,$$

By inequality (2.3) we have the result. □

Corollary 2.4. *Let A and B be two operators in $K(H)$. Then we have*

$$s_j(A+B) \leq s_j\left((|A| + |B|) \oplus (|A^*| + |B^*|)\right),$$

for $j = 1, 2, \ldots$.

Proof. Let $\alpha = \frac{1}{2}$ in Theorem 2.3. □

It is easy to see that if A and B are normal operator in $K(H)$, then we have

$$s_j(A+B) \leq s_j\left((|A| + |B|) \oplus (|A| + |B|)\right),$$

for $j = 1, 2, \ldots$.

On the other hand, for $\alpha = 1$ in Theorem 2.3, we have

$$\begin{aligned} s_j(A+B) &\leq s_j(|A|^2 + |B|^2 \oplus 2I) \\ &= s_j(|A|^2 + |B|^2) \cup s_j(2I) \\ &= s_j(A^*A + B^*B) \cup s_j(2I), \end{aligned}$$

for $j = 1, 2, \ldots$.

Theorem 2.5. *Let A,B and X be operators in $B(H)$ such that X is compact. Then we have the following*

$$s_j\left(AXB^*\right) \leq s_j\left(A^*f(|X|)^2A \oplus B^*g(|X^*|)^2B\right),$$

for $j = 1, 2, \ldots$.

Proof. Since $\begin{bmatrix} |X| & X^* \\ X & |X^*| \end{bmatrix} \geq 0$, by Lemma 2.1 we have

$$Y = \begin{bmatrix} f(|X|)^2 & X^* \\ X & g(|X^*|)^2 \end{bmatrix} \geq 0.$$

Let $Z = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Since Y is positive, we have

$$Z^*YZ = \begin{bmatrix} A^*f(|X|)^2A & A^*X^*B \\ B^*XA & B^*g(|X^*|)^2B \end{bmatrix} \geq 0.$$

Hence, by inequality (2.3), we have the desired result. □

In above theorem, let X be a normal operator. Then we have

$$s_j\left(AXB^*\right) \leq s_j\left(A^*f(|X|)^2A \oplus B^*g(|X|)^2B\right),$$

for $j = 1, 2, \ldots$.

Corollary 2.6. *Let A, B and X be operators in $B(H)$ such that X is compact. Then we have*

$$s_j\left(AXB^*\right) \leq s_j\left(A^*|X|A \oplus B^*|X^*|B\right),$$

for $j = 1, 2, \ldots$.

Proof. Let $f(t) = t^{\frac{1}{2}}$ and $g(t) = t^{\frac{1}{2}}$ in Theorem 2.5. □

Here, we apply above corollary to show that singular values of AXB^* are dominated by singular values of $\|X\|(A \oplus B)$. For our proof we need the following lemma.

Lemma 2.7. [2, p. 75] *Let $A, B \in K(H)$, then*

$$s_j(AB) \leq \|A\|s_j(B),$$

or

$$s_j(AB) \leq \|B\| s_j(A),$$

for $j = 1, 2, \ldots$.

Theorem 2.8. *Let $A, B, X \in B(H)$ such that A and B are arbitrary compact. Then, we have*

$$s_j(AXB^*) \leq \|X\| s_j^2(A \oplus B),$$

for $j = 1, 2, \ldots$.

Proof. From Corollary 2.6 we have

$$\begin{aligned}
s_j(AXB^*) &\leq s_j(A^*|X|A \oplus B^*|X^*|B) \\
&= s_j\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^* \begin{bmatrix} |X| & 0 \\ 0 & |X^*| \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \\
&= s_j\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^* \begin{bmatrix} |X|^{\frac{1}{2}} & 0 \\ 0 & |X^*|^{\frac{1}{2}} \end{bmatrix}^* \begin{bmatrix} |X|^{\frac{1}{2}} & 0 \\ 0 & |X^*|^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \\
&= s_j\left(\left(\begin{bmatrix} |X|^{\frac{1}{2}} & 0 \\ 0 & |X^*|^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right)^* \left(\begin{bmatrix} |X|^{\frac{1}{2}} & 0 \\ 0 & |X^*|^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right)\right) \\
&= s_j\left(\left|\begin{bmatrix} |X|^{\frac{1}{2}} & 0 \\ 0 & |X^*|^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right|^2\right) \\
&= s_j^2\left(\begin{bmatrix} |X|^{\frac{1}{2}} & 0 \\ 0 & |X^*|^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \\
&\leq \left\|\begin{bmatrix} |X|^{\frac{1}{2}} & 0 \\ 0 & |X^*|^{\frac{1}{2}} \end{bmatrix}\right\|^2 s_j^2(A \oplus B) \\
&= \|X\| s_j^2(A \oplus B),
\end{aligned}$$

for $j = 1, 2, \ldots$. The last inequality follows by Lemma 2.7. □

In Theorem 2.8, let A and B be positive operators in $K(H)$. Then we have

$$s_j(A^{\frac{1}{2}} X B^{\frac{1}{2}}) \leq \|X\| s_j(A \oplus B),$$

for $j = 1, 2, \ldots$.

Corollary 2.9. *Let A and B be two operators in $K(H)$. Then we have*

$$s_j(AB^*) \leq s_j\left(A^*A \oplus B^*B\right), \tag{2.6}$$

for $j = 1, 2, \ldots$.

Proof. Let $X = I$ in Corollary 2.6. □

Moreover, we can write inequality (2.6) in the following form

$$\begin{aligned} s_j(AB^*) &\leq s_j(|A|^2 \oplus |B|^2) \\ &= s_j^2(|A| \oplus |B|) = s_j^2(A \oplus B), \end{aligned}$$

for $j = 1, 2, \ldots$.
We should note here that inequality (2.6) can be obtained by Theorem 1 in [4] and Corollary 2.2 in [9].
Here, we give two results of Corollary 2.9. As the first application, let $A = \begin{bmatrix} X & Y \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} Y & -X \\ 0 & 0 \end{bmatrix}$, such that $X, Y \in K(H)$ then by easy computations we have

$$s_j(XY^* - YX^*) \leq s_j((XX^* + YY^*) \oplus (XX^* + YY^*)),$$

for $j = 1, 2, \ldots$.

For obtaining second application, replace A and B in (2.6) by AX^α and $B(X^{(1-\alpha)})^*$ respectively, where X is a compact positive operator and $\alpha \in \mathbb{R}$. So, we have

$$\begin{aligned}
s_j(AXB^*) &\leq s_j\left((X^\alpha)^* A^* A X^\alpha \oplus X^{(1-\alpha)} B^* B (X^{(1-\alpha)})^*\right) \\
&= s_j\left(\begin{bmatrix} X^\alpha A^* A X^\alpha & 0 \\ 0 & X^{(1-\alpha)} B^* B X^{(1-\alpha)} \end{bmatrix}\right) \\
&= s_j\left(\begin{bmatrix} X^\alpha A^* & 0 \\ 0 & X^{(1-\alpha)} B^* \end{bmatrix}\begin{bmatrix} A X^\alpha & 0 \\ 0 & B X^{(1-\alpha)} \end{bmatrix}\right) \\
&= s_j\left(\begin{bmatrix} A X^\alpha & 0 \\ 0 & B X^{(1-\alpha)} \end{bmatrix}\begin{bmatrix} X^\alpha A^* & 0 \\ 0 & X^{(1-\alpha)} B^* \end{bmatrix}\right) \\
&= s_j\left(\begin{bmatrix} A X^{2\alpha} A^* & 0 \\ 0 & B X^{2(1-\alpha)} B^* \end{bmatrix}\right) \\
&= s_j(A X^{2\alpha} A^* \oplus B X^{2(1-\alpha)} B^*),
\end{aligned}$$

for all $j = 1, 2, \ldots$.
Finally, we have

$$s_j(AXB^*) \leq s_j(A X^{2\alpha} A^* \oplus B X^{2(1-\alpha)} B^*), \tag{2.7}$$

for all $j = 1, 2, \ldots$.
By a similar proof of Theorem 2.8 to inequality (2.7), we obtain

$$s_j(AXB^*) \leq \max\{\|X^{2\alpha}\|, \|X^{2(1-\alpha)}\|\} s_j^2(A \oplus B),$$

for all $j = 1, 2, \ldots$.
In above inequality, for positive operators A and B in $K(H)$ we have

$$s_j(A^{\frac{1}{2}} X B^{\frac{1}{2}}) \leq \max\{\|X^{2\alpha}\|, \|X^{2(1-\alpha)}\|\} s_j(A \oplus B)$$

for all $j = 1, 2, \ldots$.

3. Some singular value inequalities for $A + B$ and $A + iB$

In this section some inequalities for $A + B$ and $A + iB$ are obtained. Our main result in this section is as follows.

Theorem 3.1. *Let A, B be two operators in $K(H)$ such that A^*B is self-adjoint. Then we have*

$$\frac{1}{\sqrt{2}} s_j(A+B) \leq s_j(A+iB), \tag{3.1}$$

for $j = 1, 2, \ldots$.

Proof. It is easy to check that

$$0 \leq (A+B)^*(A+B) \leq 2(A^*A + B^*B).$$

So,

$$\sqrt{(A+B)^*(A+B)} \leq \sqrt{2}(\sqrt{A^*A + B^*B}).$$

Weyl's monotonicity principle implies

$$s_j(|A+B|) = s_j(\sqrt{(A+B)^*(A+B)}) \leq \sqrt{2} s_j(\sqrt{A^*A + B^*B})$$

for $j = 1, 2, \ldots$.
On the other hand we have

$$\begin{aligned} s_j(A+iB) &= s_j(|A+iB|) \\ &= s_j(\sqrt{(A+iB)^*(A+iB)}) \\ &= s_j(\sqrt{(A^* - iB^*)(A+iB)}) \\ &= s_j(\sqrt{A^*A + i(A^*B - B^*A) + B^*B}) \\ &= s_j(\sqrt{A^*A + B^*B}), \end{aligned}$$

for $j = 1, 2, \ldots$. So, we have inequality (3.1).

□

In above theorem we utilized $0 \leq (A+B)^*(A+B) \leq 2(A^*A + B^*B)$ for proving our inequality. One may consider $0 \leq (A-B)^*(A-B) \leq 2(A^*A + B^*B)$, by a similar proof we can obtain the following inequality for $A, B \in K(H)$ such that A^*B is self-adjoint,

$$\frac{1}{\sqrt{2}} s_j(A-B) \leq s_j(A+iB), \tag{3.2}$$

for $j = 1, 2, \ldots$.

In inequality (3.1) and (3.2), we can change B with $-B$. In this case we have the following

$$\frac{1}{\sqrt{2}} s_j(A - B) \leq s_j(A - iB), \tag{3.3}$$

for $j = 1, 2, \ldots$. and

$$\frac{1}{\sqrt{2}} s_j(A + B) \leq s_j(A - iB), \tag{3.4}$$

for $j = 1, 2, \ldots$.

The following example shows that self-adjoint condition is necessary.

Example 3.2. Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ then a calculation shows

$$s_2(A + B) = 1 > s_2(A + iB) \approx 0.41.$$

Let A and B be two operators in $K(H)$ such that A^*B and AB^* are self-adjoint operators. By this property, it is easy to check that A^*A and B^*B are commutative. So, we have

$$\sqrt{A^*A + B^*B} \leq \sqrt{A^*A} + \sqrt{B^*B}.$$

Now Weyl's monotonicity principle and the proof of Theorem 3.1 imply that

$$s_j(A + iB) \leq s_j(|A| + |B|) \tag{3.5}$$

for $j = 1, 2, \ldots$.

Corollary 3.3. *Let A be in $K(H)$, then*

$$\sqrt{2} s_j(A) \leq s_j(A + iA) \leq 2 s_j(A)$$

for $j = 1, 2, \ldots$.

Proof. It is an immediate consequence of Theorem 3.1 and inequality (3.5), by assuming $B = A$. □

Corollary 3.4. *Let $k \geq 1$ be an integer. If A and B be two self-adjoint operators in $K(H)$, then*

$$\frac{1}{\sqrt{2}} s_j(A + (BA)^k) \leq s_j(|A| + |(BA)^k|)$$

and

$$\frac{1}{\sqrt{2}} s_j(A - (BA)^k) \leq s_j(|A| + |(BA)^k|)$$

for $j = 1, 2, \ldots$.

Proof. By inequalities (3.1), (3.2), (3.5) and the fact that if A and B are self-adjoint operators then $A^*(BA)^k = ((BA)^k)^* A$ and $A((BA)^k)^* = (BA)^k A^*$, which implies the desired results. □

4. Some singular value inequalities for normal operators

Here we give some results for compact normal operators. For every operator A, the Cartesian decomposition is to write $A = \Re(A) + i\Im(A)$, where $\Re(A) = \frac{A+A^*}{2}$ and $\Im(A) = \frac{A-A^*}{2i}$. If A is normal operator then $\Re(A)$ and $\Im(A)$ commute together and vice versa.

Theorem 4.1. *Let $A_1, A_2, \ldots, A_n$ be normal operators in $K(H)$. Then we have*

$$\begin{aligned} \frac{1}{\sqrt{2}} s_j\left(\oplus_{i=1}^n (\Re(A_i) + \Im(A_i))\right) &\leq s_j(\oplus_{i=1}^n A_i) \\ &\leq s_j\left(\oplus_{i=1}^n (|\Re(A_i)| + |\Im(A_i)|)\right), \end{aligned}$$

for $j = 1, 2, \ldots$.

Proof. Let $A_1, A_2, \ldots, A_n$ be normal operators, then

$$\oplus_{i=1}^n A_i = \begin{pmatrix} A_1 & & 0 & \\ & A_2 & & \\ & & \ddots & \\ & 0 & & A_n \end{pmatrix}$$

is normal, so we have

$$(\oplus_{i=1}^n \Re(A_i))(\oplus_{i=1}^n \Im(A_i)) = (\oplus_{i=1}^n \Im(A_i))(\oplus_{i=1}^n \Re(A_i))).$$

By above equation, we obtain the following

$$\sqrt{(\oplus_{i=1}^n A_i)^*(\oplus_{i=1}^n A_i)} = \sqrt{(\oplus_{i=1}^n \Re(A_i))^2 + (\oplus_{i=1}^n \Im(A_i))^2}.$$

So

$$\begin{aligned} s_j(\oplus_{i=1}^n A_i) &= s_j(| \oplus_{i=1}^n A_i|) \\ &= s_j\left(\sqrt{(\oplus_{i=1}^n A_i)^*(\oplus_{i=1}^n A_i)}\right) \\ &= s_j\left(\sqrt{(\oplus_{i=1}^n \Re(A_i))^2 + (\oplus_{i=1}^n \Im(A_i))^2}\right), \end{aligned}$$

for $j = 1, 2, \ldots$.
By using Weyl's monotonicity principle [2] and the inequality

$$\sqrt{(\oplus_{i=1}^n \Re(A_i))^2 + (\oplus_{i=1}^n \Im(A_i))^2} \leq | \oplus_{i=1}^n \Re(A_i)| + | \oplus_{i=1}^n \Im(A_i)|,$$

we have the following

$$s_j\left(\sqrt{(\oplus_{i=1}^n \Re(A_i))^2 + (\oplus_{i=1}^n \Im(A_i))^2}\right) \leq s_j(|\oplus_{i=1}^n \Re(A_i)|+|\oplus_{i=1}^n \Im(A_i)|),$$

for $j = 1, 2, \ldots$. Now for proving left side inequality, we recall the following inequality

$$0 \leq (\Re(A_i) + \Im(A_i)^*(\Re(A_i) + \Im(A_i) \leq 2(\Re(A_i)^2 + \Im(A_2)^2).$$

Therefore, by using the Weyl's monotonicity principle we can write

$$s_j\left(\sqrt{((\oplus_{i=1}^n \Re(A_i)) + (\oplus_{i=1}^n \Im(A_i)))^* ((\oplus_{i=1}^n \Re(A_i)) + (\oplus_{i=1}^n \Im(A_i)))}\right),$$

which is less than

$$\sqrt{2} s_j\left(\sqrt{(\oplus_{i=1}^n \Re(A_i))^2 + (\oplus_{i=1}^n \Im(A_i))^2}\right).$$

for $j = 1, 2, \ldots$. Therefore,

$$\begin{aligned} s_j((\oplus_{i=1}^n \Re(A_i)) + (\oplus_{i=1}^n \Im(A_i))) &= s_j(|(\oplus_{i=1}^n \Re(A_i)) + (\oplus_{i=1}^n \Im(A_i))|) \\ &\leq \sqrt{2} s_j(\sqrt{(\oplus_{i=1}^n \Re(A_i))^2 + (\oplus_{i=1}^n \Im(A_i))^2}). \end{aligned}$$

for $j = 1, 2, \ldots$. □

The following example shows that normal condition is necessary.

Example 4.2. Let $A = \begin{bmatrix} -1+i & 1 \\ i & 1+2i \end{bmatrix}$, then a calculation shows

$$s_2(\Re(A) + i\Im(A)) \approx 1.34 > s_2(|\Re(A)| + |\Im(A)|) \approx 1.27.$$

Corollary 4.3. *Let A be a normal operator in $K(H)$. Then we have*

$$(1/\sqrt{2})s_j(\Re(A) + \Im(A)) \leq s_j(A) \leq s_j(|\Re(A)| + |\Im(A)|),$$

for $j = 1, 2, \ldots$.

For each complex number $x = a+ib$, we know the following inequality holds

$$\frac{1}{\sqrt{2}}|a+b| \leq |x| \leq |a| + |b|. \tag{4.1}$$

Now, by applying Corollary 4.3, we can obtain operator version of inequality (4.1).

As mentioned in Introduction, in the following theorem we determine the upper and lower bound for $A + iA^*$.
In [12], we have proved the following theorem for an arbitrary matrix.

Theorem 4.4. *Let A be a $n \times n$ complex matrix. Then*

$$\sqrt{2}s_1(\Re(A) + \Im(A)) \leq s_1(A + iA^*) \leq 2s_1(\Re(A) + \Im(A)).$$

Here, we prove the above theorem for compact operators. The proof of the following theorem is similar to Theorem 4.4, but for reader's convenience we provide the proof.

Theorem 4.5. *Let $A_1, A_2, \ldots, A_n$ be in $K(H)$. Then*

$$\begin{aligned}\sqrt{2}s_j\left(\oplus_{i=1}^n(\Re(A_i) + \Im(A_i))\right) &\leq s_j(\oplus_{i=1}^n(A_i + iA_i^*)) \\ &\leq 2s_j(\oplus_{i=1}^n(\Re(A_i) + \Im(A_i))),\end{aligned}$$

for $j = 1, 2, \ldots$.

Proof. Note that $A_i + iA_i^*$ is normal operator for $i = 1, \dots, n$, so $T = \oplus_{i=1}^n (A_i + iA_i^*)$ is normal. On the other hand, we can write $T = \Re(T) + i\Im(T)$ where

$$\Re(T) = (\oplus_{i=1}^n (A_i + A_i^*) + i \oplus_{i=1}^n (A_i^* - A_i))/2,$$

$$\Im(T) = (\oplus_{i=1}^n (A_i - A_i^*) + i \oplus_{i=1}^n (A_i^* + A_i))/2i.$$

It is enough to compare $\Re(T)$ and $\Im(T)$ to see $\Re(T) = \Im(T)$. So

$$\Re(T) + \Im(T) = \oplus_{i=1}^n (A_i + A_i^*) + i \oplus_{i=1}^n (A_i^* - A_i). \tag{4.2}$$

Now apply Theorem 4.1, we have

$$\begin{aligned} (1/\sqrt{2}) s_j(\Re(T) + \Im(T)) &\leq s_j(\Re(T) + i\Im(T)) \\ &\leq s_j(|\Re(T)| + |\Im(T)|), \end{aligned} \tag{4.3}$$

for $j = 1, 2, \dots$. Put (4.2), $\Re(T) + i\Im(T) = \oplus_{i=1}^n (A_i + iA_i^*)$ and $\Re(T)$ in (4.3) to obtain

$$(1/\sqrt{2}) s_j(\oplus_{i=1}^n (A_i + A_i^*) + i \oplus_{i=1}^n (A_i^* - A_i)) \leq s_j(\oplus_{i=1}^n (A_i + iA_i^*)), \tag{4.4}$$

and

$$\begin{aligned} s_j(\oplus_{i=1}^n (A_i + iA_i^*)) &\leq 2 s_j(\oplus_{i=1}^n (A_i + A_i^*)/2 + i \oplus_{i=1}^n (A_i^* - A_i)/2) \\ &= s_j(\oplus_{i=1}^n (A_i + A_i^*) + i \oplus_{i=1}^n (A_i^* - A_i)), \end{aligned}$$

for $j = 1, 2, \dots$. By writing $\Re(\oplus_{i=1}^n A_i) = \oplus_{i=1}^n (A_i + A_i^*)/2$ and $\Im(\oplus_{i=1}^n A_i) = \oplus_{i=1}^n (A_i - A_i^*)/2i$ we have

$$\begin{aligned} (1/\sqrt{2}) s_j(2\Re(\oplus_{i=1}^n A_i) + 2\Im(\oplus_{i=1}^n A_i)) &\leq s_j(\oplus_{i=1}^n (A_i + iA_i^*)) \\ &\leq s_j(2\Re(\oplus_{i=1}^n A_i) + 2\Im(\oplus_{i=1}^n A_i)), \end{aligned}$$

for $j = 1, 2, \dots$. Finally

$$\begin{aligned} \sqrt{2} s_j\left(\oplus_{i=1}^n (\Re(A_i) + \Im(A_i))\right) &\leq s_j(\oplus_{i=1}^n (A_i + iA_i^*)) \\ &\leq 2 s_j(\oplus_{i=1}^n (\Re(A_i) + \Im(A_i))), \end{aligned}$$

for $j = 1, 2, \dots$. □

By applying inequality (2.1), Audeh and Kittaneh have proved in [1] that if $A, B \in K(H)$, such that A is self-adjoint, $B \geq 0$ and $\pm A \leq B$, then

$$2s_j(A) \leq s_j((B+A) \oplus (B-A)), \tag{4.5}$$

for $j = 1, 2, \ldots$.

Here we give another kind of above inequality.

Theorem 4.6. *Let* $\begin{bmatrix} A & B \\ B^* & A \end{bmatrix} \geq 0$, *such that* $A, B \in K(H)$ *and* $B^* = -B$, *then we have*

$$2s_j(B) \leq s_j((A+iB) \oplus (A-iB)),$$

for $j = 1, 2, \ldots$.

Proof. Let $X = \begin{bmatrix} A & B \\ B^* & A \end{bmatrix}$. If $U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & iI \\ iI & I \end{bmatrix}$ then U is unitary and

$$\begin{bmatrix} A+iB & 0 \\ 0 & A-iB \end{bmatrix} = U^* \begin{bmatrix} A & B \\ B^* & A \end{bmatrix} U.$$

So, $\begin{bmatrix} A+iB & 0 \\ 0 & A-iB \end{bmatrix}$ is positive since it is unitarily equivalent with $\begin{bmatrix} A & B \\ B^* & A \end{bmatrix}$ and have the same singular values. By the inequality (2.1) we have

$$\begin{aligned} 2s_j(B) \leq s_j(X) &= s_j\left(\begin{bmatrix} A+iB & 0 \\ 0 & A-iB \end{bmatrix}\right) \\ &= s_j((A+iB) \oplus (A-iB)), \end{aligned}$$

for $j = 1, 2, \ldots$. □

Note that if $A^* = -A$ then $\Re(A) = 0$.

Theorem 4.7. *Let $A \in K(H)$ and* $\begin{bmatrix} A & A \\ A^* & A \end{bmatrix} \geq 0$ *such that $A^* = -A$, then*

$$s_j(A) \leq s_j\left(\Im(A) \oplus \Im(A)\right) \tag{4.6}$$

for $j = 1, 2, \ldots$.
In particular,

$$|||A||| \leq |||\Im(A) \oplus \Im(A)|||,$$

$$\|A\|_p \leq (\|\Im(A)\|_p^p + \|\Im(A)\|_p^p)^{1/p} = 2^{\frac{1}{p}} \|\Im(A)\|_p,$$

and

$$\|A\| \leq \max\{\|\Im(A)\|, \|\Im(A)\|\} = \|\Im(A)\|.$$

Proof. By using Theorem 4.6, we have

$$\begin{aligned} 2s_j(A) &\leq s_j((A + iA) \oplus (A - iA)) \\ &= s_j((A - iA^*) \oplus (A + iA^*)), \end{aligned}$$

for $j = 1, 2, \ldots$. Let $X = \begin{bmatrix} A - iA^* & 0 \\ 0 & A + iA^* \end{bmatrix}$, since $A - iA^*$ and $A + iA^*$ are normal for $A \in K(H)$, by applying Theorem 4.1 we have the following

$$\begin{aligned} s_j((A - iA^*) \oplus (A + iA^*)) &= s_j\left(\begin{bmatrix} A - iA^* & 0 \\ 0 & A + iA^* \end{bmatrix}\right) \\ &\leq s_j(|\Re(X)| + |\Im(X)|), \end{aligned}$$

for $j = 1, 2, \ldots$. where

$$\Re(X) = \begin{bmatrix} -\Im(A) & 0 \\ 0 & \Im(A) \end{bmatrix}$$

and

$$\Im(X) = \begin{bmatrix} \Im(A) & 0 \\ 0 & \Im(A) \end{bmatrix}.$$

So,

$$\begin{aligned} s_j(|\Re(X)|+|\Im(X)|) &= s_j\left(\begin{bmatrix} |\Im(A)| & 0 \\ 0 & |\Im(A)| \end{bmatrix} + \begin{bmatrix} |\Im(A)| & 0 \\ 0 & |\Im(A)| \end{bmatrix}\right) \\ &= 2s_j(|\Im(A)| \oplus |\Im(A)|) \\ &= 2s_j(\Im(A) \oplus \Im(A)), \end{aligned}$$

for $j = 1, 2, \ldots$. we have

$$s_j(A) \leq s_j\left(\Im(A) \oplus \Im(A)\right).$$

Other inequalities follow from (4.6), (1.1) and (1.2). □

In [6] Bhatia and Kittaneh have proved for two $n \times n$ complex matrices A and B we have

$$s_j(A^*B + B^*A) \leq s_j((A^*A + B^*B) \oplus (A^*A + B^*B)), \quad 1 \leq j \leq n.$$

In particular for Hermitian A and B we have the following

$$s_j(AB + BA) \leq s_j((A^2 + B^2) \oplus (A^2 + B^2)), \quad 1 \leq j \leq n.$$

Also Hirzallah and Kittaneh have shown in [9] that we have

$$s_j(AB^* + BA^*) \leq s_j((A^*A + B^*B) \oplus (A^*A + B^*B)), \quad 1 \leq j \leq n.$$

Here we establish some results for compact operators in $B(H)$.

Theorem 4.8. *Let A and B be in $K(H)$. Then*

$$s_j(AB + BA) \leq s_j((A^*A + B^*B) \oplus (AA^* + BB^*)),$$

for $j = 1, 2, \ldots$.

Proof. Suppose $X = \begin{bmatrix} A & B \\ B^* & A^* \end{bmatrix}$. So we have,

$$XX^* = \begin{bmatrix} AA^* + BB^* & AB + BA \\ B^*A^* + A^*B^* & A^*A + B^*B \end{bmatrix}.$$

By using inequality (2.3), we obtain

$$s_j(AB + BA) \leq s_j((A^*A + B^*B) \oplus (AA^* + BB^*)),$$

for $j = 1, 2, \ldots$. □

Corollary 4.9. *Let A and B be two normal operators in $K(H)$. Then*

$$s_j(AB + BA) \leq s_j(AA^* + BB^* \oplus AA^* + BB^*),$$

for $j = 1, 2, \ldots$.

In the operator theory, several extensions of the notion of the normality are known [11]. One of the most important and most widely studied classes among them is the hyponormality (i.e., $TT^* \leq T^*T$)[8]. Recall that a compact hyp-normal operator in $B(H)$ is normal [14], so we can restate the results of Section 4 for compact hyponormal operators.

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